Math 224 **Spring 2018**
Homework 2 **Drew Armstrong**

---

1. Suppose that a fair s-sided die is rolled n times.

(a) If the i-th side is labeled a_i then we can think of the sample space S as the set of all words of length n from the alphabet $\{a_1, \ldots, a_s\}$. Find $\#S$.
(b) Let E be the event that "the 1st side shows up k_1 times, and ... and the s-th side shows up k_s times. Find $\#E$. [Hint: The elements of E are words of length n in which the letter a_i appears k_i times.]
(c) Compute the probability $P(E)$. [Hint: Since the die is fair you can assume that the outcomes in S are equally likely.]

2. In a certain state lottery four numbers are drawn (one and at a time and with replacement) from the set $\{1, 2, 3, 4, 5, 6\}$. You win if any permutation of your selected numbers is drawn. Rank the following selections in order of how likely each is to win.

(a) You select $1, 2, 3, 4$.
(b) You select $1, 3, 3, 5$.
(c) You select $4, 4, 6, 6$.
(d) You select $3, 5, 5, 5$.
(e) You select $4, 4, 4, 4$.

3. A bridge hand consists of 13 (unordered) cards taken (at random and without replacement) from a standard deck of 52. Recall that a standard deck contains 13 hearts and 13 diamonds (which are red cards), 13 clubs and 13 spades (which ard black cards). Find the probabilities of the following hands.

(a) 4 hearts, 3 diamonds, 2 spades and 4 clubs.
(b) 4 hearts, 3 diamonds and 6 black cards.
(c) 7 red cards and 6 black cards.

4. Two cards are drawn (in order and without replacement) from a standard deck of 52. Consider the events

$$A = \{\text{the first card is a heart}\}$$
$$B = \{\text{the second card is red}\}.$$

Compute the probabilities

$$P(A), \quad P(B), \quad P(B|A), \quad P(A \cap B), \quad P(A|B).$$

5. An urn contains 2 red and 2 green balls. Your friend selects two balls (at random and without replacement) and tells you that at least one of the balls is red. What is the probability that the other ball is also red?

6. There are two bowls on a table. The first bowl contains 3 chips and 3 green chips. The second bowl contains 2 red chips and 4 green chips. Your friend walks up to the table and chooses one chip at random. Consider the events

$$B_1 = \{\text{the chip comes from the first bowl}\},$$
$$B_2 = \{\text{the chip comes from the second bowl}\},$$
$$R = \{\text{the chip is red}\}.$$

(a) Compute the probabilities $P(R|B_1)$ and $P(R|B_2)$.
(b) Assuming that your friend is equally likely to choose either bowl (i.e., $P(B_1) = P(B_2) = 1/2$), compute the probability $P(R)$ that the chip is red.
(c) Compute $P(B_1|R)$. That is, assuming that your friend chose a red chip, what is the probability that they got it from the first bowl?

7. A diagnostic test is administered to a random person to determine if they have a certain disease. Consider the events

$$T = \{\text{the test returns positive}\},$$
$$D = \{\text{the person has the disease}\}.$$

Suppose that the test has the following "false positive" and "false negative" rates:

$$P(T|D') = 2\% \quad \text{and} \quad P(T'|D) = 3\%.$$

(a) For any events A, B recall that the Law of Total Probability says

$$P(A) = P(A \cap B) + P(A \cap B').$$

Use this to prove that

$$1 = P(B|A) + P(B'|A).$$

(b) Use part (a) to compute the probability $P(T|D)$ of a "true positive" and the probability $P(T'|D')$ of a "true negative."
(c) Assume that 10% of the population has the disease, so that $P(D) = 10\%$. In this case compute the probability $P(T)$ that a random person tests positive. [Hint: The Law of Total Probability says $P(T) = P(T \cap D) + P(T \cap D')$.]
(d) Suppose that a random person is tested and the test returns positive. Compute the probability $P(D|T)$ that this person actually has the disease. Is this a good test?

8. Consider a classroom containing n students. We ask each student for their birthday, which we record as a number from the set $\{1, 2, \ldots, 365\}$ (i.e., we ignore leap years). Let S be the sample space.

(a) Explain why $\#S = 365^n$.
(b) Let E be the event that {no two students have the same birthday}. Compute $\#E$.
(c) Assuming that all birthdays are equally likely, compute the probability of the event

$$E' = \{\text{at least two students have the same birthday}\}.$$

(d) Find the smallest value of n such that $P(E') > 50\%$.

9. It was not easy to find a formula for the entries of Pascal's Triangle. However, once we've found the formula it is not difficult to check that the formula is correct.

(a) Explain why $n! = n \times (n-1)!$.
(b) Use part (a) and the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ to prove that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$