Math 224 **Exam 2**
Spring 2018 **Thurs Mar 29**

Version A

Problem 1. Let X, Y be discrete random variables with joint pmf defined as follows:

$X \setminus Y$	1	2
1	2/9	1/9
2	4/9	2/9

(a) Fill in the following tables:

k	1	2
$P(X = k)$	$\frac{3}{9} = \boxed{\frac{1}{3}}$	$\frac{6}{9} = \boxed{\frac{2}{3}}$

ℓ	1	2
$P(Y = \ell)$	$\frac{6}{9} = \boxed{\frac{2}{3}}$	$\frac{3}{9} = \boxed{\frac{1}{3}}$

(b) Compute the expected values $E[X]$ and $E[Y]$.

$$E[X] = (1)\frac{1}{3} + (2)\frac{2}{3} = \boxed{\frac{5}{3}} \quad \text{and} \quad E[Y] = (1)\frac{2}{3} + (2)\frac{1}{3} = \boxed{\frac{4}{3}}$$

(c) Compute the mixed moment $E[XY]$ and the covariance $\operatorname{Cov}(X, Y)$.

We use the formulas

$$E[XY] = \sum_{k,\ell} k \cdot \ell \cdot P(X = k, Y = \ell),$$
$$\operatorname{Cov}(X, Y) = E[XY] - E[X] \cdot E[Y],$$

to compute

$$E[XY] = (1)(1)\frac{2}{9} + (1)(2)\frac{1}{9} + (2)(1)\frac{4}{9} + (2)(2)\frac{2}{9} = \frac{2+2+8+8}{9} = \boxed{\frac{20}{9},}$$
$$\operatorname{Cov}(X, Y) = E[XY] - E[X] \cdot E[Y] = \frac{20}{9} - \frac{5}{3} \cdot \frac{4}{3} = \frac{20}{9} - \frac{20}{9} = \boxed{0.}$$

In fact, the random variables X, Y are independent.

Problem 2. Let X, Y be random variables with the following moments:

$$E[X] = 1, \quad E[X^2] = 2, \quad E[Y] = 1, \quad E[Y^2] = 3, \quad E[XY] = 2.$$

(a) Compute $E[X+5]$ and $\mathrm{Var}(X+5)$.

$$E[X+5] = E[X] + 5 = 1 + 5 = \boxed{6}$$
$$\mathrm{Var}(X+5) = \mathrm{Var}(X) = E[X^2] - E[X]^2 = 2 - 1^2 = \boxed{1}$$

(b) Compute $\mathrm{Cov}(X,Y)$ and $\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$.

First we have

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= E[XY] - E[X] \cdot E[Y] = 2 - 1^2 = \boxed{1,} \\
\mathrm{Var}(X) &= E[X^2] - E[X]^2 = 2 - 1^2 = 1, \\
\mathrm{Var}(Y) &= E[Y^2] - E[Y]^2 = 3 - 1^2 = 2.
\end{aligned}$$

Then we have

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}(Y)}} = \frac{1}{\sqrt{1} \cdot \sqrt{2}} = \boxed{\frac{1}{\sqrt{2}}.}$$

(c) Compute $E[X+Y]$ and $\mathrm{Var}(X+Y)$.

$$E[X+Y] = E[X] + E[Y] = 1 + 1 = \boxed{2}$$
$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2 \cdot \mathrm{Cov}(X,Y) = 1 + 2 + 2 \cdot 1 = \boxed{5}$$

Problem 3. Start rolling a fair 6-sided die and stop when you see "1" for the first time. Let X be the number of rolls you did.

(a) Write down a formula for the probability mass function $P(X = k)$.

This is a geometric random variable with $p = 1/6$. Hence the pmf is

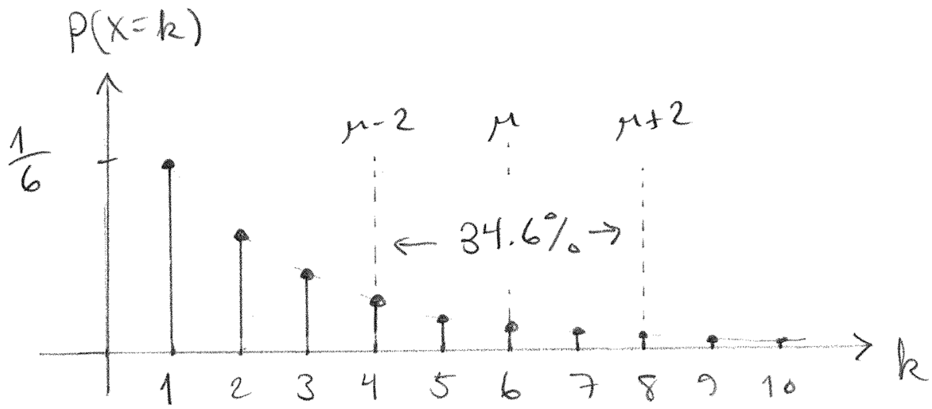
$$P(X = k) = q^{k-1} p = \left(\frac{5}{6}\right)^{k-1} \left(\frac{1}{6}\right).$$

(b) Let $\mu = E[X]$ and compute the probability $P(\mu - 2 \leq X \leq \mu + 2)$.

The expected value is $\mu = E[X] = 1/p = 6$. Then we have

$$\begin{aligned}
P(\mu - 2 \leq X \leq \mu + 2) &= P(4 \leq X \leq 8) \\
&= q^3 - q^8 \\
&= \left(\frac{5}{6}\right)^3 - \left(\frac{5}{6}\right)^8 = 34.6\%.
\end{aligned}$$

(c) Draw the line graph of the pmf $P(X = k)$. Show the interval $\mu \pm 2$ in your picture.



Problem 4. Suppose that an urn contains 3 red and 5 green balls. Draw 4 balls **with replacement** from the urn and let X be the number of **red** balls you get.

(a) Write down a formula for the probability mass function $P(X = k)$.

If the balls are mixed after replacement then this random variable is binomial:

$$P(X = k) = \binom{4}{k}\left(\frac{3}{8}\right)^k\left(\frac{5}{8}\right)^{4-k}.$$

(b) Tell me the expected value $E[X]$ and the variance $\mathrm{Var}(X)$. [Hint: Don't do a big calculation.]

We know the expected value and the variance of a binomial:

$$E[X] = np = 4 \cdot \frac{3}{8} = \frac{12}{8} = \boxed{\frac{3}{2}},$$

$$\mathrm{Var}(X) = npq = 4 \cdot \frac{3}{8} \cdot \frac{5}{8} = \frac{60}{64} = \boxed{\frac{15}{16}}.$$

(c) Use part (b) to compute the second moment $E[X^2]$.

$$\begin{aligned} E[X^2] - E[X]^2 &= \mathrm{Var}(X) \\ E[X^2] &= \mathrm{Var}(X) + E[X]^2 \\ &= \frac{15}{16} + \left(\frac{3}{2}\right)^2 = \boxed{\frac{51}{16}} \end{aligned}$$

Problem 5. Suppose that an urn contains 2 red and 3 green balls. Draw 2 balls **without replacement** from the urn and consider the following random variables:

$$X_1 = \begin{cases} 1 & \text{if 1st ball is red,} \\ 0 & \text{if 1st ball is green,} \end{cases} \qquad X_2 = \begin{cases} 1 & \text{if 2nd ball is red,} \\ 0 & \text{if 2nd ball is green.} \end{cases}$$

Let $X = X_1 + X_2$ be the total number of red balls that you get.

(a) Compute the variances $\mathrm{Var}(X_1)$ and $\mathrm{Var}(X_2)$.

Each of these is Bernoulli with $p = 2/5$. Hence

$$\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = pq = \frac{2}{5} \cdot \frac{3}{5} = \boxed{\frac{6}{25}}.$$

(b) Compute the probability mass function of X.

This is a hypergeometric random variable.

k	0	1	2
$P(X=k)$	$\frac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}} = \boxed{\frac{3}{10}}$	$\frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} = \boxed{\frac{6}{10}}$	$\frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}} = \boxed{\frac{1}{10}}$

(c) Use parts (a) and (b) to compute the covariance $\mathrm{Cov}(X_1, X_2)$. [Hint: It's not zero.]

First solution. From (b) we have

$$\begin{aligned}
E[X] &= (0)\frac{3}{10} + (1)\frac{6}{10} + (2)\frac{1}{10} = 8/10, \\
E[X^2] &= (0)^2\frac{3}{10} + (1)^2\frac{6}{10} + (2)^2\frac{1}{10} = 1, \\
\mathrm{Var}(X) &= E[X^2] - E[X]^2 = 1 - (8/10)^2 = 36/100 = 9/25.
\end{aligned}$$

and then

$$\begin{aligned}
\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2 \cdot \mathrm{Cov}(X_1, X_2) &= \mathrm{Var}(X) \\
2 \cdot \mathrm{Cov}(X_1, X_2) &= \mathrm{Var}(X) - \mathrm{Var}(X_1) - \mathrm{Var}(X_2) \\
2 \cdot \mathrm{Cov}(X_1, X_2) &= 9/25 - 6/25 - 6/25 = -3/25 \\
\mathrm{Cov}(X_1, X_2) &= \boxed{-3/50.}
\end{aligned}$$

Second solution. The expected value of a Bernoulli is $E[X_1] = E[X_2] = p = 2/5$ and the joint distribution of X_1 and X_2 is

$X_1 \setminus X_2$	0	1
0	3/10	(3/5)(2/4)
1	(2/5)(3/4)	1/10

This implies that $E[X_1 X_2] = 0 + 0 + 0 + 1/10$ and hence

$$\mathrm{Cov}(X_1, X_2) = E[X_1 X_2] - E[X_1] \cdot E[X_2] = \frac{1}{10} - \frac{2}{5} \cdot \frac{2}{5} = \boxed{-\frac{3}{50}}.$$

Version B

Problem 1. Let X, Y be discrete random variables with joint pmf defined as follows:

$X \setminus Y$	1	2
1	2/12	1/12
2	6/12	3/12

(a) Fill in the following tables:

k	1	2
$P(X = k)$	$\frac{3}{12} = \boxed{\frac{1}{4}}$	$\frac{9}{12} = \boxed{\frac{3}{4}}$

ℓ	1	2
$P(Y = \ell)$	$\frac{8}{12} = \boxed{\frac{2}{3}}$	$\frac{4}{12} = \boxed{\frac{1}{3}}$

(b) Compute the expected values $E[X]$ and $E[Y]$.

$$E[X] = (1)\frac{1}{4} + (2)\frac{3}{4} = \boxed{\frac{7}{4}} \qquad \text{and} \qquad E[Y] = (1)\frac{2}{3} + (2)\frac{1}{3} = \boxed{\frac{4}{3}}$$

(c) Compute the mixed moment $E[XY]$ and the covariance $\mathrm{Cov}(X, Y)$.

We use the formulas

$$E[XY] = \sum_{k,\ell} k \cdot \ell \cdot P(X = k, Y = \ell),$$
$$\mathrm{Cov}(X, Y) = E[XY] - E[X] \cdot E[Y],$$

to compute

$$E[XY] = (1)(1)\frac{2}{12} + (1)(2)\frac{1}{12} + (2)(1)\frac{6}{12} + (2)(2)\frac{3}{12} = \frac{2 + 2 + 12 + 12}{12} = \boxed{\frac{7}{3},}$$
$$\mathrm{Cov}(X, Y) = E[XY] - E[X] \cdot E[Y] = \frac{7}{3} - \frac{7}{4} \cdot \frac{4}{3} = \frac{7}{3} - \frac{7}{3} = \boxed{0.}$$

In fact, the random variables X, Y are independent.

Problem 2. Let X, Y be random variables with the following moments:

$$E[X] = 1, \quad E[X^2] = 3, \quad E[Y] = 1, \quad E[Y^2] = 2, \quad E[XY] = 2.$$

(a) Compute $E[X + 4]$ and $\mathrm{Var}(X + 4)$.

$$E[X + 5] = E[X] + 4 = 1 + 4 = \boxed{5}$$
$$\mathrm{Var}(X + 5) = \mathrm{Var}(X) = E[X^2] - E[X]^2 = 3 - 1^2 = \boxed{2}$$

(b) Compute $\mathrm{Cov}(X,Y)$ and $\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$.

First we have

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= E[XY] - E[X]\cdot E[Y] = 2 - 1^2 = \boxed{1,} \\
\mathrm{Var}(X) &= E[X^2] - E[X]^2 = 3 - 1^2 = 2, \\
\mathrm{Var}(Y) &= E[Y^2] - E[Y]^2 = 2 - 1^2 = 1.
\end{aligned}$$

Then we have

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\cdot\sqrt{\mathrm{Var}(Y)}} = \frac{1}{\sqrt{2}\cdot\sqrt{1}} = \boxed{\frac{1}{\sqrt{2}}.}$$

(c) Compute $E[X+Y]$ and $\mathrm{Var}(X+Y)$.

$$\begin{aligned}
E[X+Y] &= E[X] + E[Y] = 1 + 1 = \boxed{2} \\
\mathrm{Var}(X+Y) &= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\cdot\mathrm{Cov}(X,Y) = 2 + 1 + 2\cdot 1 = \boxed{5}
\end{aligned}$$

Problem 3. Start rolling a fair 4-sided die and stop when you see "1" for the first time. Let X be the number of rolls you did.

(a) Write down a formula for the probability mass function $P(X = k)$.

This is a geometric random variable with $p = 1/4$. Hence the pmf is

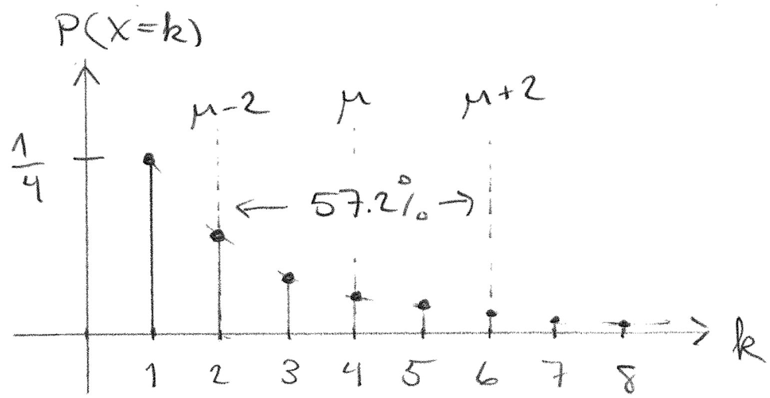
$$P(X = k) = q^{k-1}p = \left(\frac{3}{4}\right)^{k-1}\left(\frac{1}{4}\right).$$

(b) Let $\mu = E[X]$ and compute the probability $P(\mu - 2 \leq X \leq \mu + 2)$.

The expected value is $\mu = E[X] = 1/p = 4$. Then we have

$$\begin{aligned}
P(\mu - 2 \leq X \leq \mu + 2) &= P(2 \leq X \leq 6) \\
&= q^1 - q^6 \\
&= \left(\frac{3}{4}\right)^1 - \left(\frac{3}{4}\right)^6 = 57.2\%.
\end{aligned}$$

(c) Draw the line graph of the pmf $P(X = k)$. Show the interval $\mu \pm 2$ in your picture.



Problem 4. Suppose that an urn contains 5 red and 3 green balls. Draw 4 balls **with replacement** from the urn and let X be the number of **red** balls you get.

(a) Write down a formula for the probability mass function $P(X = k)$.

If the balls are mixed after replacement then this random variable is binomial:

$$P(X = k) = \binom{4}{k} \left(\frac{5}{8}\right)^k \left(\frac{3}{8}\right)^{4-k}.$$

(b) Tell me the expected value $E[X]$ and the variance $\mathrm{Var}(X)$. [Hint: Don't do a big calculation.]

We know the expected value and the variance of a binomial:

$$E[X] = np = 4 \cdot \frac{5}{8} = \frac{20}{8} = \boxed{\frac{5}{2}},$$

$$\mathrm{Var}(X) = npq = 4 \cdot \frac{5}{8} \cdot \frac{3}{8} = \frac{60}{64} = \boxed{\frac{15}{16}}.$$

(c) Use part (b) to compute the second moment $E[X^2]$.

$$\begin{aligned} E[X^2] - E[X]^2 &= \mathrm{Var}(X) \\ E[X^2] &= \mathrm{Var}(X) + E[X]^2 \\ &= \frac{15}{16} + \left(\frac{5}{2}\right)^2 = \boxed{\frac{115}{16}} \end{aligned}$$

Problem 5. Suppose that an urn contains 3 red and 2 green balls. Draw 2 balls **without replacement** from the urn and consider the following random variables:

$$X_1 = \begin{cases} 1 & \text{if 1st ball is red,} \\ 0 & \text{if 1st ball is green,} \end{cases} \qquad X_2 = \begin{cases} 1 & \text{if 2nd ball is red,} \\ 0 & \text{if 2nd ball is green.} \end{cases}$$

Let $X = X_1 + X_2$ be the total number of red balls that you get.

(a) Compute the variances $\mathrm{Var}(X_1)$ and $\mathrm{Var}(X_2)$.

Each of these is Bernoulli with $p = 3/5$. Hence

$$\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = pq = \frac{3}{5} \cdot \frac{2}{5} = \boxed{\frac{6}{25}}.$$

(b) Compute the probability mass function of X.

This is a hypergeometric random variable.

k	0	1	2
$P(X = k)$	$\frac{\binom{3}{0}\binom{2}{2}}{\binom{5}{2}} = \boxed{\frac{1}{10}}$	$\frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}} = \boxed{\frac{6}{10}}$	$\frac{\binom{3}{2}\binom{2}{0}}{\binom{5}{2}} = \boxed{\frac{3}{10}}$

(c) Use parts (a) and (b) to compute the covariance $\mathrm{Cov}(X_1, X_2)$. [Hint: It's not zero.]

First solution. From (b) we have

$$\begin{aligned}
E[X] &= (0)\frac{1}{10} + (1)\frac{6}{10} + (2)\frac{3}{10} = 12/10,\\
E[X^2] &= (0)^2\frac{1}{10} + (1)^2\frac{6}{10} + (2)^2\frac{3}{10} = 18/10,\\
\mathrm{Var}(X) &= E[X^2] - E[X]^2 = 18/10 - (12/10)^2 = 36/100 = 9/25.
\end{aligned}$$

and then

$$\begin{aligned}
\mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2 \cdot \mathrm{Cov}(X_1, X_2) &= \mathrm{Var}(X)\\
2 \cdot \mathrm{Cov}(X_1, X_2) &= \mathrm{Var}(X) - \mathrm{Var}(X_1) - \mathrm{Var}(X_2)\\
2 \cdot \mathrm{Cov}(X_1, X_2) &= 9/25 - 6/25 - 6/25 = -3/25\\
\mathrm{Cov}(X_1, X_2) &= \boxed{-3/50.}
\end{aligned}$$

Second solution. The expected value of a Bernoulli is $E[X_1] = E[X_2] = p = 3/5$ and the joint distribution of X_1 and X_2 is

$X_1 \setminus X_2$	0	1
0	1/10	(2/5)(3/4)
1	(3/5)(2/4)	3/10

This implies that $E[X_1 X_2] = 0 + 0 + 0 + 3/10$ and hence

$$\mathrm{Cov}(X_1, X_2) = E[X_1 X_2] - E[X_1] \cdot E[X_2] = \frac{3}{10} - \frac{3}{5} \cdot \frac{3}{5} = \boxed{-\frac{3}{50}}.$$