Math 224 **Exam 1**
Spring 2018 **Tues Feb 20**

Version A

Problem 1. Let S be the sample space of an experiment with ten equally likely outcomes, so that $\#S = 10$. Now consider two events $E, F \subseteq S$ such that

$$\#E = 5, \qquad \#F = 6 \qquad \text{and} \qquad \#(E \cap F) = 3.$$

(a) Find the number of outcomes in the union: $\#(E \cup F)$.

$$\#(E \cup F) = \#E + \#F - \#(E \cap F) = 5 + 6 - 3 = 8.$$

(b) Find the probability that E or F happens.

$$P(E \cup F) = \frac{\#(E \cup F)}{\#S} = \frac{8}{10} = 80\%.$$

(c) Find the conditional probability that E happens, assuming that F happens.

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\#(E \cap F)/\#S}{\#F/\#S} = \frac{\#(E \cap F)}{\#F} = \frac{3}{6} = 50\%.$$

Problem 2.

(a) Draw Pascal's Triangle down to the 4th row. (The top row is the 0th row.)

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

(b) Suppose a fair coin is flipped 4 times and let X be the number of heads that appear. Use part (a) to fill in the following table:

k	0	1	2	3	4
$P(X = k)$	$\binom{4}{0}/2^4$	$\binom{4}{1}/2^4$	$\binom{4}{2}/2^4$	$\binom{4}{3}/2^4$	$\binom{4}{4}/2^4$
	1/16	4/16	6/16	4/16	1/16

(c) Consider a coin with $P(\text{heads}) = 1/3$. Suppose the coin is flipped 4 times and let Y be the number of heads that appear. Use part (a) to fill in the following table:

k	0	1	2	3	4
$P(Y=k)$	$\binom{4}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4$	$\binom{4}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3$	$\binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2$	$\binom{4}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^1$	$\binom{4}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^0$
	16/81	32/81	24/81	8/81	1/81

Problem 3. A standard deck contains 13 hearts and 13 diamonds (called red cards), 13 spades and 13 clubs (called black cards). Suppose that 5 cards are drawn without replacement from a standard deck.

(a) Find the probability of getting 3 hearts and 2 spades.

$$P(\text{3 hearts, 2 spades}) = \frac{\binom{13}{3}\binom{13}{2}}{\binom{52}{5}} = 0.9\%.$$

(b) Find the probability of getting 3 hearts, 1 spade and 1 club.

$$P(\text{3 hearts, 1 spade, 1 club}) = \frac{\binom{13}{3}\binom{13}{1}\binom{13}{1}}{\binom{52}{5}} = 1.9\%.$$

(c) Find the probability of getting 3 hearts and 2 black cards.

$$P(\text{3 hearts, 2 black cards}) = \frac{\binom{13}{3}\binom{26}{2}}{\binom{52}{5}} = 3.6\%.$$

Problem 4. A fair 6-sided die is rolled 4 times.

(a) What is the probability that the numbers $2, 3, 4, 5$ show up, in some order?

$$P(2,3,4,5 \text{ in some order}) = \frac{\binom{4}{1,1,1,1}}{6^4} = \frac{4!/(1!1!1!1!)}{6^4} = \frac{24}{6^4} = 1.9\%.$$

(b) What is the probability that the numbers $2, 2, 3, 3$ show up, in some order?

$$P(2,2,3,3 \text{ in some order}) = \frac{\binom{4}{2,2}}{6^4} = \frac{4!/(2!2!)}{6^4} = \frac{6}{6^4} = 0.5\%.$$

(c) What is the probability that the same number shows up four times? [Hint: How many ways could this happen?]

$$P(\text{all the same}) = \frac{6}{6^4} = 0.5\%.$$

Problem 5. There are two bowls on a table. The first bowl contains 1 red chip and 2 white chips. The second bowl contains 2 red chips and 3 white chips. Your friend walks up to the table and chooses one chip. Consider the events:

$$\begin{aligned} R &= \{\text{the chip is red}\}, \\ B_1 &= \{\text{the chip comes from the first bowl}\}, \\ B_2 &= \{\text{the chip comes from the second bowl}\}. \end{aligned}$$

(a) Compute the probabilities $P(R|B_1)$ and $P(R|B_2)$.

$$P(R|B_1) = \frac{1}{1+2} = \frac{1}{3} \qquad \text{and} \qquad P(R|B_2) = \frac{2}{2+3} = \frac{2}{5}.$$

(b) Assume that the bowls have probabilities $P(B_1) = 1/3$ and $P(B_2) = 2/3$. In this case, compute the probability $P(R)$ that your friend gets a red chip.

$$\begin{aligned} P(R) &= P(B_1 \cap R) + P(B_2 \cap R) \\ &= P(B_1) \cdot P(R|B_1) + P(B_2) \cdot P(R|B_2) \\ &= \frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{5} = \frac{17}{45} = 37.8\%. \end{aligned}$$

(c) After performing the experiment in secret, your friend shows you that the chip is **red**. Compute the probability $P(B_1|R)$ that this red chip came from the first bowl.

$$P(B_1|R) = \frac{P(B_1 \cap R)}{P(R)} = \frac{P(B_1) \cdot P(R|B_1)}{P(R)} = \frac{(1/3)(1/3)}{17/45} = \frac{5}{17} = 29.4\%.$$

Version B

Problem 1. Let S be the sample space of an experiment with ten equally likely outcomes, so that $\#S = 10$. Now consider two events $E, F \subseteq S$ such that

$$\#E = 4, \qquad \#F = 5 \qquad \text{and} \qquad \#(E \cap F) = 3.$$

(a) Find the number of outcomes in the union: $\#(E \cup F)$.

$$\#(E \cup F) = \#E + \#F - \#(E \cap F) = 4 + 5 - 3 = 6.$$

(b) Find the probability that E or F happens.

$$P(E \cup F) = \frac{\#(E \cup F)}{\#S} = \frac{6}{10} = 60\%.$$

(c) Find the conditional probability that E happens, assuming that F happens.

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\#(E \cap F)/\#S}{\#F/\#S} = \frac{\#(E \cap F)}{\#F} = \frac{3}{5} = 60\%.$$

Problem 2.

(a) Draw Pascal's Triangle down to the 4th row. (The top row is the 0th row.)

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

(b) Suppose a fair coin is flipped 4 times and let X be the number of heads that appear. Use part (a) to fill in the following table:

k	0	1	2	3	4
$P(X = k)$	$\binom{4}{0}/2^4$	$\binom{4}{1}/2^4$	$\binom{4}{2}/2^4$	$\binom{4}{3}/2^4$	$\binom{4}{4}/2^4$
	$1/16$	$4/16$	$6/16$	$4/16$	$1/16$

(c) Consider a coin with $P(\text{heads}) = 1/4$. Suppose the coin is flipped 4 times and let Y be the number of heads that appear. Use part (a) to fill in the following table:

k	0	1	2	3	4
$P(Y = k)$	$\binom{4}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^4$	$\binom{4}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^3$	$\binom{4}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2$	$\binom{4}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^1$	$\binom{4}{4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^0$
	$81/256$	$108/256$	$54/256$	$12/256$	$1/256$

Problem 3. A standard deck contains 13 hearts and 13 diamonds (called red cards), 13 spades and 13 clubs (called black cards). Suppose that 5 cards are drawn without replacement from a standard deck.

(a) Find the probability of getting 2 hearts and 3 spades.

$$P(\text{2 hearts, 3 spades}) = \frac{\binom{13}{2}\binom{13}{3}}{\binom{52}{5}} = 0.9\%.$$

(b) Find the probability of getting 2 hearts, 2 spades and 1 club.

$$P(\text{2 hearts, 2 spades, 1 club}) = \frac{\binom{13}{2}\binom{13}{2}\binom{13}{1}}{\binom{52}{5}} = 3.0\%.$$

(c) Find the probability of getting 2 hearts and 3 black cards.

$$P(\text{2 hearts, 3 black cards}) = \frac{\binom{13}{2}\binom{26}{3}}{\binom{52}{5}} = 7.8\%.$$

Problem 4. A fair 6-sided die is rolled 4 times.

(a) What is the probability that the numbers $2, 3, 4, 5$ show up, in some order?

$$P(2, 3, 4, 5 \text{ in some order}) = \frac{\binom{4}{1,1,1,1}}{6^4} = \frac{4!/(1!1!1!1!)}{6^4} = \frac{24}{6^4} = 1.9\%.$$

(b) What is the probability that the numbers $2, 2, 3, 4$ show up, in some order?

$$P(2, 2, 3, 4 \text{ in some order}) = \frac{\binom{4}{2,1,1}}{6^4} = \frac{4!/(2!1!1!)}{6^4} = \frac{12}{6^4} = 0.9\%.$$

(c) What is the probability that the same number shows up four times? [Hint: How many ways could this happen?]

$$P(\text{all the same}) = \frac{6}{6^4} = 0.5\%.$$

Problem 5. There are two bowls on a table. The first bowl contains 1 red chip and 3 white chips. The second bowl contains 2 red chips and 4 white chips. Your friend walks up to the table and chooses one chip. Consider the events:

$$\begin{aligned} R &= \{\text{the chip is red}\}, \\ B_1 &= \{\text{the chip comes from the first bowl}\}, \\ B_2 &= \{\text{the chip comes from the second bowl}\}. \end{aligned}$$

(a) Compute the probabilities $P(R|B_1)$ and $P(R|B_2)$.

$$P(R|B_1) = \frac{1}{1+3} = \frac{1}{4} \qquad \text{and} \qquad P(R|B_2) = \frac{2}{2+4} = \frac{2}{6}.$$

(b) Assume that the bowls have probabilities $P(B_1) = 1/3$ and $P(B_2) = 2/3$. In this case, compute the probability $P(R)$ that your friend gets a red chip.

$$\begin{aligned} P(R) &= P(B_1 \cap R) + P(B_2 \cap R) \\ &= P(B_1) \cdot P(R|B_1) + P(B_2) \cdot P(R|B_2) \\ &= \frac{1}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \frac{2}{6} = \frac{11}{36} = 30.6\%. \end{aligned}$$

(c) After performing the experiment in secret, your friend shows you that the chip is **red**. Compute the probability $P(B_1|R)$ that this red chip came from the first bowl.

$$P(B_1|R) = \frac{P(B_1 \cap R)}{P(R)} = \frac{P(B_1) \cdot P(R|B_1)}{P(R)} = \frac{(1/3)(1/4)}{11/36} = \frac{3}{11} = 27.3\%.$$