Math 224 **Fall 2017**
Homework 1 **Drew Armstrong**

Problems from *Probability and Statistical Inference (9th ed.)* by Hogg, Tanis and Zimmerman.

- Section 1.1, Exercises 4,5,6,7,9,12.

Solutions to Book Problems.

1.1-4. A fair coin is tossed four times and the sequence of heads and tails is observed.

(a) The sample space is

$$\begin{aligned} S = \{&TTTT, HTTT, THTT, TTHT, TTTH, HHTT, HTHT, HTTH,\\ &THHT, THTH, TTHH, THHH, HTHH, HHTH, HHHT, HHHH\} \end{aligned}$$

(b) Consider the following events:

$$\begin{aligned} A &= \{ \text{ at least 3 heads } \}\\ &= \{THHH, HTHH, HHTH, HHHT, HHHH\}\\ B &= \{ \text{ at most 2 heads } \}\\ &= \{TTTT, HTTT, THTT, TTHT, TTTH, HHTT, HTHT, HTTH, THHT, THTH, TTHH\}\\ C &= \{ \text{ heads on the third toss } \}\\ &= \{TTHT, HTHT, THHT, TTHH, THHH, HTHH, HHHT, HHHH\}\\ D &= \{ \text{ 1 head and 3 tails } \}\\ &= \{HTTT, THTT, TTHT, TTTH\}. \end{aligned}$$

We note that $\#S = 16$, $\#A = 5$, $\#B = 11$, $\#C = 8$ and $\#D = 4$ so that

$$P(A) = \frac{5}{16}, \qquad P(B) = \frac{11}{16}, \qquad P(C) = \frac{8}{16} \quad \text{and} \quad P(D) = \frac{4}{16}.$$

We are also asked to consider the following events:

$$\begin{aligned} A \cap B &= \{ \text{ at least 3 heads AND at most 2 heads } \} = \emptyset\\ A \cap C &= \{ \text{ at least 3 heads AND heads on the third toss } \}\\ &= \{THHH, HTHH, HHHT, HHHH\}\\ A \cup C &= \{ \text{ at least 3 heads OR heads on the third toss } \} = \text{ "never mind"}\\ B \cap D &= \{ \text{ at most 2 heads AND 1 head and 3 tails } \} = \{ \text{ 1 head and 3 tails } \} = D. \end{aligned}$$

We observe that $\#(A \cap B) = 0$, $\#(A \cap C) = 4$ and $\#(B \cap D) = \#D = 4$ so that

$$P(A \cap B) = \frac{0}{16}, \qquad P(A \cap C) = \frac{4}{16} \quad \text{and} \quad P(B \cap D) = \frac{4}{16}.$$

I said "never mind" for the set $A \cup C$ because we don't need to list all the elements. Indeed, we already know that $P(A) = 5/16$, $P(C) = 8/16$ and $P(A \cap C) = 4/16$, so that

$$P(A \cup C) = P(A) + P(C) - P(A \cap C) = \frac{5 + 8 - 4}{16} = \frac{9}{16}.$$

1.1-5. We roll a fair six-sided die until we see a 3. Consider the events

$$A = \{ \text{ we get a 3 on the first roll } \}$$
$$B = \{ \text{ at least two rolls are required to see a 3 } \}.$$

If the die is fair then we observe that $P(A) = 1/6$. We also observe that the events A and B are complementary because they are mutually exclusive and they exhaust all the possible outcomes. Therefore we conclude that

$$P(A \cup B) = 1 \qquad \text{and} \qquad P(B) = 1 - P(A) = 1 - \frac{1}{6} = \frac{5}{6}.$$

1.1-6. Consider two events A, B such that

$$P(A) = 0.4, \qquad P(B) = 0.5 \quad \text{and} \quad P(A \cap B) = 0.3.$$

(a) Then we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.5 - 0.3 = 0.6.$$

(b) Note that we can decompose the event A into two disjoint pieces by looking at the stuff that's **inside** B or **outside** B:

$$A = (A \cap B) \cup (A \cap B').$$

[You should draw a Venn diagram to get a feeling for this.] Thus we have

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B') \\ P(A) - P(A \cap B) &= P(A \cap B') \\ 0.4 - 0.3 &= P(A \cap B') \\ 0.1 &= P(A \cap B'). \end{aligned}$$

(c) De Morgan's law says that $(A \cap B)' = A' \cup B'$. That is, the stuff that is not in (A AND B) is the same as the stuff that is (not in A) OR (not in B). [You should draw a Venn diagram to get a feeling for this.] Thus we have

$$P(A' \cup B') = 1 - P(A \cap B) = 1 - 0.3 = 0.7.$$

1.1-7. Consider two events A, B such that

$$P(A \cup B) = 0.76 \qquad \text{and} \qquad P(A \cup B') = 0.87.$$

We want to find $P(A)$. How can we do this? Well, we saw in the previous problem that A can be decomposed by the stuff inside/outside of B:

$$A = (A \cap B) \cup (A \cap B').$$

And we can do the same trick to divide up A' in terms of B and B':

$$A' = (A' \cap B) \cup (A' \cap B').$$

It follows from this that

$$P(A) = 1 - P(A') = 1 - [P(A' \cap B) + P(A' \cap B')].$$

So what? Well, de Morgan's law also tells us that $(A' \cap B) = (A \cup B')'$ and $(A' \cap B') = (A \cup B)'$, so that

$$\begin{aligned} P(A' \cap B) &= 1 - P(A \cup B'), \\ P(A' \cap B') &= 1 - P(A \cup B). \end{aligned}$$

Finally, putting everything together gives

$$\begin{aligned} P(A) &= 1 - [P(A' \cap B) + P(A' \cap B')] \\ &= 1 - \big[[1 - P(A \cup B')] + [1 - P(A \cup B)]\big] \\ &= P(A \cup B') + P(A \cup B) - 1 \\ &= 0.87 + 0.76 - 1 \\ &= 0.63. \end{aligned}$$

1.1-9. We roll a fair six-sided die 3 times. Consider the following events:

$$\begin{aligned} A_1 &= \{ \text{ 1 or 2 on the first roll } \} \\ A_2 &= \{ \text{ 3 or 4 on the second roll } \} \\ A_3 &= \{ \text{ 5 or 6 on the third roll } \}. \end{aligned}$$

Luckily we don't have to analyze this experiment ourselves because the book just tells us that:

- $P(A_i) = 1/3$ for all i.
- $P(A_i \cap A_j) = (1/3)^2$ for all $i \neq j$.
- $P(A_1 \cap A_2 \cap A_3) = (1/3)^3$.

Now we are asked to find $P(A_1 \cup A_2 \cup A_3)$. At this point you can just quote Theorem 1.1-6 from the book. However, I'll do it myself from scratch. First we have

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1 \cup (A_2 \cup A_3)) \\ &= P(A_1) + P(A_2 \cup A_3) - P(A_1 \cap (A_2 \cup A_3)) \end{aligned}$$

and

$$P(A_2 \cup A_3) = P(A_2) + P(A_3) - P(A_2 \cap A_3).$$

Then by rewriting $A_1 \cap (A_2 \cup A_3) = (A_1 \cap A_2) \cup (A_1 \cap A_3)$ we have

$$\begin{aligned} P(A_1 \cap (A_2 \cup A_3)) &= P((A_1 \cap A_2) \cup (A_1 \cap A_3)) \\ &= P(A_1 \cap A_2) + P(A_1 \cap A_3) - P((A_1 \cap A_2) \cap (A_1 \cap A_3)) \\ &= P(A_1 \cap A_2) + P(A_1 \cap A_3) - P(A_1 \cap A_2 \cap A_3). \end{aligned}$$

Putting everything together gives

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3) &= P(A_1) + P(A_2 \cup A_3) - P(A_1 \cap (A_2 \cup A_3)) \\ &= P(A_1) + [P(A_2) + P(A_3) - P(A_2 \cap A_3)] \\ &\quad - [P(A_1 \cap A_2) + P(A_1 \cap A_3) - P(A_1 \cap A_2 \cap A_3)], \end{aligned}$$

or, in other words, $P(A_1 \cup A_2 \cup A_3)$ equals

$$P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3).$$

Finally, since we know value of each term in this sum, we get

$$P(A_1 \cup A_2 \cup A_3) = 3 \cdot \left(\frac{1}{3}\right) - 3 \cdot \left(\frac{1}{3}\right)^2 + 1 \cdot \left(\frac{1}{3}\right)^3.$$

Wow, that really reminds me of the third row of Pascal's triangle. Indeed, note that

$$\begin{aligned}\left(1-\frac{1}{3}\right)^3 &= 1+3\cdot\left(\frac{-1}{3}\right)+3\cdot\left(\frac{-1}{3}\right)^2+1\cdot\left(\frac{-1}{3}\right)^3\\ &= 1-3\cdot\left(\frac{1}{3}\right)+3\cdot\left(\frac{1}{3}\right)^2-1\cdot\left(\frac{1}{3}\right)^3\end{aligned}$$

and hence we have

$$P(A_1\cup A_2\cup A_3)=1-\left(1-\frac{1}{3}\right)^3=1-\left(\frac{2}{3}\right)^3=1-\frac{8}{27}=\frac{19}{27}.$$

[Remark: Maybe there's a shorter way to do this, but it was good practice to do it the long way.]

1.1-12. This one is just a "thinking problem," since we aren't told precisely what "selected randomly" means in this case. Just use your intuition.

Suppose a real number x is "selected randomly" from the closed interval $[0,1]$. We are supposed to assume that all possible choices are "equally likely," whatever that means. Since there are infinitely many possible choices, this suggests that the probability of any **particular** x is $1/\infty$, or 0. Ok, I guess.

We know that $P([0,1])=1$ because $[0,1]$ is the whole sample space. It also seems intuitively clear that $P([0,1/2])=P([1/2,1])=1/2$. (The number is equally likely to be in the left half or the right half of the interval.) More generally, it seems that the probability that x lies in a particular line segment is just the **length** of the line segment (whether or not the endpoints of the line segment are included).

So here are my answers:

(a) $P(\{x:0\leq x\leq 1/3\})=1/3$,

(b) $P(\{x:1/3\leq x\leq 1\})=2/3$,

(c) $P(\{x:x=1/3\})=0$,

(d) $P(\{x:1/2<x<5\})=P(\{x:1/2<x\leq 1\})+P(\{x:1\leq 5\})=1/2+0=1/2$.

[Remark: We will discuss "continuous probability distributions" in detail later.]

Additional Problems.

1. Consider a biased coin with $P(\text{"heads"})=p$ and $P(\text{"tails"})=1-p$. Suppose that you flip the coin n times and let X be the number of heads that you get. Compute $P(X\geq 1)$. [Hint: Observe that $P(X\geq 1)+P(X=0)=1$.]

Solution: Recall from the course notes that the probability of getting exactly k heads is

$$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}.$$

Thus we are asked to compute the following sum:

$$P(X\geq 1)=\sum_{k=1}^{n}\binom{n}{k}p^k(1-p)^{n-k}.$$

That seems really hard so instead we use the formula

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{n}{0} p^0 (1-p)^n = 1 - (1-p)^n.$$

Alternatively, we can compute $P(X = 0)$ by observing that "$X = 0$" corresponds to the event $\{TTT \cdots T\}$. Since the probability of each "tail" is $(1-p)$ and since the coin flips are "independent" we see that

$$P(X = 0) = P(TTT \cdots T) = P(T)P(T) \cdots P(T) = (1-p)(1-p) \cdots (1-p) = (1-p)^n.$$

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[Remark: Compare the formula $P(X = 0) = 1 - (1-p)^n$ to Exercise 1.1-9 above. Can you see how to obtain the answer to 1.1-9 by plugging in $p = 1/3$ and $n = 3$? We can think of each die roll as a fancy coin flip with $P(\text{"heads"}) = 1/3$. The definition of "heads" changes on each roll, but I guess that doesn't matter.]

2. Suppose that you roll a pair of fair six-sided dice.

(a) Write down the elements of the sample space S. What is $\#S$? Are the outcomes equally likely?

(b) Compute the probability of getting a "double six." [Hint: Let $E \subseteq S$ be the subset of outcomes that correspond to getting a "double six." Compute $P(E) = \#E/\#S$.]

Solution: (a) The sample space is

$$\begin{aligned} S = \{&11, 12, 13, 14, 15, 16 \\ &21, 22, 23, 24, 25, 26 \\ &31, 32, 33, 34, 35, 36 \\ &41, 42, 43, 44, 45, 46 \\ &61, 62, 63, 64, 65, 66\}. \end{aligned}$$

Therefore we have $\#S = 6 \times 6 = 36$. If the dice are fair I guess that all 36 possible outcomes are equally likely. Therefore we can use the formula $P(E) = \#E/\#S$.

(b) The event "double six" corresponds to $E = \{66\}$, so that $\#E = 1$. Thus

$$P(\text{"double six"}) = P(E) = \#E/\#S = 1/36.$$

3. The Chevalier de Méré considered the following two games/experiments:

(1) Roll a fair six-sided die 4 times.

(2) Roll a pair of fair six-sided dice 24 times.

For the first experiment, let X be the number of "sixes" that you get. Apply counting and axioms of probability to compute $P(X \geq 1)$. For the second experiment let Y be the number of "double sixes" that you get. Apply similar ideas to compute $P(Y \geq 1)$. Which of these two events is more likely? [Hint: You can think of a fair six-sided die as a bised coin with "heads"="six" and "tails"="not six," so that $P(\text{"heads"}) = 1/6$ and $P(\text{"tails"}) = 5/6$. You will find that it is easier to compute $P(X = 0)$ and $P(Y = 0)$.]

Solution: All of the work has been done. For game (1) we think of "rolling a fair six-sided die" as a fancy coin flip with "heads"="six" and "tails"="not six." Roll the die $n = 4$ times and let $X =$"number of sixes we get." Since $p = P(\text{"heads"}) = 1/6$, Problem 1 tells us that

$$P(\text{"at least one six"}) = P(X \geq 1) = 1 - P(X = 0) = 1 - (1-p)^n = 1 - \left(\frac{5}{6}\right)^4 \approx 0.5177 = 51.77\%.$$

For game (2) we think of "rolling a pair of fair six-sided dice" as a fancy coin flip with "heads"="double six" and "tails"="not double six." Roll the fancy coin $n = 24$ times and let $Y =$"number of double sixes we get." From Problem 2 we know that $p = P(\text{"double six"}) = 1/36$, hence Problem 1 tells us that

$$P(\text{"at least one double six"}) = P(Y \geq 1) = 1 - P(Y = 0) = 1 - \left(\frac{35}{36}\right)^{24} \approx 0.4914 = 49.14\%.$$

[Remark: The Chevalier's mathematical intuition told him that these two events should be equally likely, but his gambling experience told him that "$X \geq 1$" happened more often than "$Y \geq 1$." The subject of mathematical probability was born when Fermat and Pascal came up with a mathematical theory (the one we just used) that does agree with experience. The most important thing is to make accurate predictions.]