Math 224 Q **Exam 1**
Fall 2017 **Tues Oct 10**

Version A

Problem 1. Consider a coin with $P(H) = 1/3$ and $P(T) = 2/3$. Suppose that the coin is flipped 8 times in sequence and suppose that the coin has no memory.

(a) What is the probability of getting the sequence $THTTTHTT$?

Since the coin has no memory, we can multiply the probabilities:

$$\begin{aligned} P(THTTTHTT) &= P(T)P(H)P(T)P(T)P(T)P(H)P(T)P(T) \\ &= P(H)^2P(T)^6 \\ &= \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^6 = \frac{2^6}{3^8} \approx 0.97\%. \end{aligned}$$

(b) What is the probability that H shows up **exactly twice**?

We use the formula for binomial probability:

$$\begin{aligned} P(\text{we get } H \text{ twice}) &= \binom{8}{2}P(H)^2P(T)^6 \\ &= \frac{8!}{2!6!}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^6 \\ &= \frac{8\cdot 7}{2}\cdot\frac{2^6}{3^8} \approx 27.31\%. \end{aligned}$$

(c) What is the probability that H shows up **at least once**?

$$\begin{aligned} P(\text{at least one head}) &= 1 - P(\text{all tails}) \\ &= 1 - \left(\frac{2}{3}\right)^8 \\ &= \frac{3^8 - 2^8}{3^8} \approx 96.1\%. \end{aligned}$$

Problem 2. There are two bowls on a table. The first bowl contains 1 red chip and 5 white chips. The second bowl contains 3 red chips and 3 white chips. Your friend walks up to the table and chooses one chip at random. Consider the following events:

$$\begin{aligned} B_1 &= \text{"the chip comes from the first bowl,"} \\ B_2 &= \text{"the chip comes from the second bowl,"} \\ R &= \text{"the chip is red."} \end{aligned}$$

(a) Compute the probabilities $P(R|B_1)$ and $P(R|B_2)$.

The first bowl has 1 red chips and 5 white chips, so $P(R|B_1) = \frac{1}{1+5} = \frac{1}{6}$.

The second bowl has 3 red chips and 3 white chips, so $P(R|B_2) = \frac{3}{3+3} = \frac{3}{6}$.

(b) Suppose that the bowls are equally likely, i.e., $P(B_1) = P(B_2) = 1/2$. In this case, what is the probability that the chip is red?

First method: If the bowls are equally likely then we just have 12 chips, 4 of which are red. So the probability of red is $P(R) = 4/12$.

Second method: We use the "law of total probability" to get

$$\begin{aligned} R &= (R \cap B_1) \sqcup (R \cap B_2) \\ P(R) &= P(R \cap B_1) + P(R \cap B_2) \\ P(R) &= P(B_1)P(R|B_1) + P(B_2)P(R|B_2) \\ &= \frac{1}{2} \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{3}{6} = \frac{4}{12} \approx 33.33\%. \end{aligned}$$

(c) On the other hand, suppose that $P(B_1) = 1/3$ and $P(B_2) = 2/3$. Now what is the probability of getting a red chip?

This time we have to use the law of total probability:

$$\begin{aligned} R &= (R \cap B_1) \sqcup (R \cap B_2) \\ P(R) &= P(R \cap B_1) + P(R \cap B_2) \\ P(R) &= P(B_1)P(R|B_1) + P(B_2)P(R|B_2) \\ &= \frac{1}{3} \cdot \frac{1}{6} + \frac{2}{3} \cdot \frac{3}{6} = \frac{7}{18} \approx 38.89\%. \end{aligned}$$

Problem 3. Suppose that a certain state labels each license plate with a sequence of 7 letters taken from the standard alphabet $\{A, B, C, \ldots, Z\}$.

(a) How many license plates are possible if letters are **not** allowed to be repeated?

$$\underbrace{26}_{\text{1st}} \times \underbrace{25}_{\text{2nd}} \times \underbrace{24}_{\text{3rd}} \times \underbrace{23}_{\text{4th}} \times \underbrace{22}_{\text{5th}} \times \underbrace{21}_{\text{6th}} \times \underbrace{20}_{\text{7th}} = \frac{26!}{19!}$$

(b) How many ways are there to arrange the letters P, I, Z, Z, A, Z, Z?

There are 7! ways to arrange the labeled letters $P, I, Z_1, Z_2, A, Z_3, Z_4$. Then we have to divide by 1!1!4!1! to remove the labels:

$$\binom{7}{1,1,4,1} = \frac{7!}{1!1!4!1!} = 210.$$

(c) Suppose that license plates **are** allowed to contain repeated letters, and that all license plates are equally likely. In this case, what is the probability that a random license plate contains the letters P, I, Z, Z, A, Z, Z, in some order?

If letters can be repeated then the total number of license plates is 26^7. Since these are equally likely the probability of getting P, I, Z, Z, A, Z, Z in some order is

$$\frac{\#(\text{ways to get } P, I, Z, Z, A, Z, Z)}{26^7} = \frac{\binom{7}{1,1,4,1}}{26^7} = \frac{210}{26^7} \approx 0\%.$$

Problem 4. An urn contains 2 orange balls and 5 purple balls. You reach into the urn and pull out a collection of 3 balls (unordered, and without replacement). Assume that all possible outcomes are equally likely.

(a) What is the size of the sample space?

The sample space S is the set of all possible collections of 3 balls. Since there are $2 + 5 = 7$ balls to choose from we have

$$\binom{7}{3} = \frac{7!}{3!4!} = 35.$$

(b) What is the probability of getting **exactly one** orange ball?

The number of ways to get exactly one orange ball is

$$\underbrace{\binom{2}{1}}_{\text{choose 1 orange}} \times \underbrace{\binom{5}{2}}_{\text{choose 2 purple}} = 2 \times 10 = 20,$$

so the probability is $P(\text{ 1 orange }) = \binom{2}{1}\binom{5}{2}/\binom{7}{3} = 20/35 \approx 57.14\%$.

(c) What is the probability of getting **at least one** orange ball?

The number of ways to get **zero** orange balls is $\binom{2}{0}\binom{5}{3} = 10$, so the probability of getting **at least one** orange ball is

$$P(\geq 1 \text{ orange}) = 1 - P(\text{ 0 orange}) = 1 - \frac{\binom{2}{0}\binom{5}{3}}{\binom{7}{3}} = 1 - \frac{10}{35} = \frac{25}{35} \approx 71.43\%.$$

Problem 5. A diagnostic test is administered to a random person to determine if they have a certain disease. Consider the events:

$$T = \text{“the test is positive,”}$$
$$D = \text{“the person has the disease.”}$$

Suppose that the test has the following “false positive” and “false negative” probabilities:

$$P(T|D') = 0.03 \text{ (i.e., 3\%)} \quad \text{and} \quad P(T'|D) = 0.02 \text{ (i.e., 2\%)}.$$

(a) Compute the probabilities $P(T|D)$ and $P(T'|D')$.

Assuming that the person has the disease, we have $P(T'|D) = 0.04$ and hence

$$P(T|D) = 1 - P(T'|D) = 1 - 0.02 = 0.98.$$

Assuming that the person does **not** have the disease, we have $P(T|D') = 0.02$ and hence

$$P(T'|D') = 1 - P(T|D') = 1 - 0.03 = 0.97.$$

(b) Assume that 10% of the population has this disease, i.e., $P(D) = 0.1$. What is the probability that a random person will test positive?

Using the "law of total probability" gives

$$\begin{aligned} T &= (T \cap D) \sqcup (T \cap D') \\ P(T) &= P(T \cap D) + P(T \cap D') \\ &= P(D)P(T|D) + P(D')P(T|D') \\ &= (0.1)(0.98) + (0.9)(0.03) = 12.5\%. \end{aligned}$$

(c) Suppose that a random person is tested and the test returns **positive**. What is the probability that this person actually has the disease?

We are looking for the probability $P(D|T)$. Using Bayes' theorem gives

$$\begin{aligned} P(D|T) &= P(D \cap T) \,/\, P(T) \\ &= P(D)P(T|D) \,/\, P(T) \\ &= \frac{P(D)P(T|D)}{P(D)P(T|D) + P(D')P(T|D')} \\ &= \frac{(0.1)(0.98)}{(0.1)(0.98) + (0.9)(0.03)} = 78.4\%. \end{aligned}$$

Version B

Problem 1. Consider a coin with $P(H) = 3/4$ and $P(T) = 1/4$. Suppose that the coin is flipped 8 times in sequence and suppose that the coin has no memory.

(a) What is the probability of getting the sequence $THTTTHTT$?

Since the coin has no memory, we can multiply the probabilities:

$$\begin{aligned} P(THTTTHTT) &= P(T)P(H)P(T)P(T)P(T)P(H)P(T)P(T) \\ &= P(H)^2 P(T)^6 \\ &= \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^6 = \frac{3^2}{4^8} \approx 0.013\%. \end{aligned}$$

(b) What is the probability that H shows up **exactly twice**?

We use the formula for binomial probability:

$$\begin{aligned} P(\text{we get } H \text{ twice}) &= \binom{8}{2} P(H)^2 P(T)^6 \\ &= \frac{8!}{2!6!}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^6 \\ &= \frac{8\cdot 7}{2}\cdot\frac{3^2}{4^8} \approx 0.38\%. \end{aligned}$$

(c) What is the probability that H shows up **at least once**?

$$\begin{aligned} P(\text{at least one head}) &= 1 - P(\text{all tails}) \\ &= 1 - \left(\frac{1}{4}\right)^8 \\ &= \frac{4^8 - 1}{4^8} \approx 100\%. \end{aligned}$$

Problem 2. There are two bowls on a table. The first bowl contains 3 red chips and 3 white chips. The second bowl contains 2 red chips and 4 white chips. Your friend walks up to the table and chooses one chip at random. Consider the following events:

$$\begin{aligned} B_1 &= \text{"the chip comes from the first bowl,"} \\ B_2 &= \text{"the chip comes from the second bowl,"} \\ R &= \text{"the chip is red."} \end{aligned}$$

(a) Compute the probabilities $P(R|B_1)$ and $P(R|B_2)$.

The first bowl has 3 red chips and 3 white chips, so $P(R|B_1) = \frac{3}{3+3} = \frac{3}{6}$.

The second bowl has 2 red chips and 4 white chips, so $P(R|B_2) = \frac{2}{2+4} = \frac{2}{6}$.

(b) Suppose that the bowls are equally likely, i.e., $P(B_1) = P(B_2) = 1/2$. In this case, what is the probability that the chip is red?

First method: If the bowls are equally likely then we just have 12 chips, 5 of which are red. So the probability of red is $P(R) = 5/12$.

Second method: We use the "law of total probability" to get

$$\begin{aligned} R &= (R \cap B_1) \sqcup (R \cap B_2) \\ P(R) &= P(R \cap B_1) + P(R \cap B_2) \\ P(R) &= P(B_1)P(R|B_1) + P(B_2)P(R|B_2) \\ &= \frac{1}{2}\cdot\frac{3}{6} + \frac{1}{2}\cdot\frac{2}{6} = \frac{5}{12} \approx 41.67\%. \end{aligned}$$

(c) On the other hand, suppose that $P(B_1) = 2/3$ and $P(B_2) = 1/3$. Now what is the probability of getting a red chip?

This time we have to use the law of total probability:

$$\begin{aligned} R &= (R \cap B_1) \sqcup (R \cap B_2) \\ P(R) &= P(R \cap B_1) + P(R \cap B_2) \\ P(R) &= P(B_1)P(R|B_1) + P(B_2)P(R|B_2) \\ &= \frac{2}{3} \cdot \frac{3}{6} + \frac{1}{3} \cdot \frac{2}{6} = \frac{8}{18} \approx 44.44\%. \end{aligned}$$

Problem 3. Suppose that a certain state labels each license plate with a sequence of 6 letters taken from the standard alphabet $\{A, B, C, \ldots, Z\}$.

Remark: $\#\{A, B, C, \ldots, Z\} = 26$. Sorry if this confused anyone.

(a) How many license plates are possible if letters are **not** allowed to be repeated?

$$\underbrace{26}_{\text{1st}} \times \underbrace{25}_{\text{2nd}} \times \underbrace{24}_{\text{3rd}} \times \underbrace{23}_{\text{4th}} \times \underbrace{22}_{\text{5th}} \times \underbrace{21}_{\text{6th}} = \frac{26!}{20!}$$

(b) How many ways are there to arrange the letters B, A, N, A, N, A?

There are 6! ways to arrange the labeled letters $B, A_1, N_1, A_2, N_2, A_3$. Then we have to divide by 1!2!3! to remove the labels:

$$\binom{6}{1,2,3} = \frac{6!}{1!2!3!} = 60.$$

(c) Suppose that license plates **are** allowed to contain repeated letters, and that all license plates are equally likely. In this case, what is the probability that a random license plate contains the letters B, A, N, A, N, A, in some order?

If letters can be repeated then the total number of license plates is 26^6. Since these are equally likely the probably of getting B, A, N, A, N, A in some order is

$$\frac{\#(\text{ways to get } B, A, N, A, N, A)}{26^6} = \frac{\binom{6}{1,2,3}}{26^6} = \frac{60}{26^6} \approx 0\%.$$

Problem 4. An urn contains 2 orange balls and 4 purple balls. You reach into the urn and pull out a collection of 3 balls (unordered, and without replacement). Assume that all possible outcomes are equally likely.

(a) What is the size of the sample space?

The sample space S is the set of all possible collections of 3 balls. Since there are $2 + 4 = 6$ balls to choose from we have

$$\binom{6}{3} = \frac{6!}{3!3!} = 20.$$

(b) What is the probability of getting **exactly one** orange ball?

The number of ways to get exactly one orange ball is

$$\underbrace{\binom{2}{1}}_{\text{choose 1 orange}} \times \underbrace{\binom{4}{2}}_{\text{choose 2 purple}} = 2 \times 6 = 12,$$

so the probability is $P(\,1 \text{ orange}\,) = \binom{2}{1}\binom{4}{2}/\binom{6}{3} = 12/20 = 60\%$.

(c) What is the probability of getting **at least one** orange ball?

The number of ways to get **zero** orange balls is $\binom{2}{0}\binom{4}{3} = 4$, so the probability of getting **at least one** orange ball is

$$P(\geq 1 \text{ orange}) = 1 - P(\,0 \text{ orange}) = 1 - \frac{\binom{2}{0}\binom{4}{3}}{\binom{6}{3}} = 1 - \frac{4}{20} = \frac{16}{20} = 80\%.$$

Problem 5. A diagnostic test is administered to a random person to determine if they have a certain disease. Consider the following events:

$$T = \text{"the test returns positive,"}$$
$$D = \text{"the person has the disease."}$$

Suppose that the test has the following "false positive" and "false negative" probabilities:

$$P(T|D') = 0.02 \text{ (i.e., 2\%)} \quad \text{and} \quad P(T'|D) = 0.04 \text{ (i.e., 4\%)}.$$

(a) Compute the probabilities $P(T|D)$ and $P(T'|D')$.

Assuming that the person has the disease, we have $P(T'|D) = 0.04$ and hence

$$P(T|D) = 1 - P(T'|D) = 1 - 0.04 = 0.96.$$

Assuming that the person does **not** have the disease, we have $P(T|D') = 0.02$ and hence

$$P(T'|D') = 1 - P(T|D') = 1 - 0.02 = 0.98.$$

(b) Assume that 10% of the population has this disease, i.e., $P(D) = 0.1$. What is the probability that a random person will test positive?

Using the "law of total probability" gives

$$\begin{aligned} T &= (T \cap D) \sqcup (T \cap D') \\ P(T) &= P(T \cap D) + P(T \cap D') \\ &= P(D)P(T|D) + P(D')P(T|D') \\ &= (0.1)(0.96) + (0.9)(0.02) = 11.4\%. \end{aligned}$$

(c) Suppose that a random person is tested and the test returns **positive**. What is the probability that this person actually has the disease?

We are looking for the probability $P(D|T)$. Using Bayes' theorem gives

$$\begin{aligned}
P(D|T) &= P(D \cap T) \,/\, P(T) \\
&= P(D)P(T|D) \,/\, P(T) \\
&= \frac{P(D)P(T|D)}{P(D)P(T|D) + P(D')P(T|D')} \\
&= \frac{(0.1)(0.96)}{(0.1)(0.96) + (0.9)(0.02)} \approx 84.21\%.
\end{aligned}$$