Math 210 F **Spring 2016**
Homework 5 **Drew Armstrong**

Problem 1. Continued from HW3.1. Any linear system of m equations in n unknowns can be written in the form

$$A\vec{x} = \vec{b}$$

where A is an $m \times n$ matrix, $\vec{x}$ is an $n \times 1$ column, and $\vec{b}$ is an $m \times 1$ column. Use matrix algebra to give a short proof of the following statement: If a linear system has **two solutions** then it must have **a whole line of solutions**. [Hint: Let $\vec{x}$ and $\vec{y}$ be two solutions. Now consider the line $t\vec{x} + (1-t)\vec{y}$.]

Problem 2. Definition of Matrix Multiplication. Let A and B be matrices such that the number of columns in A equals the number of rows in B. Then the product matrix AB exists and has the following properties:

$$\begin{aligned} (i,j)^{\text{th}} \text{ entry of } AB &= (i^{\text{th}} \text{ row of } A)\,(j^{\text{th}} \text{ column of } B) \\ i^{\text{th}} \text{ row of } AB &= (i^{\text{th}} \text{ row of } A)\, B \\ j^{\text{th}} \text{ column of } AB &= A\,(j^{\text{th}} \text{ column of } B)\,. \end{aligned}$$

Now consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 0 & -1 \\ 3 & 1 & 2 \end{pmatrix}.$$

(a) Compute the 2^{nd} column of $A^2 = AA$ without computing the full matrix A^2.
(b) Compute the 2^{nd} column of $A^3 = AA^2$ without computing the full matrix A^3.
(c) Compute the 2^{nd} column of $A^4 = AA^3$ without computing the full matrix A^4.

Problem 3. Rotation Matrices. The following matrix rotates the plane $\mathbb{R}^2$ counterclockwise by angle θ:

$$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

(a) Find the matrices R_{30°, R_{45°, R_{60° and R_{90°.
(b) Give a geometric explanation why for all angles α and β we have

$$R_\alpha R_\beta = R_{\alpha+\beta}.$$

[Hint: Don't do any calculations.]
(c) Now use the result of part (b) to prove the trigonometric angle sum identities:

$$\begin{cases} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \cos\alpha\sin\beta + \sin\alpha\cos\beta. \end{cases}$$

[Hint: Do a calculation.]

Problem 4. Projection Matrices. We say that P is a projection matrix if

- $P^T = P$ (i.e., P is "symmetric"), and
- $P^2 = P$ (i.e., P is "idempotent").

(a) If P is a projection, show that $I - P$ is also a projection.
(b) Show that the projections P and $I - P$ satisfy $P(I-P) = 0$.

Problem 5. How to Compute an Inverse. Let A be a matrix and suppose that we have

$$A\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad A\vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad A\vec{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Explain why the matrix A has three rows.
(b) Let $X = \begin{pmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{pmatrix}$ be the matrix with columns $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$. Compute AX.

Problem 6. Actually Doing It. Now consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solve the three linear systems from Problem 5 to find the vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ and the matrix X. Then compute the products AX and XA to make sure that your answer is correct.