MTH 210, Spring 2016
HW3 Solutions

Problem 1:

(a) Let a, b, c, d be constants and consider the linear equation

$$ax + by + cz = d.$$

[This represents a plane in 3D]. Now suppose we have two points (x_1, y_1, z_1) and (x_2, y_2, z_2) such that

- $ax_1 + by_1 + cz_1 = d$
- $ax_2 + by_2 + cz_2 = d.$

Then we must also have

$$a\left(\frac{x_1+x_2}{2}\right) + b\left(\frac{y_1+y_2}{2}\right) + c\left(\frac{z_1+z_2}{2}\right)$$

$$= \frac{1}{2}\left[a(x_1+x_2) + b(y_1+y_2) + c(z_1+z_2)\right]$$

$$= \frac{1}{2}\left[ax_1 + ax_2 + by_1 + by_2 + cz_1 + cz_2\right]$$

$$= \frac{1}{2}\left[(ax_1 + by_1 + cz_1) + (ax_2 + by_2 + cz_2)\right]$$

$$= \frac{1}{2}\left[d + d\right]$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2}d\right] = d.$$

Hence the midpoint is also a solution.
[Geometrically, if two points lie on a plane in 3D then their midpoint also lies on the plane. This is why we say a plane is "flat".]

(b) If 25 hyperplanes in 12-dimensional space meet at two given points then they must also meet at

the midpoint of those two points.

The reason is the same as in part (a), i.e., because hyperplanes are "flat".

[We will give a quick algebraic proof later in the language of "matrix algebra".]

Problem 2: Consider the linear system

$$\begin{cases} x + y + z = 2 & ① \\ x + 2y + z = 3 & ② \\ x + 3y + 2z = 5 & ③ \end{cases}$$

(a) We write the system as an augmented matrix and then perform Gaussian elimination as follows:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 3 & 2 & 5 \end{array}\right) \quad \begin{array}{l} ① \\ ② \\ ③ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 3 \end{array}\right) \quad \begin{array}{l} ① \to ① \\ ② \to ② - ① \\ ③ \to ③ - ① \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \begin{array}{l} ① \to ① \\ ② \to ② \\ ③ \to ③ - 2② \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \begin{array}{l} ① \to ① - ③ \\ ② \to ② \\ ③ \to ③ \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \begin{array}{l} ① \to ① - ② \\ ② \to ② \\ ③ \to ③ \end{array}$$

Now we translate this back into a linear system

$$\begin{cases} x + 0 + 0 = 0 \\ 0 + y + 0 = 1 \\ 0 + 0 + z = 1 \end{cases}$$

(b) Row Picture: The three original planes intersect at the single point

$$(x, y, z) = (0, 1, 1).$$

(c) Column Picture: We can reach the target vector $(2, 3, 5)$ with the following linear combination:

$$0\begin{pmatrix}1\\1\\1\end{pmatrix} + 1\begin{pmatrix}1\\2\\3\end{pmatrix} + 1\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}2\\3\\5\end{pmatrix}.$$

[Pictures omitted]

Problem 3. Let c be a constant and consider the linear system

$$\begin{cases} x + y + z = 2 & (1) \\ x + 2y + z = 3 & (2) \\ 2x + 3y + 2z = c & (3) \end{cases}$$

(a) To put the system in staircase form we first eliminate below the x pivot to get

$$\begin{cases} x + y + z = 2 & (1) \to (1) \\ 0 + y + 0 = 1 & (2) \to (2) - (1) \\ 0 + y + 0 = c - 4 & (3) \to (3) - 2(1) \end{cases}$$

Then we eliminate below the y pivot to get

$$\begin{cases} x + y + z = 2 & (1) \to (1) \\ 0 + y + 0 = 1 & (2) \to (2) \\ 0 + 0 + 0 = c - 5 & (3) \to (3) - (2) \end{cases}$$

Note that we obtain the equation $0 = c - 5$. If $c \neq 5$ then the system will have NO SOLUTION.

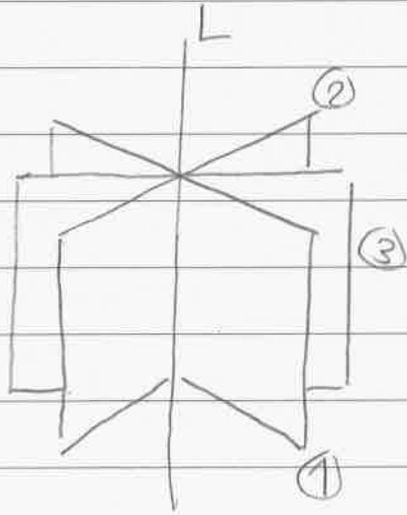
(b) The first two planes meet in a line
call it L. When $c = 5$ then we have
∞ many solutions because the third
plane

contains the line L,

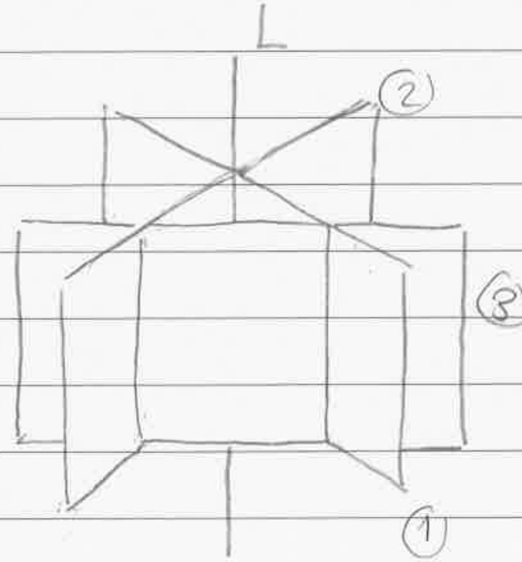
but when $c = 6$ (more generally for
any $c \neq 5$) we have 0 solutions
because the third plane

does not intersect (i.e. is
parallel to) the line L.

Picture:



$c = 5$



$c \neq 5$.

(c) It is impossible for the system to have exactly 1 solution because if we have one solution

$$(*) \quad x_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + y_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + z_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix},$$

then we also have another solution,

$$(x_1 + k, y_1, z_1 - k) \text{ for any } k.$$

Proof: Assume $(*)$ is true. Then we have

$$(x_1 + k) \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + y_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + (z_1 - k) \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$= x_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + y_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + z_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \cancel{k \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} - \cancel{k \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}$$

$$= \begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ c \end{pmatrix}.$$

Problem 4. Consider the linear system

$$\begin{cases} 0 + x_2 + 0 + x_4 - x_5 - 4x_6 = -1 & (1) \\ x_1 + 2x_2 - x_3 + 4x_4 - x_5 - 4x_6 = 3 & (2) \\ x_1 + 2x_2 - x_3 + 4x_4 + 0 - x_6 = 5 & (3) \end{cases}$$

(a) We write the system as an augmented matrix and then perform Gaussian elimination to obtain the RREF

$$\begin{array}{cc} \left(\begin{array}{cccccc|c} 0 & 1 & 0 & 1 & -1 & -4 & -1 \\ 1 & 2 & -1 & 4 & -1 & -4 & 3 \\ 1 & 2 & -1 & 4 & 0 & -1 & 5 \end{array}\right) & \begin{array}{l} (1) \\ (2) \\ (3) \end{array} \end{array}$$

$$\begin{array}{cc} \left(\begin{array}{cccccc|c} 1 & 2 & -1 & 4 & -1 & -4 & 3 \\ 0 & 1 & 0 & 1 & -1 & -4 & -1 \\ 1 & 2 & -1 & 4 & 0 & -1 & 5 \end{array}\right) & \begin{array}{l} (1) \to (2) \\ (2) \to (1) \\ (3) \to (3) \end{array} \end{array}$$

$$\begin{array}{ccc} \text{REF} & \left(\begin{array}{cccccc|c} 1 & 2 & -1 & 4 & -1 & -4 & 3 \\ 0 & 1 & 0 & 1 & -1 & -4 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 \end{array}\right) & \begin{array}{l} (1) \to (1) \\ (2) \to (2) \\ (3) \to (3) - (1) \end{array} \end{array}$$

$$\begin{array}{cc} \left(\begin{array}{cccccc|c} 1 & 2 & -1 & 4 & 0 & -1 & 5 \\ 0 & 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 \end{array}\right) & \begin{array}{l} (1) \to (1) + (3) \\ (2) \to (2) + (3) \\ (3) \to (3) \end{array} \end{array}$$

$$\text{RREF} \quad \left(\begin{array}{cccccc|c} 1 & 0 & -1 & 2 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 \end{array}\right) \quad \begin{array}{l} ① \to ① - 2② \\ ② \to ② \\ ③ \to ③ \end{array}$$

Then we turn this back into a linear system.

$$\begin{cases} x_1 + 0 - x_3 + 2x_4 + 0 + x_6 = 3 \\ \quad x_2 + 0 + x_4 + 0 - x_6 = 1 \\ \qquad\qquad x_5 + 3x_6 = 2 \end{cases}$$

The pivot variables are x_1, x_2, x_5 and the free variables are x_3, x_4, x_6. Let's define parameters $r := x_3$, $s := x_4$, $t := x_6$. Then the solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 + r - 2s - t \\ 1 - s + t \\ r \\ s \\ 2 - 3t \\ t \end{pmatrix}$$

We get a better of the solution if we write it like this:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} + r \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ -3 \\ 1 \end{pmatrix}$$

This is a 3-dimensional plane living in 6-dimensional space, which is what we expected because

$$\begin{array}{ccccc} \# \text{ variables} & - & \# \text{ equations} & & \\ 6 & - & 3 & = & 3 \ \checkmark \end{array}$$