2/3/16

HW2 due NOW.

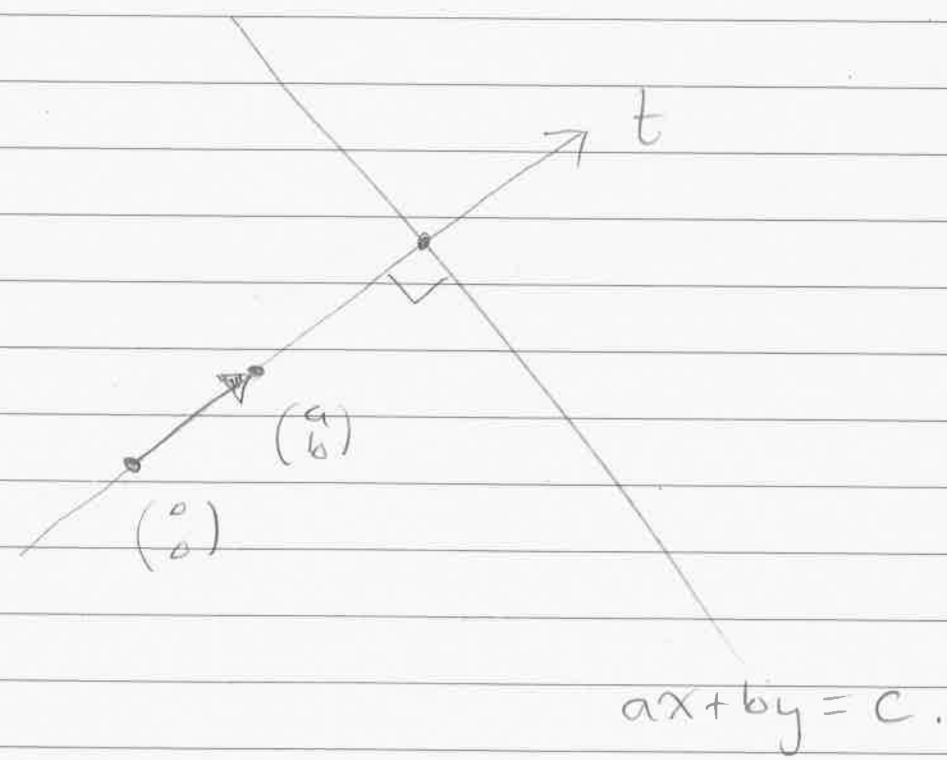
Today: HW2 Discussion.

Problem 1′: Let a, b & c be constants and let x, y be variables [this convention goes back to Descartes]. The line

$$ax + by = c$$

is perpendicular to the vector (a, b).

Picture:



There are infinitely many points on the line. Are any of them special?

Well, there is one point on the line that is closest to $(0,0)$. I think that's pretty special. For geometric reasons, this point is the intersection of the line $ax+by=c$ with the perpendicular line $(x,y)=t(a,b)=(ta,tb)$.

Let's compute it. We have a system of 3 equations in 3 unknowns x, y, t:

$$\begin{cases} x = ta \\ y = tb \\ ax+by=c. \end{cases}$$

But it's pretty easy to solve so we don't need any fancy technique. Just substitute:

$$\begin{aligned} ax+by &= c \\ a(ta)+b(tb) &= c \\ ta^2+tb^2 &= c \\ t(a^2+b^2) &= c \\ t &= c/(a^2+b^2). \end{aligned}$$

The point of intersection is

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} a \\ b \end{pmatrix} = \frac{c}{a^2+b^2} \begin{pmatrix} a \\ b \end{pmatrix}.$$

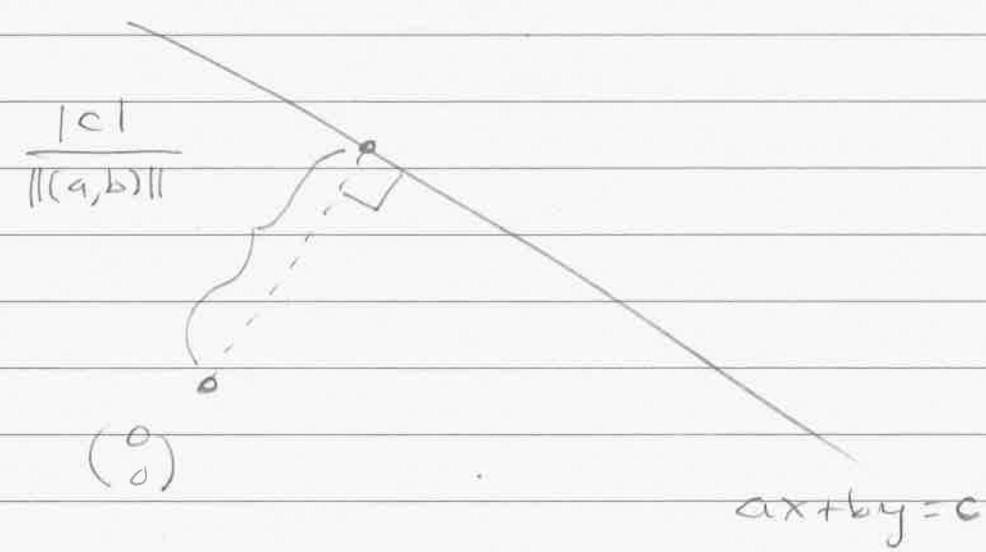
So this is the <u>closest</u> point on the line $ax+by=c$ to the origin.

HOW close is it?

After a bit of calculation (omitted for todays discussion) you will find that the distance between $(0,0)$ & $\frac{c}{a^2+b^2}(a,b)$ is

$$|c| \,/\, \|(a,b)\|.$$

Picture:



Later in the course we will be all about computing the minimum distance from a certain point to a certain "d-plane".

Problem 3': The single vector equation

(*) $$x\begin{pmatrix}-1\\1\end{pmatrix} + y\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}$$

is equivalent to the system of two simultaneous number equations

(**) $$\begin{cases} -x + 2y = 3 \\ x = 2 \end{cases}.$$

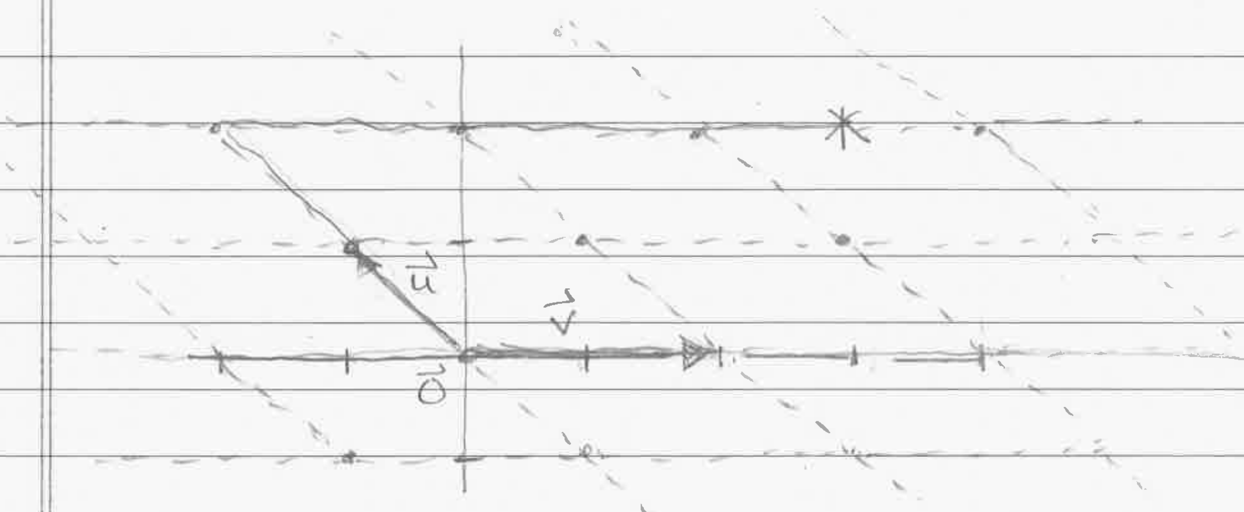
Therefore they have the same solution, which happens to be

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2\\5/2\end{pmatrix} \quad \text{or} \quad \begin{cases} x = 2 \\ y = 5/2 \end{cases}.$$

But our mental pictures of the problems (*) and (**) are quite different.

In (*) we have a "target vector" $(3,2)$ and we want to get there but we are only allowed to travel in the directions $\vec{u} := (-1,1)$ & $\vec{v} := (2,0)$.

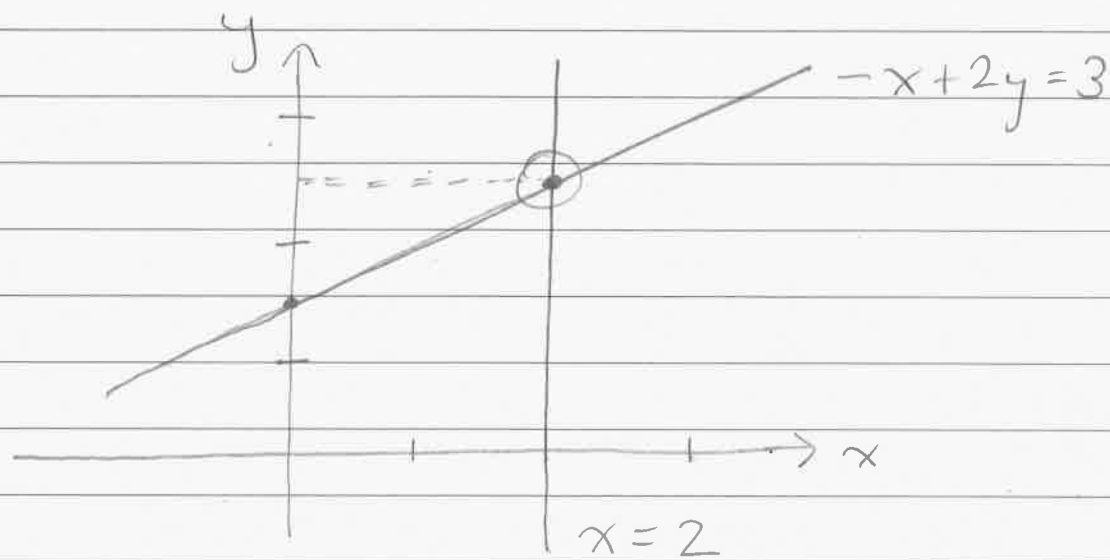
Let's think of $\vec{u}$ & $\vec{v}$ as a new "coordinate system" for the plane:



We start at the point $\vec{0}$. To get to the restaurant at $(3,2)$ we must travel
2 blocks in the $\vec{u}$ direction and
5/2 blocks in the $\vec{v}$ direction.

Gilbert Strang calls this the "column picture" of the system.

In (**) we have two equations representing lines in the plane. The solution of the system is interpreted as the point of intersection. So let's draw the lines:



Note that $-x + 2y = 3$ is the same as $y = 1/2\ x + 3/2$ in "slope/y-intercept" form.

The point of intersection is $(x,y) = (2, 5/2)$.

Gilbert Strang calls this the "row picture" of the system.

The point I want to emphasize is that the "row" & "column" pictures are just two different visualizations for the

SAME MATHEMATICAL PROBLEM.

In general it is a strength when we have multiple ways to visualize a mathematical problem because it gives us more opportunities to use our intuition.

2/5/16

HW3 will appear soon.

So far we have been building intuition and learning the background material. Today, we are finally ready to state the central problem of linear algebra.

Recall that a system of m equations in n unknowns (most likely) represents an $(n-m)$-dimensional shape living in n-dimensional space. [That is, each new equation probably reduces the dimension of the solution by 1.]

Unfortunately, the problem of general equations is mostly impossible so in this course we will focus on a very special kind of equations.

☆ Definition: A <u>linear equation</u> in the n unknowns $x_1, x_2, \ldots, x_n$ has the form

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b}$$

where $a_1, a_2, \ldots, a_n$ and b are constants.
We can also express this equation as

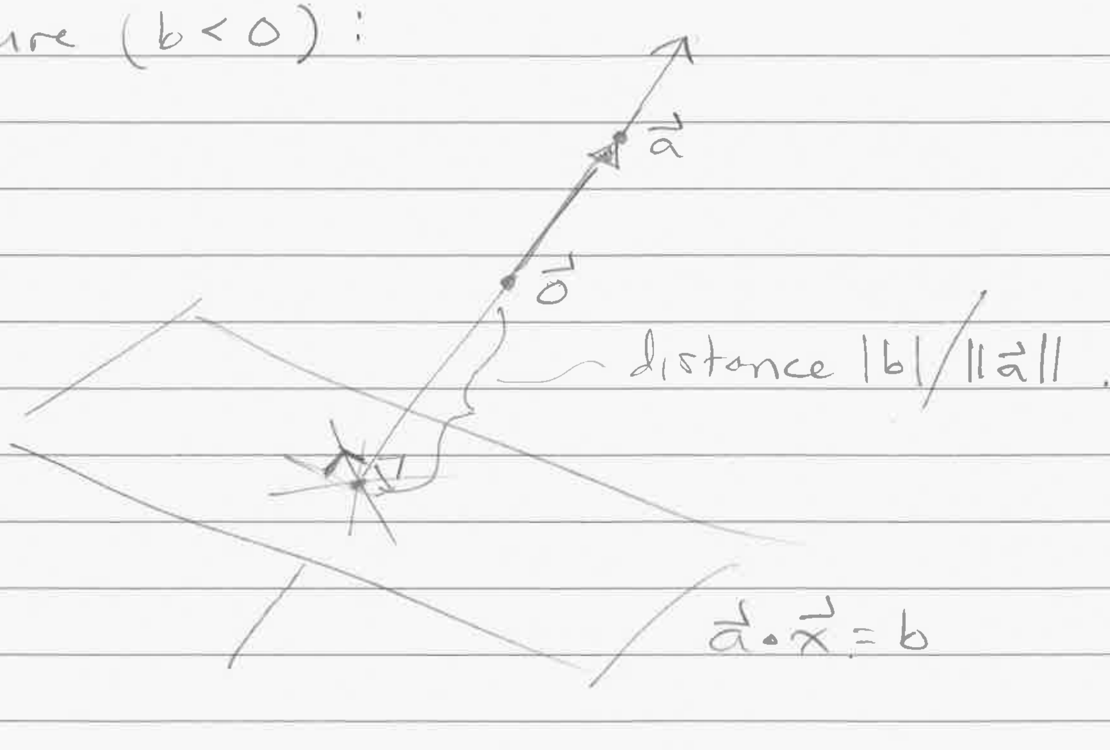
$$\boxed{\vec{a} \cdot \vec{x} = b}$$

where $\vec{a} = (a_1, a_2, \ldots, a_n)$ & $\vec{x} = (x_1, \ldots, x_n)$.

Geometrically, this is the "hyperplane" perpendicular to $\vec{a}$ that has distance $|b| / \|\vec{a}\|$ from the origin

[Recall that a "hyperplane" is an $(n-1)$-dimensional flat shape living in n-dimensional space.]

Picture $(b<0)$:



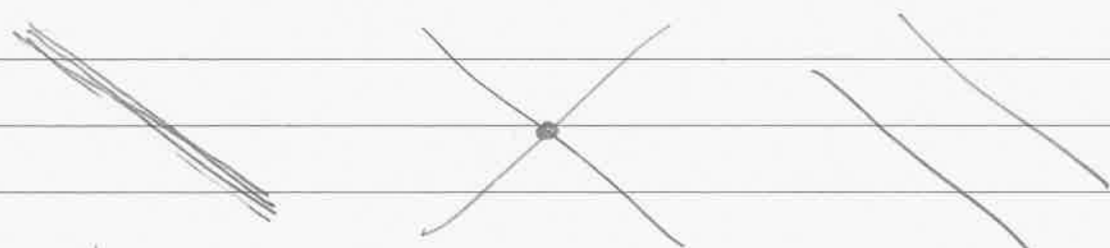
Special Case: If $b = 0$ then the hyperplane contains the point $\vec{0}$; otherwise not.

Finally, here it is:

☆ The Central Problem of Linear Algebra is to solve a system of m linear equations in n unknowns.

Geometrically, this means we want to compute the intersection of m hyperplanes living in n-dimensional space. We know that the answer will be a "d-plane" [flat d-dimensional shape] for some d. Probably we will have $d = n - m$, but funny things can happen sometimes.

Example ($m = 2$, $n = 2$): The intersection of two lines can be a line (1-plane), a point (0-plane) or empty.



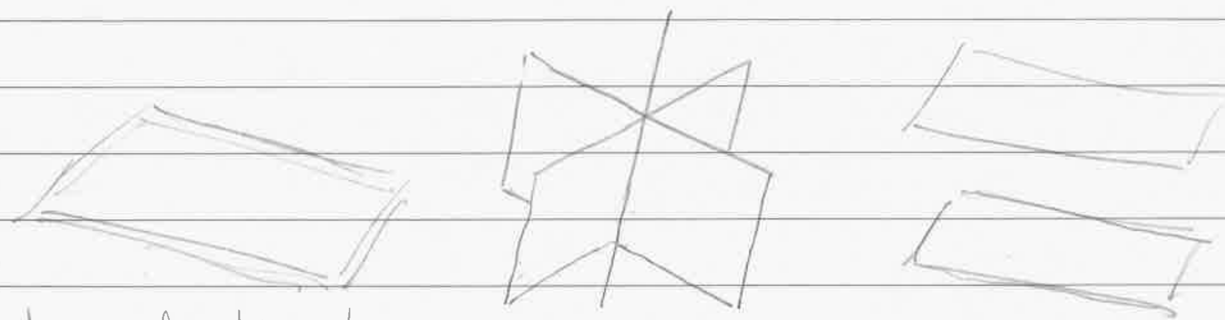
two lines on top of each other

two parallel lines

Intersecting at a point is most likely.

[Remark: Some people call an empty intersection a "(−1)-plane"! You do not need to call it that.]

Example ($m=2$, $n=3$): Two planes in 3D can intersect in a plane (2-plane), a line (1-plane) or have empty intersection.



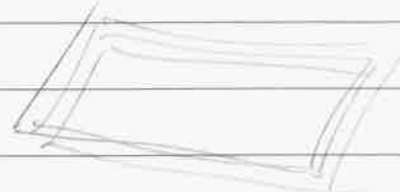
two identical planes

two parallel planes.

The line is most likely.

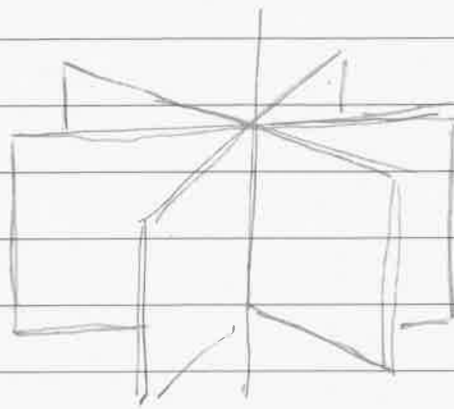
Example ($m=3$, $n=3$): Three planes in 3D can intersect in a plane (2-plane), a line (1-plane), a point (0-plane); or have empty intersection.

2-plane

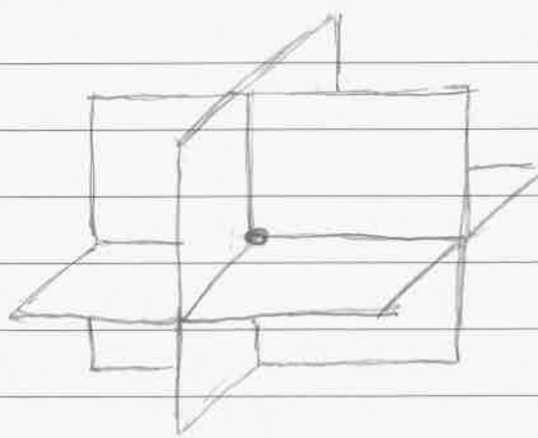


three identical planes

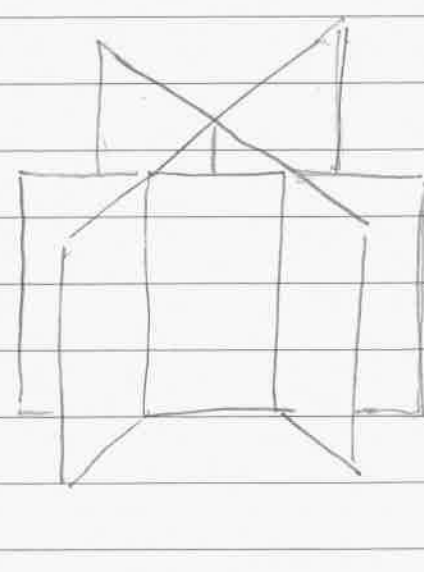
1-plane



0-plane



empty



Intersecting in a point is most likely.

In higher dimensions the number of possible configurations explodes and the pictures are impossible to draw. Nevertheless, humans have a complete and satisfactory way to solve this problem.

Q: How do they do it?

A: With algebra!

Example: Solve the linear system

$$\begin{cases} x + y + z = 2 & (1) \\ x + 2y + z = 3 & (2) \\ 2x + 3y + 2z = 5 & (3) \end{cases}$$

We will use the method of elimination. First eliminate x from (2) & (3) using (1):

$$\begin{array}{ll} (2) & x + 2y + z = 3 \\ (1) & x + y + z = 2 \end{array}$$

$$(2)-(1) \qquad y = 1 \qquad (2)'$$

Hey, that was lucky; z went away too!

$$\begin{array}{lr} (3) & 2x + 3y + 2z = 5 \\ (1) & x + y + z = 2 \end{array}$$

$$(3) - 2(1) \qquad y = 1 \qquad (3)'$$

Our new equivalent system is

$$\begin{cases} x + y + z = 2 & (1) \\ y = 1 & (2)' \\ y = 1 & (3)' \end{cases}$$

Next we eliminate y from $(3)'$ using $(2)'$:

$$\begin{array}{ll} (3)' & y = 1 \\ (2)' & y = 1 \end{array}$$

$$(3)' - (2)' \qquad 0 = 0. \qquad (3)''$$

Oops, we got the true (but uninteresting) equation "$0 = 0$". Our new system is

$$\begin{cases} x + y + z = 2 & (1) \\ y = 1 & (2)' \\ 0 = 0 & (3)'' \end{cases}$$

Finally we eliminate y from ① using ②$'$:

$$\begin{array}{llcl} ① & x + y + z & = & 2 \\ ②' & \quad\; y & = & 1 \end{array}$$

$$①-②' \quad x \qquad + z = 1 \qquad ①'$$

Our final equivalent system is

$$\begin{cases} x \qquad + z = 1 & ①' \\ \quad y \qquad\;\; = 1 & ②' \\ \qquad\quad 0 = 0 & ③'' \end{cases}$$

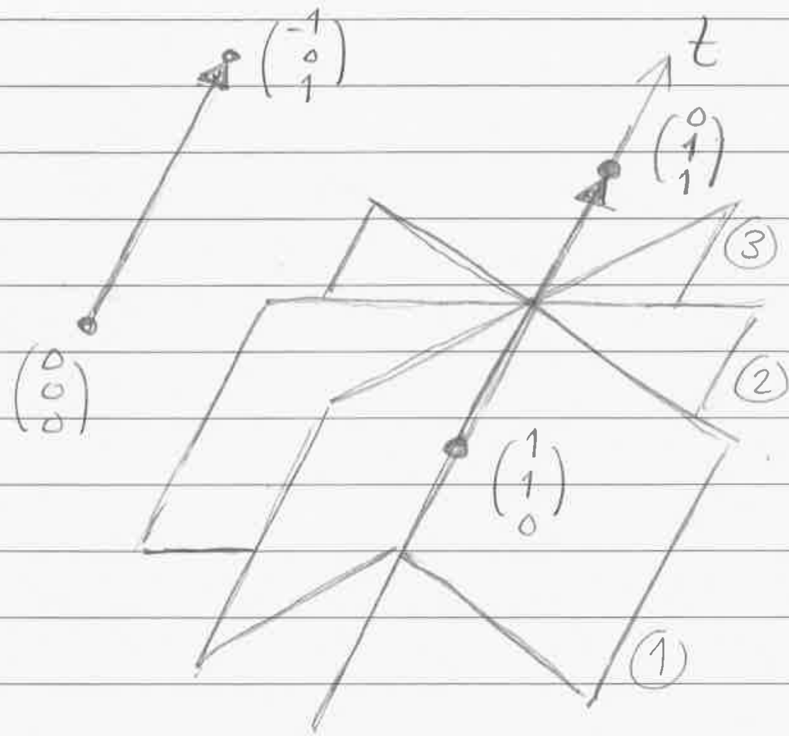
There is nothing left to do but read off the answer. We use $z = t$ as a parameter to get

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - z \\ 1 \\ z \end{pmatrix} = \begin{pmatrix} 1 - 1z \\ 1 + 0z \\ 0 + 1z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

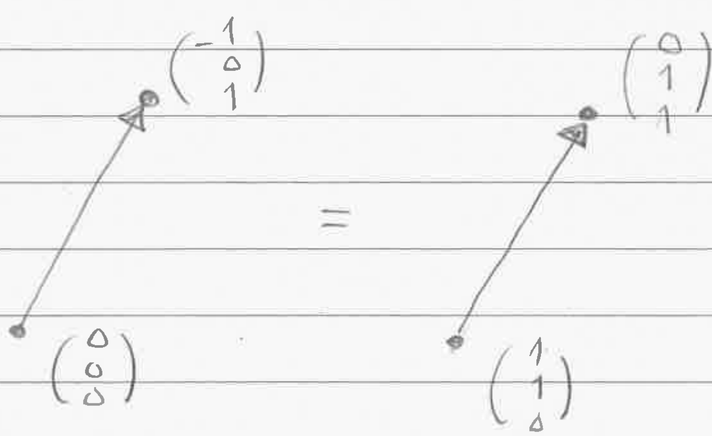
$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

This is a line.

Picture: The three original planes (1), (2), (3) meet in the line that contains the point $(1,1,0)$ and is parallel to the vector $(-1,0,1)$.



[Note that



.]

2/8/16

I'll distribute HW3 on Wed.

Last time I stated the "Central Problem of Linear Algebra":

☆ To solve a system of m simultaneous linear equations in n unknowns.

We will write the general system as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where $x_1, \ldots, x_n$ are variables and $a_{11}, \ldots, a_{mn}$ & $b_1, \ldots, b_m$ are constants.

There are two different ways to visualize a linear system. Gilbert Strang calls them the "row picture" and the "column picture".

(1) The Row Picture.

The m simultaneous linear equations in n variables represent the intersection of m hyperplanes in n-dimensional space.

Intuition: If the equations are "random" or "generic" then the solution will be an

(n-m)-dimensional plane.

If m>n then there is "probably"

NO SOLUTION.

Example: 3 planes in 3D probably meet at a point. 4 planes in 3D probably don't meet anywhere [the first 3 probably meet at a point and then the 4th probably doesn't contain this point].

↓

(2) The Column Picture.

We can rewrite the original system of m linear equations as <u>one vector equation</u>:

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\boxed{x_1 \vec{a}_{*1} + x_2 \vec{a}_{*2} + \cdots + x_n \vec{a}_{*n} = \vec{b}}$$

Problem: Combine n given vectors in m-dimensional space to reach a given target vector. In other words: starting at $\vec{0}$, how far do you have to travel in the directions $\vec{a}_{*1}, \vec{a}_{*2}, \ldots, \vec{a}_{*n}$ in order to reach the restaurant at $\vec{b}$?

Since this problem is mathematically <u>equivalent</u> to (1) we can transfer some of our intuition.

Example: If I give you 3 direction vectors $\vec{a}_{*1}, \vec{a}_{*2}, \vec{a}_{*3}$ and a target vector $\vec{b}$ in 4D space, can you find numbers x_1, x_2, x_3 such that

$$x_1 \vec{a}_{*1} + x_2 \vec{a}_{*2} + x_3 \vec{a}_{*3} = \vec{b} \ ?$$

Probably not! There are two reasons:

1. The vectors $x_1 \vec{a}_{*1} + x_2 \vec{a}_{*2} + x_3 \vec{a}_{*3}$ probably form a 3-plane in 4D and the point $\vec{b}$ is probably not in this 3-plane.

2. The row picture of this problem is the intersection of 4 planes in 3D and we already agreed that this probably has no solution.

The pictures ① & ② are quite valuable for interpreting solutions or guessing the form of the solution. But they don't help us to actually compute the solution. For that we need a specific algebraic technique; and we are lucky to have one.

Our technique is called "Gaussion Elimination" and we've already seen it in action. I'll be a bit more explicit today.

Example of Gaussian Elimination:

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 & (1) \\ 0 + 0 + x_3 + 4x_4 = 6 & (2) \\ x_1 + 3x_2 + x_3 + 6x_4 = 7 & (3) \end{cases}$$

Use the "pivot" x_1 in (1) to eliminate x_1 from (2) & (3). Luckily, equation (2) already has no x_1.

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 & (1)' = (1) \\ 0 + 0 + x_3 + 4x_3 = 6 & (2)' = (2) \\ 0 + 0 + x_3 + 4x_3 = 6 & (3)' = (3) - 1(1) \end{cases}$$

Now we look for a pivot in the x_2 column but there isn't one! So we move on the x_3 column. We will use the pivot x_3 in (2)' to eliminate the x_3 from (3)'.

$$\begin{cases} \boxed{x_1} + 3x_2 + 0 + 2x_4 = 1 & ①'' = ①' \\ 0 + 0 + \boxed{x_3} + 4x_4 = 6 & ②'' = ②' \\ 0 + 0 + 0 + 0 = 0 & ③'' = ③' - 1② ' \end{cases}$$

Now we look for a pivot in the x_4 column but there isn't one! Oh well. Now our system is in "row echelon form" (REF).

[Note : echelon = staircase].

The final step is to multiply equations by numbers so that the pivot terms have coefficient 1. Then we perform "backwards elimination" to eliminate the terms above our pivots. Since both of these steps are already done (luckily) we can say that our system is in "reduced row echelon form" (RREF).

Once the system is in RREF it becomes easy to read off the solution. In our case we have

pivot variables : x_1, x_3
free variables : x_2, x_4

The solution is

$$
\begin{aligned}
x_1 &= 1 - 3x_2 - 2x_4 \\
x_2 &= x_2 \\
x_3 &= 6 - 4x_4 \\
x_4 &= x_4
\end{aligned}
$$

which can be written in vector form as

$$
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 - 3x_2 - 2x_4 \\ 0 + 1x_2 + 0x_4 \\ 6 + 0x_2 - 4x_4 \\ 0 + 0x_2 + 1x_4 \end{pmatrix}
$$

$$
= \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 4 \\ 1 \end{pmatrix}.
$$

Note that the solution is "2-dimensional" because it has two free variables.

2/10/16

HW 3 due next Wed Feb 17.

HW 2	
Total	26
Ave	21
Med	22

My Office Hours : Mon 12-1
Thurs 2-3

Exam 1 is Fri Mar 4 in class.

Recall from last time: We considered the linear system

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 & ① \\ 0 + 0 + x_3 + 4x_4 = 6 & ② \\ x_1 + 3x_2 + x_3 + 6x_4 = 7 & ③ \end{cases}$$

We performed "Gaussian elimination" to put the system in the form

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 \\ 0 + 0 + x_3 + 4x_4 = 6 \\ 0 + 0 + 0 + 0 = 0 \end{cases}$$

We called this the "reduced row echelon form" (RREF) of the system.

The variables in the corners of the staircase (i.e. x_1 & x_3) are called pivot variables and all other variables (i.e. x_2 & x_4) are called free variables.

Finally we can write down the solution in terms of the free variables:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 - 3x_2 - 2x_4 \\ x_2 \\ 6 - 4x_4 \\ x_4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 3x_2 - 2x_4 \\ 0 + 1x_2 + 0x_4 \\ 6 + 0x_2 - 4x_4 \\ 0 + 0x_2 + 1x_4 \end{pmatrix}$$

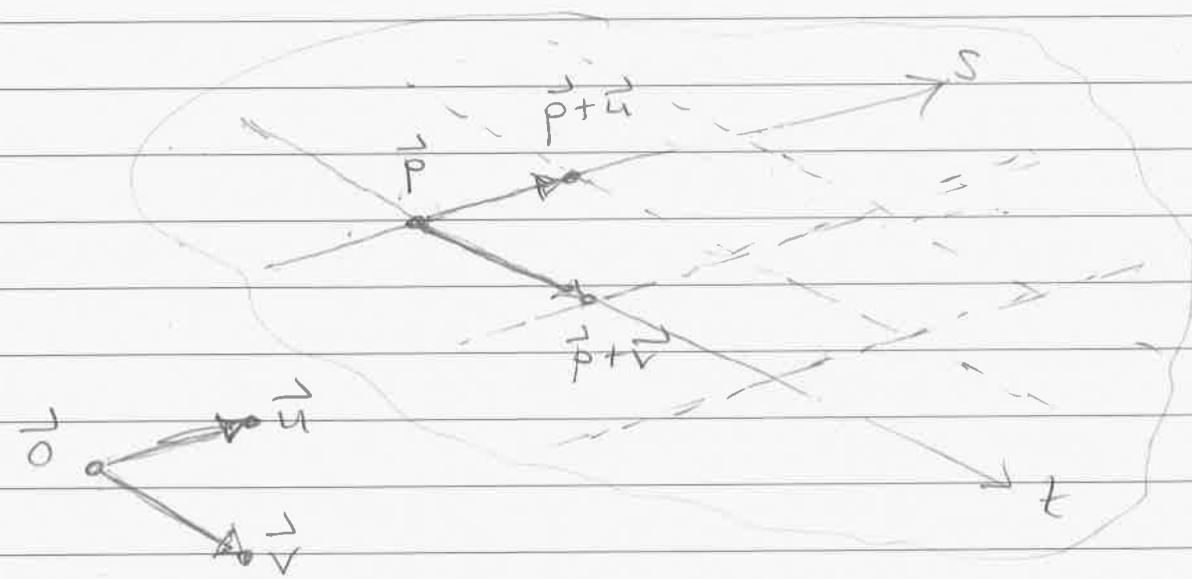
$$= \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix}.$$

To clean this up let's define $\vec{p} = (1, 0, 6, 0)$, $\vec{u} = (-3, 1, 0, 0)$, $\vec{v} = (-2, 0, -4, 1)$, $x_2 = s$ & $x_4 = t$.

Then the solution is

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \vec{p} + s\vec{u} + t\vec{v}}$$

Picture: This is the 2-dimensional plane in 4D containing the point $\vec{p}$ and spanned by the vectors $\vec{u}$ & $\vec{v}$.



This plane does not contain the origin $\vec{0}$.

We can think of the 2-plane $\vec{p}+s\vec{u}+t\vec{v}$ as the interesection of the three hyperplanes defined by equations (1), (2) & (3).

This is the row picture

Q: We expect three hyperplanes in 4D to intersect in a line (1-plane). What went wrong?

A: While performing elimination we found the relationship

$$① + ② = ③.$$

This means that any solution to the first two equations is also a solution to the third. Geometrically, the intersection hyperplanes ① & ② is accidentally contained in the hyperplane ③.

That means there must also be something wrong with the column picture. Let's see what it is. The system (*) becomes one vector equation:

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 7 \end{pmatrix}.$$

AHA, I see the problem! The vectors

$$\begin{pmatrix}1\\0\\1\end{pmatrix} \ \& \ \begin{pmatrix}3\\0\\3\end{pmatrix}$$

are in the same direction, so one of them was completely unnecessary.

That was a human introduction to Gaussian elimination. Now let me show you how I would tell it to a computer.

Given a system of equations, there are three operations we can do that will yield an equivalent system (i.e. a system with the same solution):

(A) Swap two equations

(B) Replace equation (i) by c (i) where c is a nonzero constant.

(C) Replace equation (i) by (i) − c (j) where c is any constant and (j) is any other equation.

We call A, B, C the elementary row operations (EROs). The goal of Gaussian elimination is to perform a sequence of EROs to put a linear system in a nice, standard form (the RREF).

[Most computers have a button to do this.]

Here's (one version of) the algorithm:

- Do A to get a nonzero pivot in the top left corner. If this is impossible, move one column to the right. If that's impossible, STOP.
- Do B to turn the pivot into a 1.
- Do C to eliminate all entries below the pivot.
- Repeat the process on the subsystem below and to the right of the pivot.

Now the system looks something like this:

$$\left\{\begin{array}{cccccccccc}
0 & 0 & 1 & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right.$$

- Finally, do (c) to eliminate all entries <u>above</u> the pivots, working from the bottom right to the top left.

NOW the system is in RREF, which looks something like this:

$$\left\{\begin{array}{cccccccccc}
0 & 0 & 1 & * & 0 & * & * & 0 & * & 0 \\
0 & 0 & 0 & 0 & 1 & * & * & 0 & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right.$$

Performing Gaussian elimination is a job best left to computers, but we <u>will</u> practice doing it by hand on some small systems. [see HW 3]

2/12/16

HW3 due next Wed.
Exam 1 is Fri Mar 4 in class.

Last time I finally defined the method of "Gaussian elimination" in all its gory details. This method was invented by Carl Friedrich Gauss around 1800 in order to compute the orbits of various celestial bodies. However, a similar method already appeared in China in the "Nine Chapters on the Mathematical Art" (263 A.D.).

The algorithm is best suited for computers but we can still compute some small systems by hand. Today we'll get some practice with this.

Example 1: Solve the system.

$$\begin{cases} 0 + y + z = 1 & (1) \\ x + y + z = 2 & (2) \\ 2x + 0 + 3z = 1 & (3) \end{cases}$$

First swap (1) & (2) to get a pivot in the top left.

$$\begin{cases} x + y + z = 2 & ①' = ② \\ 0 + y + z = 1 & ②' = ① \\ 2x + 0 + 3z = 1 & ③' = ③. \end{cases}$$

The pivot already has coefficient 1. ✓
Now eliminate below the x pivot.

$$\begin{cases} x + y + z = 2 & ①'' = ①' \\ 0 + y + z = 1 & ②'' = ②' \\ 0 - 2y + z = -3 & ③'' = ③' - 2①' \end{cases}$$

Now recursively apply the method to the subsystem ②'' & ③'' that involves only y & z.

$$\begin{cases} x + y + z = 2 & ①''' = ①'' \\ 0 + y + z = 1 & ②''' = ②'' \\ 0 + 0 + 3z = -1 & ③''' = ③'' + 2②'' \end{cases}$$

Divide equation ③''' by 3 to get the pivot 1. Now the system is in "row echelon form" (REF).

$$\begin{cases} x + y + z = 2 & ①'''' = ①''' \\ 0 + y + z = 1 & ②'''' = ②''' \\ 0 + 0 + z = -1/3 & ③'''' = \tfrac{1}{3}③'''. \end{cases}$$

To put the system in reduced row echelon form (RREF) we first eliminate above the z pivot.

$$\begin{cases} x + y + 0 = 7/3 & (1)''''' = (1)'''' - (3)'''' \\ 0 + y + 0 = 4/3 & (2)''''' = (1)'''' - (2)'''' \\ 0 + 0 + z = -1/3 & (3)''''' = (3)'''' \end{cases}$$

Finally, we eliminate above the y pivot.

$$\begin{cases} x + 0 + 0 = 1 & (1)'''''' = (1)''''' - (2)''''' \\ 0 + y + 0 = 4/3 & (2)'''''' = (2)''''' \\ 0 + 0 + z = -1/3 & (3)'''''' = (2)''''' \end{cases}$$

This is the RREF, and now the solution is obvious:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4/3 \\ -1/3 \end{pmatrix}.$$

Row Picture: The three planes (1), (2), (3) meet at the single point $(1, 4/3, -1/3)$.

$\wr$

Column Picture: We can reach the point $(1, 2, 1)$ by combining the three columns as follows.

$$1\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \frac{4}{3}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

As we see, the notation gets quite cumbersome. So in the next example let's streamline the notation by throwing away all unnecessary symbols.

Example 2: Solve the system.

$$\begin{cases} 2x_1 - 3x_2 - x_3 + 2x_4 + 3x_5 = 4 & (1) \\ 4x_1 - 4x_2 - x_3 + 4x_4 + 11x_5 = 4 & (2) \\ 2x_1 - 5x_2 - 2x_3 + 2x_4 - x_5 = 9 & (3) \\ 0 + 2x_2 + x_3 + 0 + 4x_5 = -5 & (4) \end{cases}$$

Instead we'll write it like this:

$$\begin{array}{cc} \left(\begin{array}{ccccc|c} 2 & -3 & -1 & 2 & 3 & 4 \\ 4 & -4 & -1 & 4 & 11 & 4 \\ 2 & -5 & -2 & 2 & -1 & 9 \\ 0 & 2 & 1 & 0 & 4 & 5 \end{array}\right) & \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array} \end{array}.$$

This is called the "augmented matrix" notation. Now we perform Gaussian elimination as usuall. [Actually, I'll avoid scaling the pivots to 1 until the end because I don't like fractions.]

$$\left(\begin{array}{ccccc|c} 2 & -3 & -1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 0 & 5 & -4 \\ 0 & -2 & -1 & 0 & -4 & 5 \\ 0 & 2 & 1 & 0 & 4 & 5 \end{array}\right) \qquad \begin{array}{l} 1 \to 1 \\ 2 \to 2 - 1 \\ 3 \to 3 - 1 \\ 4 \to 4 \end{array}$$

$$\left(\begin{array}{ccccc|c} 2 & -3 & -1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 0 & 5 & -4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{array}\right) \qquad \begin{array}{l} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 + 2 \\ 4 \to 4 - 2 \end{array}$$

$$\left(\begin{array}{ccccc|c} 2 & -3 & -1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 0 & 5 & -4 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \\ 4 \to 4 + 3 \end{array}$$

REF

$$\left(\begin{array}{ccccc|c} 1 & -3/2 & -1/2 & 1 & 3/2 & 2 \\ 0 & 1 & 1/2 & 0 & 5/2 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} 1 \to 1/2 \, 1 \\ 2 \to 1/2 \, 2 \\ 3 \to 3 \\ 4 \to 4 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -3/2 & -1/2 & 1 & 0 & 1/2 \\ 0 & 1 & 1/2 & 0 & 0 & -9/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} (1) \to (1) - \frac{3}{2}(3) \\ (2) \to (2) - \frac{5}{2}(3) \\ (3) \to (3) \\ (4) \to (4) \end{array}$$

RREF

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1/4 & 1 & 0 & -25/4 \\ 0 & 1 & 1/2 & 0 & 0 & -9/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} (1) \to (1) + \frac{3}{2}(2) \\ (2) \to (2) \\ (3) \to (3) \\ (4) \to (4) \end{array}$$

Translating back to old notation, the RREF of the system is

$$\begin{cases} x_1 + 0 + \frac{1}{4}x_3 + x_4 + 0 = -25/4 \\ 0 + x_2 + \frac{1}{2}x_3 + 0 + 0 = -9/2 \\ 0 + 0 + 0 + 0 + x_5 = 1 \\ 0 + 0 + 0 + 0 + 0 = 0 \end{cases}$$

Pivot variables : x_1, x_2, x_5

Free variables : x_3, x_4.

So let's define $s := x_3$, $t := x_4$ and then express the solution in terms of the parameters s & t.

The solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -25/4 - 1/4\, x_3 - x_4 \\ -9/2 - 1/2\, x_3 \\ x_3 \\ x_4 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -25/4 - 1/4\, s - 1t \\ -9/2 - 1/2\, s + 0t \\ 0 + 1s + 0t \\ 0 + 0s + 1t \\ 1 + 0s + 0t \end{pmatrix}$$

$$\begin{pmatrix} -25/4 \\ -9/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1/4 \\ -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

This is a parametrized 2D plane living in 5D space. We expected a line $(5-4=1)$ but we got a plane, so there must be some relationship among the equations.

2/15/16

HW 3 due Wed.
Exam 1 Fri Mar 4 in class.
Office Hours: Mon 12-1 & Thurs 2-3.

Last time we looked at the following system of 4 linear equations in 5 unknowns:

$$\begin{cases} 2x_1 - 3x_2 - x_3 + 2x_4 + 3x_5 = 4 & (1) \\ 4x_1 - 4x_2 - x_3 + 4x_4 + 11x_5 = 4 & (2) \\ 2x_1 - 5x_2 - 2x_3 + 2x_4 - x_5 = 9 & (3) \\ 0 + 2x_2 + x_3 + 0 + 4x_5 = -5 & (4) \end{cases}$$

We dropped all unnessary symbols to write this as an "augmented matrix":

$$\left(\begin{array}{ccccc|c} 2 & -3 & -1 & 2 & 3 & 4 \\ 4 & -4 & -1 & 4 & 11 & 4 \\ 2 & -5 & -2 & 2 & -1 & 9 \\ 0 & 2 & 1 & 0 & 4 & -5 \end{array}\right)$$

Then we performed Gaussian elimination to put the matrix in RREF:

$\wr$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 1/4 & 1 & 0 & -25/4 \\ 0 & 1 & 1/2 & 0 & 0 & -9/2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Finally, we turned this back into a system of equations

$$\begin{cases} x_1 + \frac{1}{4}x_3 + x_4 = -25/4 \\ x_2 + \frac{1}{2}x_3 = -9/2 \\ x_5 = 1 \\ 0 = 0 \end{cases}$$

and then read off the solution. Naming the free variables $x_3 = s$ & $x_4 = t$ gives us

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -\frac{25}{4} - \frac{1}{4}s - t \\ -9/2 - 1/2\,s \\ s \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} -25/4 \\ -9/2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} -1/4 \\ -1/2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

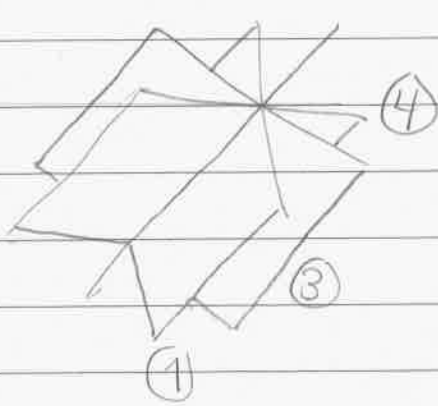
Now let's interpret the solution.

Row Picture: The intersection of the 4 hyperplanes ①, ②, ③, ④ is a 2-dimensional plane living in 5D space.

This is not what we expected. [With $m = 4$ equations in $n = 5$ unknowns we expect a $(n-m) = (5-4) = 1$ dimensional solution.] So there must have been some relationship among the equations. Sure enough, we have

$$① = ③ + ④,$$

which means that any one of these three equations can be thrown away without changing the solution. Geometrically, the intersection of any two of these hyperplanes is contained in the third. My mental picture looks like this



even though I know this picture has the wrong dimension. [The correct picture is three 4-planes meeting at a 3-plane in 5D space, which I can't draw.]

Column Picture: We are trying to hit the point $\vec{b} = (4,4,9,-5)$ in 5D space by <u>combining</u> the five vectors

$$\vec{a}_1 = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{a}_2 = \begin{pmatrix} -3 \\ -4 \\ -5 \\ 2 \end{pmatrix}, \quad \vec{a}_3 = \begin{pmatrix} -1 \\ -1 \\ -2 \\ 1 \end{pmatrix}, \quad \vec{a}_4 = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{a}_5 = \begin{pmatrix} 3 \\ 11 \\ -1 \\ 4 \end{pmatrix}.$$

We know that there must be some relationship among these vectors [because the solution doesn't have the expected dimension] and indeed there is:

$$\vec{a}_1 = \vec{a}_4 .$$

This means that <u>if</u> the problem has a solution, then it must have in<u>finite</u>ly <u>many</u> solutions.

Indeed, suppose that

(*). $x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 + x_4 \vec{a}_4 + x_5 \vec{a}_5 = \vec{b}$

is a solution. Then I claim that for any number k we have another solution

$$(x_1 + k, x_2, x_3, x_4 - k, x_5).$$

Proof: Assuming (*) is true we have

$$(x_1+k)\vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 + (x_4 - k)\vec{a}_4 + x_5 \vec{a}_5$$

$$= (x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 + x_4 \vec{a}_4 + x_5 \vec{a}_5) + (k\vec{a}_1 - k\vec{a}_4)$$

$$= \vec{b} + \vec{0} = \vec{b},$$

because $\vec{a}_1 = \vec{a}_4$ and hence

$$k\vec{a}_1 - k\vec{a}_4 = k\vec{a}_1 - k\vec{a}_1 = \vec{0}.$$

//

That's enough interpretation for today.

Let's summarize what we know about linear systems and Gaussian elimination.

1. A linear system looks like this:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

2. We can express it as an augmented matrix by dropping all the unnecessary symbols:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

3. After performing Gaussian elimination we obtain the reduced row echelon form (RREF) which looks like this:

$$\left(\begin{array}{cccccccc|c} 0 & 1 & * & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{array}\right)$$

4. If we obtain the equation $0 = *$ where $*$ is not zero, then the system has NO SOLUTION.

5. Otherwise, we let $t_1, t_2, \ldots, t_f$ be the free variables and we express the solution as

$$\vec{x} = \vec{p} + t_1 \vec{u}_1 + t_2 \vec{u}_2 + \cdots + t_f \vec{u}_f$$

for some point $\vec{p}$ and some vectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_f$. The solution is an f-dimensional plane living in n-dimensional space.

For the example in part 3, we have pivot variables

$$x_2, x_4, x_7$$

and free variables

$$x_1, x_3, x_5, x_6, x_8.$$

The solution is a 5-dimensional plane living in 8-dimensional space.

Now you have seen everything there is to see about Gaussian elimination. We'll let it sink in for a little while and then we'll move on to something else.