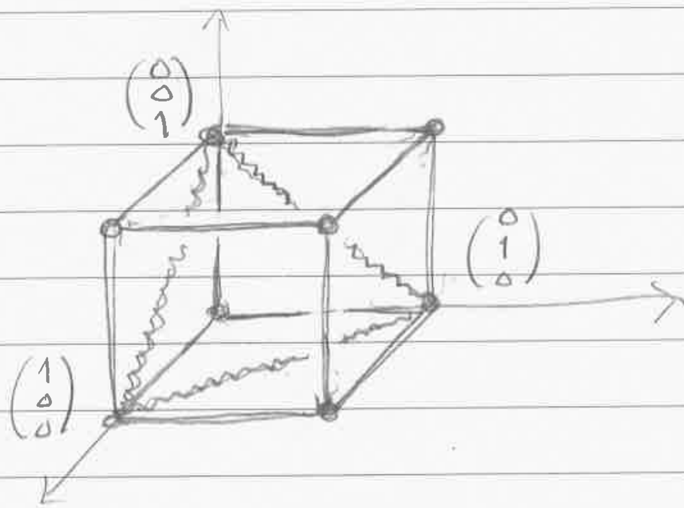
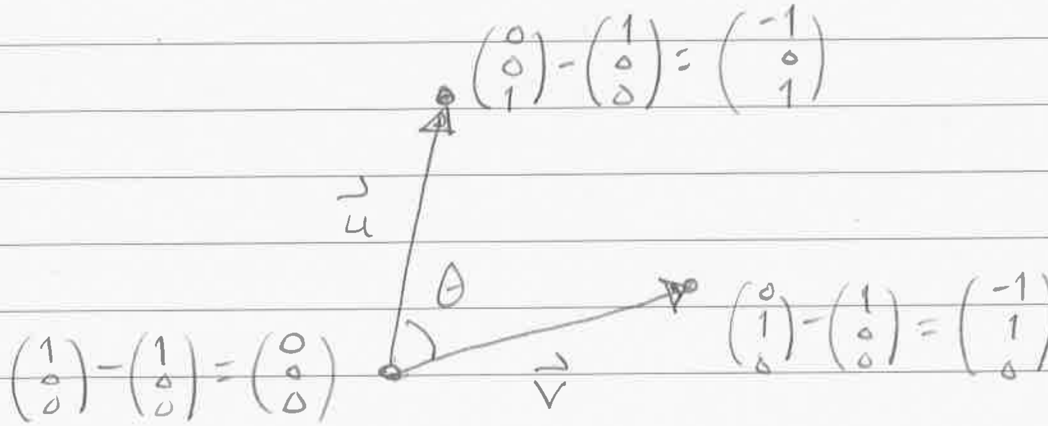
MTH 210, Spring 2016
HW1 Solutions.

Problem 1:

(a)



(b) The triangle is shown above with the squiggly lines. To compute the angle between two edges we should move the corresponding vectors to the origin. Here's one case:



To compute the angle we use the dot product.

$$\cos\theta = \vec{u}\cdot\vec{v} \Big/ \left(\|\vec{u}\|\cdot\|\vec{v}\|\right).$$

$$= \begin{pmatrix}-1\\0\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\0\end{pmatrix} \Big/ \left(\left\|\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\|\cdot\left\|\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\|\right)$$

$$= \left((-1)^2+0+0\right) \Big/ \left(\sqrt{(-1)^2+0^2+1^2}\cdot\sqrt{(-1)^2+1^2+0^2}\right)$$

$$= 1 \Big/ \left(\sqrt{2}\cdot\sqrt{2}\right) = 1/2.$$

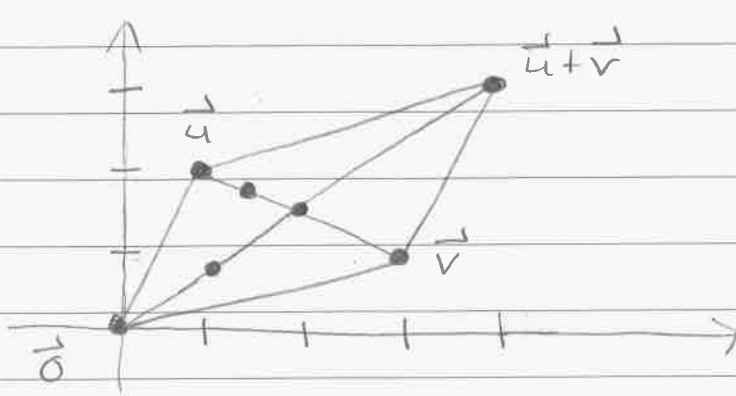
We conclude that

$$\theta = \arccos\left(1/2\right) = \pi/3 \quad (\text{or } 60^\circ).$$

It turns out that all three angles are the same [the triangle is equilateral].

Problem 2:

(a)



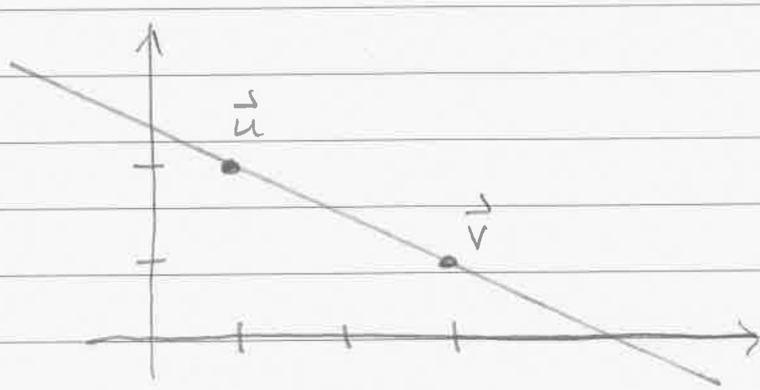
Note that

- $\frac{1}{2}\vec{u} + \frac{1}{2}\vec{v}$ is the midpoint of $\vec{u}$ & $\vec{v}$.
- $\frac{3}{4}\vec{u} + \frac{1}{2}\vec{v}$ is the midpoint of $\vec{u}$ & $\frac{1}{2}\vec{u} + \frac{1}{2}\vec{v}$.
- $\frac{1}{4}\vec{u} + \frac{1}{4}\vec{v}$ is the midpoint of $\vec{0}$ & $\frac{1}{2}\vec{u} + \frac{1}{2}\vec{v}$.

[In general, the midpoint of points $\vec{a}$ & $\vec{b}$ is

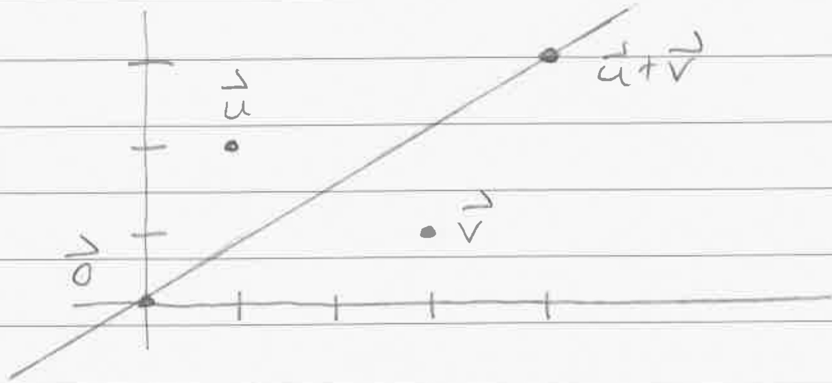
$$\frac{1}{2}(\vec{a} + \vec{b}) = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}. \quad]$$

(b) The line $a\vec{u} + b\vec{v}$ where $a+b=0$ contains the point $\vec{u}$ (when $a=1$ & $b=0$) and the point $\vec{v}$ (when $a=0$ & $b=1$). Hence it looks like this:



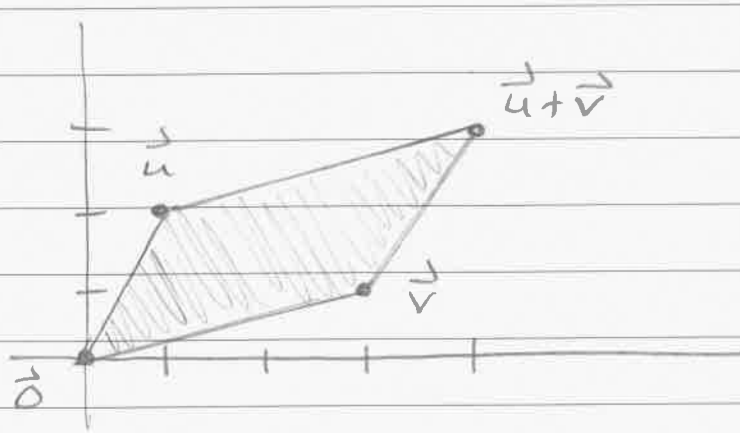
[You might try computing the equation of this line.]

(c) The line $a\vec{u} + a\vec{v}$ contains the point $\vec{0}$ (when $a=0$) and the point $\vec{u}+\vec{v}$ (when $a=1$). Hence it looks like this:

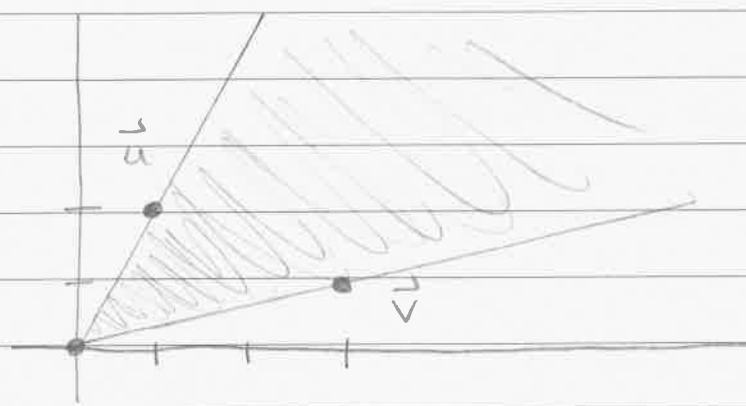


[The equation of this line is . . .]

(d) The region $a\vec{u} + b\vec{v}$ when $0 \leq a \leq 1$ and $0 \leq b \leq 1$ is the filled parallelogram with vertices $\vec{0}$, $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$:



(d) The region $a\vec{u} + b\vec{v}$ with $0 \le a$ & $0 \le b$.
We could call this region a "two-dimensional cone". It looks like this:



Problem 3: Let $\vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ & $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$.

Then for all numbers a we have

$$\vec{u} \bullet (\vec{v} + a\vec{w}) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \bullet \begin{pmatrix} v_1 + a w_1 \\ \vdots \\ v_n + a w_n \end{pmatrix}$$

$$= u_1(v_1 + a w_1) + \cdots + u_n(v_n + a w_n)$$

$$= (u_1 v_1 + a u_1 w_1) + \cdots + (u_n v_n + a v_n w_n)$$

$$= (u_1 v_1 + \cdots + u_n v_n) + a(u_1 w_1 + \cdots + u_n w_n)$$

$$= \vec{u} \circ \vec{v} + a\, \vec{u} \circ \vec{w}.$$

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Problem 4: Let $\vec{u}$ & $\vec{v}$ be two vectors in n-dimensional space such that

$$\|\vec{u}\| = \|\vec{v}\| = 1.$$

(a) Then we have

$$\vec{u} \circ (-\vec{u}) = -(\vec{u} \circ \vec{u}) = -\|\vec{u}\|^2 = -1^2 = -1.$$

$$\begin{aligned}(\vec{u}+\vec{v}) \circ (\vec{u}-\vec{v}) &= (\vec{u}+\vec{v}) \circ \vec{u} - (\vec{u}+\vec{v}) \circ \vec{v} \\ &= \vec{u} \circ \vec{u} + \cancel{\vec{v} \circ \vec{u}} - \cancel{\vec{u} \circ \vec{v}} - \vec{v} \circ \vec{v} \\ &= \vec{u} \circ \vec{u} - \vec{v} \circ \vec{v} \\ &= \|\vec{u}\|^2 - \|\vec{v}\|^2 \\ &= 1^2 - 1^2 \qquad = 0.\end{aligned}$$

$$\begin{aligned}(\vec{u}-2\vec{v}) \circ (\vec{u}+2\vec{v}) &= (\vec{u}-2\vec{v}) \circ \vec{u} + 2(\vec{u}-2\vec{v}) \circ \vec{v} \\ &= \vec{u} \circ \vec{u} - \cancel{2\vec{v} \circ \vec{u}} + \cancel{2\vec{u} \circ \vec{v}} - 4\vec{v} \circ \vec{v} \\ &= \vec{u} \circ \vec{u} - 4\vec{v} \circ \vec{v} \\ &= \|\vec{u}\|^2 - 4\|\vec{v}\|^2 \\ &= 1^2 - 4 \cdot 1^2 \qquad = -3.\end{aligned}$$

(b) Now let's also assume that $\vec{u} \circ \vec{v} = 0$ and consider the vectors

$$\vec{a} := \vec{u} + 2\vec{v} \quad \& \quad \vec{b} := 3\vec{u} + \vec{v}.$$

Let θ be the angle between $\vec{a}$ & $\vec{b}$ so that

$$\vec{a} \circ \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta .$$

To compute θ we first need to know $\vec{a} \circ \vec{b}$, $\|\vec{a}\|$, and $\|\vec{b}\|$.

$$\begin{aligned}
\vec{a} \circ \vec{b} &= (\vec{u} + 2\vec{v}) \circ (3\vec{u} + \vec{v}) \\
&= 3\vec{u} \circ \vec{u} + \vec{u} \circ \vec{v} + 6\vec{v} \circ \vec{u} + 2\vec{v} \circ \vec{v} \\
&= 3\vec{u} \circ \vec{u} + 7\vec{u} \circ \vec{v} + 2\vec{v} \circ \vec{v}. \\
&= 3(1) + 7(0) + 2(1) \quad = \quad 5
\end{aligned}$$

$$\begin{aligned}
\|\vec{a}\|^2 &= \vec{a} \circ \vec{a} \\
&= (\vec{u} + 2\vec{v}) \circ (\vec{u} + 2\vec{v}) \\
&= \vec{u} \circ \vec{u} + 2\vec{u} \circ \vec{v} + 2\vec{v} \circ \vec{u} + 4\vec{v} \circ \vec{v} \\
&= \vec{u} \circ \vec{u} + 4\vec{u} \circ \vec{v} + 4\vec{v} \circ \vec{v} \\
&= (1) + 4(0) + 4(1) \quad = \quad 5
\end{aligned}$$

$$\begin{aligned}
\|\vec{b}\|^2 &= \vec{b} \circ \vec{b} \\
&= (3\vec{u} + \vec{v}) \circ (3\vec{u} + \vec{v}) \\
&= 9\vec{u} \circ \vec{u} + 3\vec{u} \circ \vec{v} + 3\vec{v} \circ \vec{u} + \vec{v} \circ \vec{v} \\
&= 9\vec{u} \circ \vec{u} + 6\vec{u} \circ \vec{v} + \vec{v} \circ \vec{v} \\
&= 9(1) + 6(0) + (1) \quad = \quad 10.
\end{aligned}$$

We conclude that

$$\vec{a} \circ \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cos\theta$$

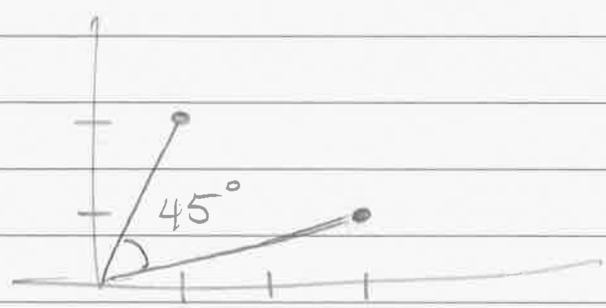
$$5 = \sqrt{5} \cdot \sqrt{10} \cos\theta$$

$$\cancel{5} = \cancel{5}\sqrt{2} \cos\theta$$

$$\cos\theta = 1/\sqrt{2}$$

$$\theta = 45^\circ \quad (\text{or } 315^\circ).$$

[Remark: We get the same result by computing the angle between the vectors $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\1\end{pmatrix}$ in 2D.



I wonder why that might be ...]