Math 210 E **Drew Armstrong**
Final Exam **Monday, May 6, 2013**

This is a closed book test. No electronic devices are allowed. If two students submit exams in which any solution has been copied, **both students will receive a score of zero**. There are 7 pages and 7 problems.

Problem 1. Consider the plane $x + 2y - z = 0$ in $\mathbb{R}^3$.

(a) Tell me a normal vector to the plane.

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

(b) Tell me a normal vector of length 1.

$$\frac{1}{\|\vec{a}\|}\vec{a} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

(c) Compute the matrix Q that projects orthogonally onto the normal line.

$$Q = \vec{a}(\vec{a}^T\vec{a})^{-1}\vec{a}^T = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\left((1\ 2\ {-1})\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right)^{-1}(1\ 2\ {-1})$$

$$= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}(6)^{-1}(1\ 2\ {-1}) = \frac{1}{6}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}(1\ 2\ {-1})$$

$$= \frac{1}{6}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

(d) Compute the matrix P that projects orthogonally onto the plane. [Hint: Use your answer from part (c) to save time.]

$$P = I - Q$$

$$= \frac{1}{6}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}$$

Problem 2. The following matrix **rotates** vectors in $\mathbb{R}^2$ **counterclockwise** by 53.13°:

$$R = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.$$

(You can just believe this. You don't have to show it.)

(a) Rotate the column vector $(1, 1)$ **counterclockwise** by 53.13°.

$$R\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

(b) Compute the matrix that rotates vectors **clockwise** by 53.13°.

$\det(R) = 9/25 + 16/25 = 25/25 = 1$, so

$$R^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$

(c) Rotate the column vector $(1, 1)$ **clockwise** by 53.13°.

$$R^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 7 \\ -1 \end{pmatrix}$$

(d) Compute the matrix that rotates **counterclockwise** by 106.26°($= 2 \times 53.13°$).

$$R^2 = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$

$$= \frac{1}{25}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$

$$= \frac{1}{25}\begin{pmatrix} 9-16 & -12-12 \\ 12+12 & -16+9 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} -7 & -24 \\ 24 & -7 \end{pmatrix}$$

Problem 3. Consider the following three planes in $\mathbb{R}^3$:

$$x + y + z = 0, \tag{1}$$
$$x + 2y - z = 0, \tag{2}$$
$$x + 0y + z = 1. \tag{3}$$

(a) Compute the intersection of the **first and second planes**. [Hint: It's a line.]

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$$

$$\longrightarrow \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \\ \begin{array}{ccc} x & y & z \end{array} \end{array} \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3s \\ 2s \\ s \end{pmatrix} = s\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}.$$

Let $z = s$

(b) Compute the intersection of **second and third planes**. [Hint: It's a line.]

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & -2 & -1 \end{array}\right)$$

$$\longrightarrow \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1/2 \end{array}\right) \\ \begin{array}{ccc} x & y & z \end{array} \end{array} \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-t \\ -1/2+t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Let $z = t$

(c) Compute the intersection of the lines from parts (a) and (b). [Hint: It's a point.]

Multiple ways to do this. I'll intersect the line from (a) with the plane (3).

$$x + z = 1$$
$$(-3s) + (s) = 1$$
$$-2s = 1$$
$$s = -1/2$$

$$\longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1 \\ -1/2 \end{pmatrix}.$$

Problem 4. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

(a) Use the Gauss-Jordan method to compute the inverse of A.

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 - R_2 \end{array} \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ R_2 \\ R_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ R_2 \\ R_3 \end{array} = (I \mid A^{-1})$$

$$\Longrightarrow A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

(b) Solve the system $A\vec{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. [Hint: Use your answer from (a) to save time.]

$$\vec{x} = A^{-1} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Problem 5. Consider the following system:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & b \\ 0 & 1 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 0 \\ c \end{pmatrix}.$$

Tell me some values for b and c such that

(a) the system has a **unique solution** $\vec{x}$.

$b = 0$ and $c =$ anything.

Then the solution is

$$\vec{x} = A^{-1}\begin{pmatrix} 1 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ c \end{pmatrix}$$ from problem 4.

(b) the system has **no solution** $\vec{x}$.

$b = 1$ and $c \neq 0$. Then

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & c \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & c \end{array}\right)$$

impossible.

(c) the system has **infinitely many solutions** $\vec{x}$.

$b = 1$ and $c = 0$. Then

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Now there are infinitely many solutions. (I won't bother finding them.)

A has shape $n \times m$

Problem 6. Consider the matrix $A = (\vec{a}_1 \;\; \vec{a}_2 \;\; \cdots \;\; \vec{a}_m)$, where $\vec{a}_i$ is the ith **column**. Suppose that A has n rows, so the columns are vectors in $\mathbb{R}^n$.

(a) Let $\vec{x}$ be a vector in $\mathbb{R}^n$. Write a single matrix equation to say that $\vec{x}$ is perpendicular to all the columns of A.

$$A^T \vec{x} = \vec{0}$$

(b) You can think of your matrix equation from part (a) as a system of **how many** linear equations, in **how many** unknowns?

A^T has shape $m \times n$, so $A^T \vec{x} = \vec{0}$

has m equations in n unknowns

(c) The solution to your equation in part (a) most likely has **how many dimensions**?

\# variables − \# equations

$$n - m$$

(d) Now let $\vec{b}$ be any point in $\mathbb{R}^n$ and let $\vec{p}$ be the point in the column space of A that is **closest** to $\vec{b}$. Write a formula for $\vec{p}$ in terms of A and $\vec{b}$.

$$\vec{p} = A(A^T A)^{-1} A^T \vec{b}$$

Problem 7. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$ with $\vec{u}^T\vec{v} \neq 0$, and consider the $n \times n$ matrix

$$A = \frac{\vec{u}\,\vec{v}^T}{\vec{u}^T\vec{v}}. = \underbrace{\left(\frac{1}{\vec{u}^T\vec{v}}\right)}_{1\times 1} \underbrace{\vec{u}\vec{v}^T}_{n\times n}$$

(a) For all vectors $\vec{x}$, show that $A\vec{x}$ is on the line generated by $\vec{u}$.

$$A\vec{x} = \frac{1}{\vec{u}^T\vec{v}}(\vec{u}\vec{v}^T)\vec{x} = \underbrace{\left(\frac{1}{\vec{u}^T\vec{v}}\right)}_{\text{number}}\vec{u}\underbrace{(\vec{v}^T\vec{x})}_{\text{number}} = \underbrace{\left(\frac{\vec{v}^T\vec{x}}{\vec{u}^T\vec{v}}\right)}_{\text{number}}\vec{u}$$

(b) The line generated by $\vec{u}$ is an eigenspace for A. What is the eigenvalue?

From part (a) we have

$$A\vec{u} = \underbrace{\left(\frac{\cancel{\vec{v}^T\vec{u}}}{\cancel{\vec{u}^T\vec{v}}}\right)}_{1}\vec{u} = 1\vec{u}$$

The eigenvalue is 1.

(c) Now let $\vec{x}$ be any vector **perpendicular** to $\vec{v}$. Show that $A\vec{x} = \vec{0}$.

Suppose $\vec{v}^T\vec{x} = 0$. Then from (a) we have

$$A\vec{x} = \left(\frac{\overset{0}{\cancel{\vec{v}^T\vec{x}}}}{\vec{u}^T\vec{v}}\right)\vec{u} = 0\vec{u} = \vec{0}.$$

(d) Is the matrix A invertible? If so, tell me its inverse.

NO, because there exists $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \vec{0}$. [If A were invertible we would get $\vec{x} = A^{-1}\vec{0} = \vec{0}$, contradiction.]