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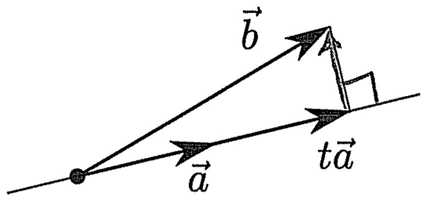
Math 210 E **Exam 2**
Spring 2013 **Friday April 19**

This is a closed book test. No electronic devices are allowed. If two students submit exams in which any solution has been copied, **both students will receive a score of zero**. There are 7 pages and 4 problems, worth a total of 30 points.

Problem 1. [5 points] We wish to **project** the vector $\vec{b}$ onto the line spanned by $\vec{a}$. The answer is $t\vec{a}$ for some number t:



(a) Write a true **equation** involving $\vec{a}$, $\vec{b}$, and t. [Hint: Dot product.]

$$\vec{a}^T(\vec{b} - t\vec{a}) = 0$$

(b) **Solve** your equation to find t.

$$\vec{a}^T\vec{b} - t\vec{a}^T\vec{a} = 0$$

$$t\vec{a}^T\vec{a} = \vec{a}^T\vec{b}$$

$$t = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}.$$

(c) Tell me the **matrix** P such that $P\vec{b} = t\vec{a}$ gives the projection.

$$P\vec{b} = \left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)\vec{a} = \vec{a}\left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)$$

$$= \left(\frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}}\right)\vec{b} \implies P = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}}$$

Problem 2. **[6 points]** Consider three data points $\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. We wish to find the **best-fit line** of the form $C + Dt = b$.

(a) Express the three equations $C + D(-1) = 0$, $C + D(0) = 1$, $C + D(1) = 3$ as a **single matrix equation** $A\vec{x} = \vec{b}$ (which, unfortunately, has no solution).

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

(b) Write down the **normal equation** $A^T A\hat{x} = A^T\vec{b}$ (which does have a solution).

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

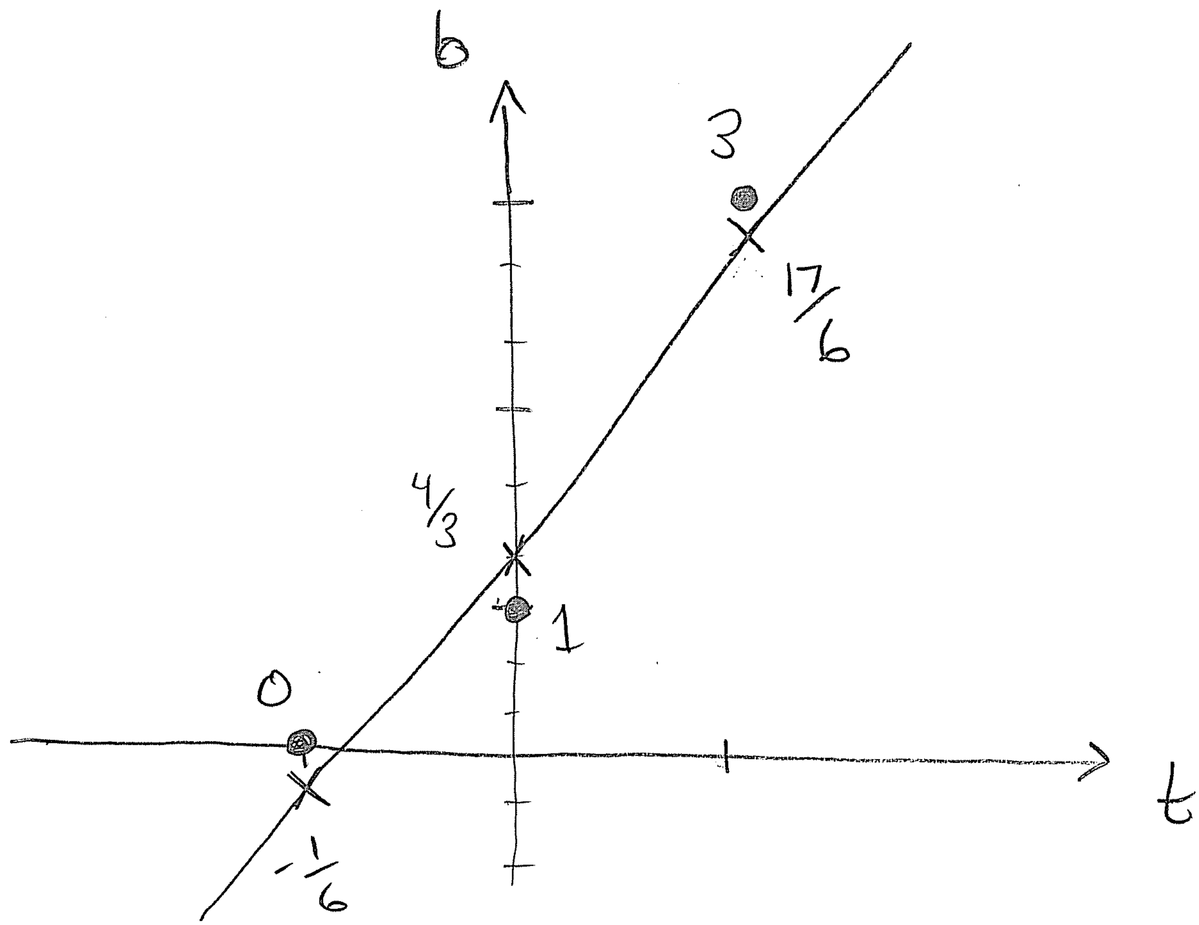
$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

(c) **Solve** the normal equation to find C and D.

$$\begin{cases} 3C + 0D = 4 \\ 0C + 2D = 3 \end{cases} \Rightarrow \quad C = 4/3, \quad D = 3/2.$$

Best fit line: $\frac{4}{3} + \frac{3}{2}t = b$

(d) **Draw** the data points and the best-fit line.



Problem 3. **[9 points]** We wish to solve the linear recurrence $\vec{v}_{n+1} = A\vec{v}_n$, with matrix

$$A = \begin{pmatrix} .2 & .8 \\ .4 & .6 \end{pmatrix}$$

and **initial condition** $\vec{v}_0 = (3, 0)$.

(a) I will tell you that the eigenvalues of A are 1 and $-.2$. Compute the eigen**vectors**.

$$\lambda = 1 \qquad \left(\begin{array}{cc|c} .2-1 & .8 & 0 \\ .4 & .6-1 & 0 \end{array}\right) \longrightarrow \begin{array}{cc|c} x & y & \\ -.8 & .8 & 0 \\ & & 0 \end{array}$$

$$\Longrightarrow \text{ The line } \begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -.2 \qquad \left(\begin{array}{cc|c} .2+.2 & .8 & 0 \\ .4 & .6+.2 & 0 \end{array}\right) \longrightarrow \begin{array}{cc|c} x & y & \\ .4 & .8 & 0 \\ & & 0 \end{array}$$

$$\Longrightarrow \text{ The line } \begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

(b) Express $\vec{v}_0 = (3, 0)$ in terms of eigenvectors.

We have

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

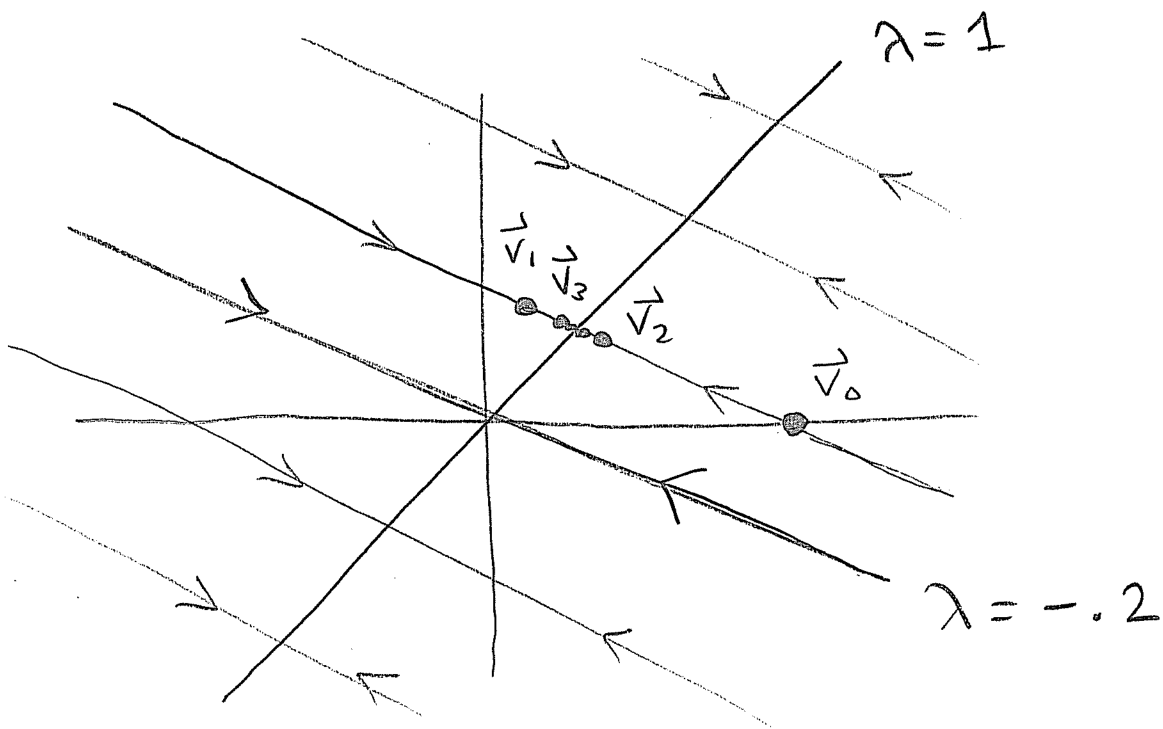
(c) Use your answer from part (b) to find an **explicit formula** for $\vec{v}_n$.

$$\vec{v}_n = A^n \begin{pmatrix} 3 \\ 0 \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + A^n \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (-.2)^n \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + (-1)^n 2/5^n \\ 1 - (-1)^n 1/5^n \end{pmatrix}.$$

(d) Draw the **phase portrait** of the system, and draw your trajectory starting from $\vec{v}_0 = (3, 0)$.



(e) Tell me the limit of $\vec{v}_n$ as $n \to \infty$

$$\vec{v}_n \longrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{as} \quad n \longrightarrow \infty$$

Problem 4. **[10 points]** Let A be any matrix with independent columns.

(a) Tell me the matrix P that **projects** orthogonally onto the column space of A.

$$P = A(A^T A)^{-1} A^T$$

(b) What are the eigenvalues of P? [Hint: One of them is 1.]

0 and 1.

(c) What are the eigenvalues of $I - P$?

$1-0$ and $1-1$

1 and 0

(d) What are the eigenvalues of $2P - I$?

$2 \cdot 0 - 1$ and $2 \cdot 1 - 1$

1 and -1.

(e) We know that the matrix from part (a) satisfies $P^2 = P$ (you **don't** need to show this). Use this fact to show that $(2P - I)^2 = I$.

$$(2P-I)(2P-I) = 4P^2 - 2PI - 2IP + I^2$$

$$= 4P - 2P - 2P + I$$

$$= I$$

(f) Is the matrix $2P - I$ invertible? If so, tell me its inverse.

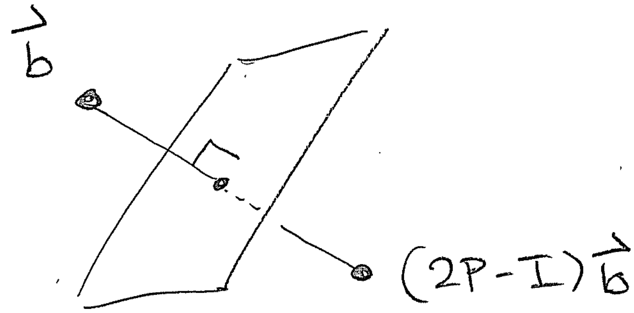
Yes. $(2P-I)(2P-I) = I$

$$\Longrightarrow (2P-I)^{-1} = 2P - I$$

(g) [**1 bonus point**] Describe the function $2P - I$ geometrically.

It performs a reflection across the column space of A.

Example:



Math 210 O
Spring 2010

Exam 2
Tuesday April 13

This is a closed book test. No electronic devices are allowed. Persons caught cheating will receive zero score. There are 6 pages and 6 problems. The 6th problem is for bonus points.

Problem 1. Consinder the following three data points:

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(a) If the line $C + Dt = b$ passed through all three data points, what matrix equation would C and D satisfy? This equation does not have a solution.

$$\begin{cases} C + (-1)D = 0 \\ C + (0)D = 0 \\ C + (1)D = 2 \end{cases} \rightsquigarrow \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

(b) Write down the associated **normal equation**, which does have a solution.

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$$

(c) Solve the normal equation for C and D.

Multiplying from part (b) gives

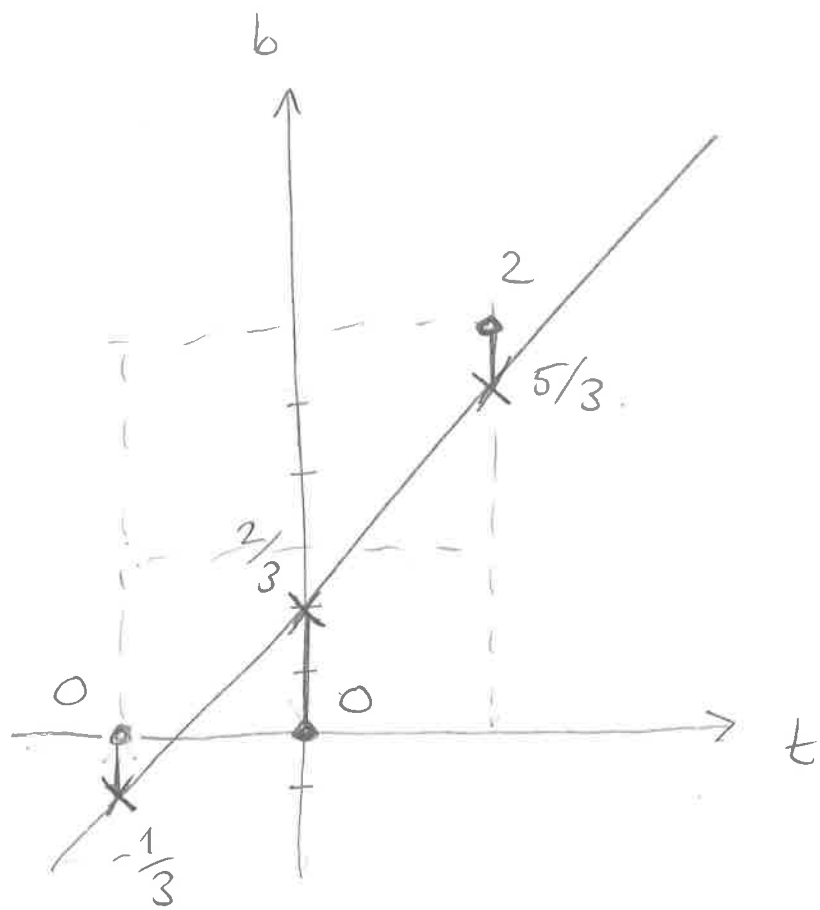
$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}.$$

The best fit line is

$$\frac{2}{3} + t = b.$$

(d) On the back of the page, **draw** the data points and the best fit line $C + Dt = b$.

OK.



Bonus Info: The vertical errors are the entries of the "error vector"

$$\vec{e} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0-(-1/3) \\ 0-(2/3) \\ 2-(5/3) \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

Problem 2. Let P be the matrix that projects onto the plane $x - 2y + z = 0$.

(a) Compute the matrix P.

I know that $\vec{a} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ is the vector orthogonal to the plane, hence

$$P = I_3 - \frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} = I_3 - \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}(1\ -2\ 1)}{(1\ -2\ 1)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}.$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 4 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$

(b) What does the matrix $I - P$ do?

It projects onto the line $t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

(c) Compute the projection of the vector $(0, 0, 2)$ onto the plane $x - 2y + z = 0$.

$$P\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 5 & 2 & -1 \\ 2 & 4 & 2 \\ -1 & 2 & 5 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -2 \\ 4 \\ 10 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 2/3 \\ 5/3 \end{pmatrix}.$$

(d) Add this information to your picture in **Problem 1**.

It is done. These are the heights of the "x" points in the picture.