3/2/16

Exam 1 Friday in class.

- no books
- no electronic devices
- no collaboration!

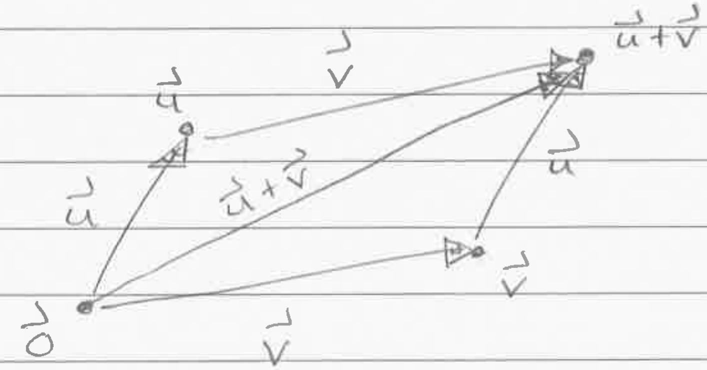
Today: Review for Exam 1.

Following Descartes (1637), we think of an ordered list of n real numbers as a point in n-dimensional space.

$$\mathbb{R}^n = n\text{-dimensional Cartesian space}$$

Subtle Idea: We can also view the point $\vec{u}$ in $\mathbb{R}^n$ as the arrow / vector with head at $\vec{u}$ and tail at $\vec{0}$.

Then vectors add head-to-tail:



We define the dot product in $\mathbb{R}^n$ by

$$\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$$:= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n .$$

and this allows us to discuss lengths, distances & angles in $\mathbb{R}^n$:

$$\text{length}(\vec{u})^2 = \|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$$

$$\text{dist}(\vec{u}, \vec{v})^2 = \|\vec{v} - \vec{u}\|^2 = (\vec{v} - \vec{u}) \cdot (\vec{v} - \vec{u}) .$$

$$\text{angle}(\vec{u}, \vec{v}) = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} \right) .$$

WHY?

By taking "linear combinations" of vectors we can describe flat shapes in $\mathbb{R}^n$:

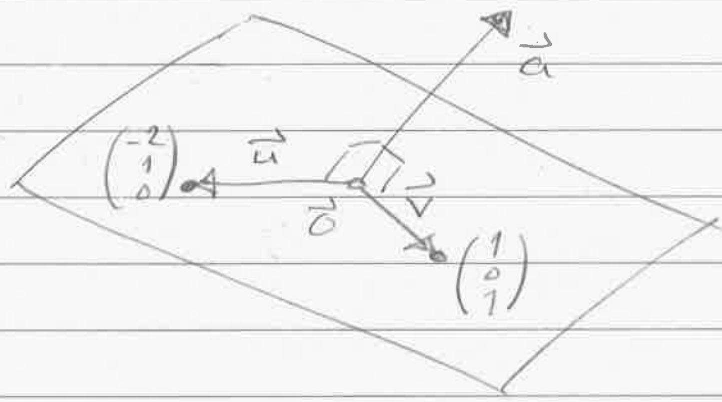
0-planes	1-planes	2-planes	$\cdots$	$(n-1)$-planes
points	lines	planes	$\cdots$	"hyperplanes"

Example from 2010 practice exam:

Let $\vec{u} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. The set of points

$$s\vec{u} + t\vec{v} = s\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2s+t \\ s+0 \\ 0+t \end{pmatrix}$$

forms a plane (i.e. 2-plane) in $\mathbb{R}^3$, which contains the point $\vec{0}$ (set $s=t=0$).



We know that a plane in $\mathbb{R}^3$ is described by one linear equation. Find the equation.

It helps to know a normal (i.e. "perpendicular") vector for the plane so let's try to find a vector

$$\vec{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

such that $\vec{a} \perp \vec{u}$ & $\vec{a} \perp \vec{v}$, i.e.

$$\vec{a} \cdot \vec{u} = 0 \qquad \vec{a} \cdot \vec{v} = 0$$

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = 0 \qquad \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$-2a + b = 0 \qquad a + c = 0.$$

Thus we have a system of 2 linear equations in 3 unknowns:

$$\begin{cases} a + c = 0 & (1) \\ -2a + b = 0 & (2) \end{cases}$$

[Since $3-2=1$, I expect the solution will have 1 free parameter.].

First do Gaussian elimination to get

RREF $$\begin{cases} a + 0 + c = 0 & (1) \\ 0 + b + 2c = 0. & (2) + 2(1) \end{cases}$$

That was fast. The pivot variables are a & b and the free variable is c.

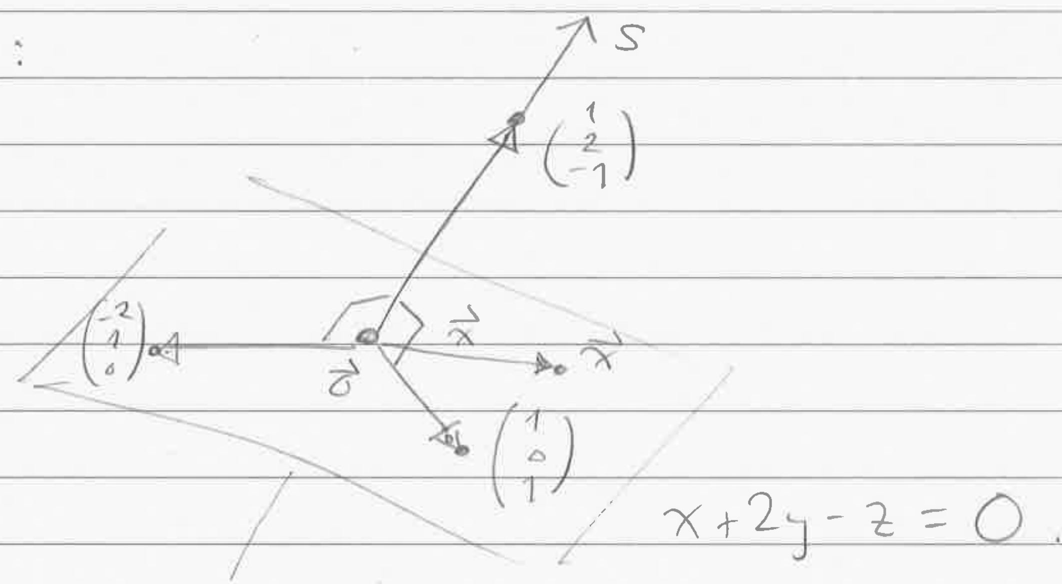
Let $c = t$ to get the general solution

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -t \\ -2t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}.$$

These are the normal vectors to the plane. In fact it's a whole normal line. For no good reason let me define a new coordinate system $s := -t$ so we can express the line as

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = t\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = s\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

Picture:



Now if $\vec{x}$ is any point in the plane (hence the vector $\vec{x}$ is in the plane) we must have

$$\vec{a} \circ \vec{x} = 0.$$

$$\begin{pmatrix} 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$x + 2y - z = 0.$$

This is the equation of the plane. [Actually just an equation; we can multiply both sides by any nonzero s and still get the same plane: $sx + 2sy - sz = 0$.]

Exercise: Find the (an) equation of the plane that is parallel to this one and contains the point $(x, y, z) = (3, 2, 1)$.

The linear system we recently solved can be written as a matrix equation

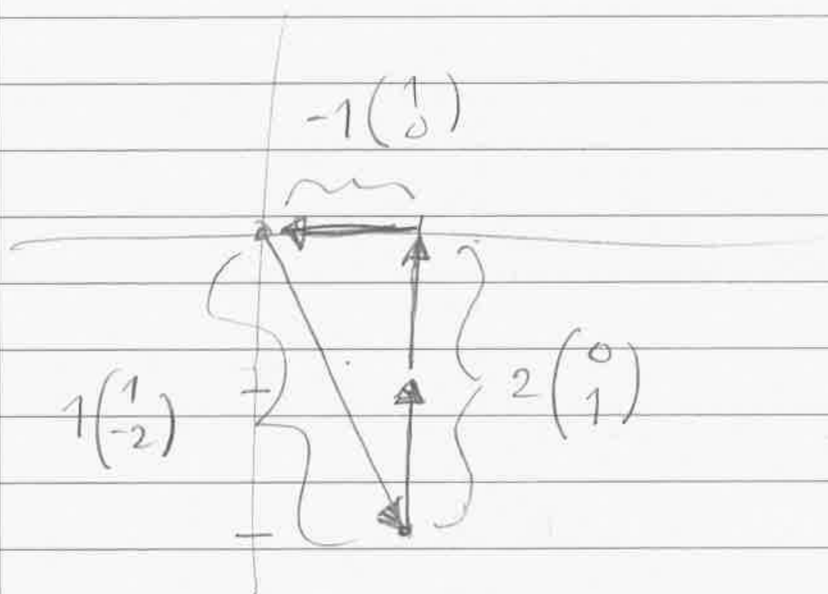
$$\begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Expanding this using the "column" definition gives a vector equation in two-dimensional space $\mathbb{R}^2$:

$$a\begin{pmatrix}1\\-2\end{pmatrix}+b\begin{pmatrix}0\\1\end{pmatrix}+c\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

What's the solution? It's the same problem so it has the same solution: $(a,b,c)=(s,2s,-s)$.

Picture when $s=1$:



$$1\begin{pmatrix}1\\-2\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}-1\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

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The general story:

A system of m linear equations in n unknowns can be written as a single matrix equation

$$A\vec{x} = \vec{b}$$

where A is the "coefficient matrix" of shape $m \times n$ (i.e. with m rows & n columns) and where $\vec{x}$ & $\vec{b}$ are the vectors of variables and constants:

$$\underset{n \times 1}{\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}, \qquad \underset{m \times 1}{\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}.$$

We can solve the system using Gaussian elimination. The solution (if it exists) will be a d-plane living in $\mathbb{R}^n$ and we <u>expect</u> that

$$d = n - m.$$

dim of solution = # variables − # equations

If B is any matrix of shape $n \times p$ (i.e. with n rows and p columns) then we will define the product matrix so that the following equation is true for all vectors $\vec{u}$ in $\mathbb{R}^p$:

$$(*) \qquad (AB)\vec{u} = A(B\vec{u}).$$

Note that AB has shape $m \times p$. [What does the matrix product have to do with solving linear systems? Just wait.]

It's tedious to compute AB from the definition so we often use the following trick.

TRICK for computing AB:

The entry of AB in the i^{th} row and j^{th} column equals the dot product

$$(i^{th} \text{ row of } A) \cdot (j^{th} \text{ column of } B).$$

This dot product exists because we assumed that both of these vectors have n components.