Mon Apr 15

Exam 2 Friday.
Wednesday Review
Today: HW 9 Solutions

A.1. The Gibonacci numbers are defined by

$$G_{k+2} = \frac{1}{2} G_{k+1} + \frac{1}{2} G_k \quad \text{for all } k \geq 0$$

with initial conditions

$$G_0 = 0, \quad G_1 = 1.$$

They start like this:

$$0, 1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}, \frac{21}{32}, \frac{43}{64}, \ldots$$

Problem: Find a formula for G_n and show that

$$G_n \to \frac{2}{3} \quad \text{as} \quad n \to \infty.$$

Solution: Write the recurrence as a matrix equation

$$\begin{pmatrix} G_{k+2} \\ G_{k+1} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} G_{k+1} \\ G_k \end{pmatrix}$$

Let $\vec{V}_k = \begin{pmatrix} G_{k+1} \\ G_k \end{pmatrix}$ and $A = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$.

Then we have.

Recurrence: $\vec{V}_{k+1} = A\vec{V}_k$

Initial Condition: $\vec{V}_0 = \begin{pmatrix} G_1 \\ G_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Hence

$$\vec{V}_n = \begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix} = A^n \vec{V}_0 = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

To solve: Express $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in terms of eigenvectors. First, Eigenvalues:

$$\left(\frac{1}{2} - \lambda\right)(0 - \lambda) - 1 \cdot \frac{1}{2} = 0$$

$$-\frac{1}{2}\lambda + \lambda^2 - \frac{1}{2} = 0.$$

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$$

$$\lambda = \frac{1/2 \pm \sqrt{1/4 + 4/2}}{2} = \frac{1/2 \pm \sqrt{9/4}}{2}$$

$$= \frac{1/2 \pm 3/2}{2} = 1 \text{ OR } -1/2.$$

Eigenvectors:

$\lambda = 1$: $$\begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\begin{pmatrix} 1/2 - 1 & 1/2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\begin{array}{cc|c} \cancel{-1/2} & \cancel{1/2} & \cancel{0} \\ 1 & -1 & 0 \end{array} \longrightarrow x - y = 0.$$

The line $t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

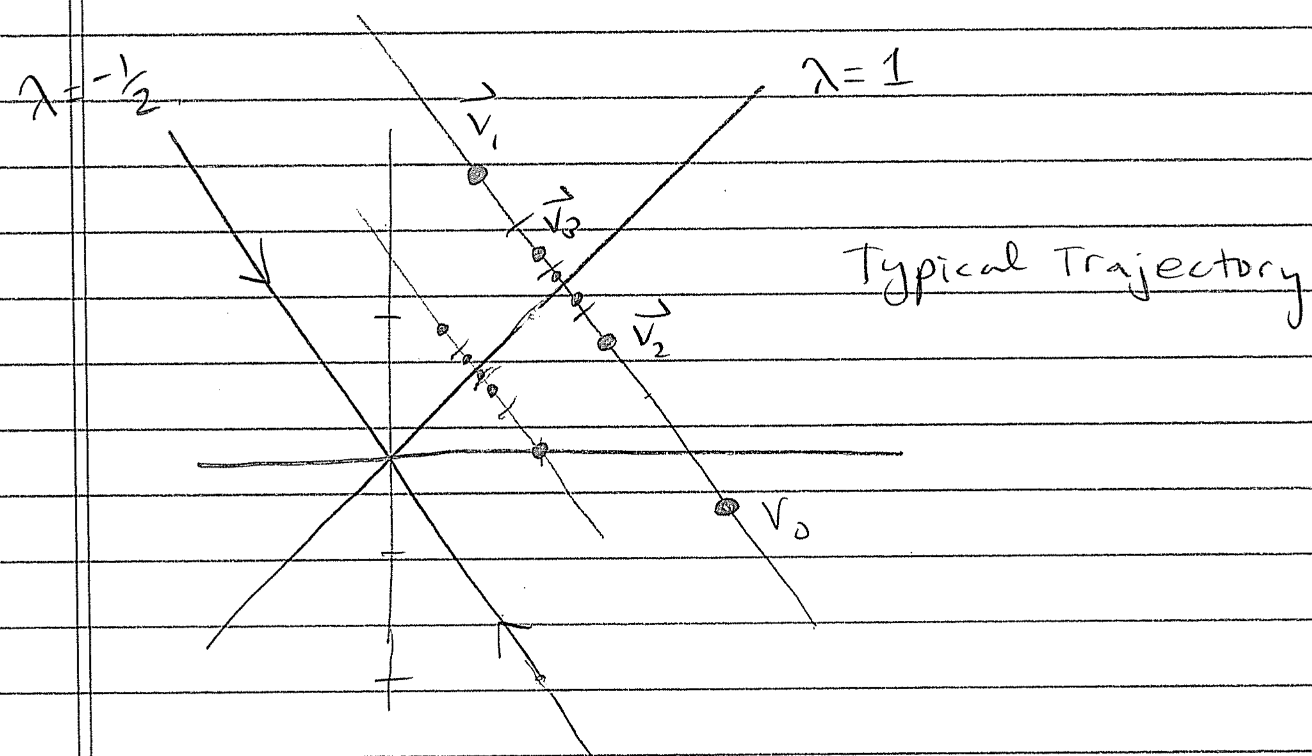
$\lambda = -\frac{1}{2}$: $$\begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}.$$

$$\begin{pmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} \\ 1 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ \cancel{1} & \cancel{\frac{1}{2}} & \cancel{0} \end{array}\right) \longrightarrow \begin{aligned} x + \tfrac{1}{2}y &= 0 \\ x &= -\tfrac{1}{2}y. \\ -2x &= y. \end{aligned}$$

The line $\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Phase Portrait:



Our Trajectory: Express

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = s\begin{pmatrix} 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Solve $\left.\begin{aligned} s+t &= 1 \\ s-2t &= 0 \end{aligned}\right\} \longrightarrow \left.\begin{aligned} s+t &= 1 \\ 0-3t &= -1 \end{aligned}\right\}$

$$\Longrightarrow \quad t = \frac{1}{3} \quad \Longrightarrow \quad s = \frac{2}{3}.$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Finally:

$$\begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A^n \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix} + A^n \begin{pmatrix} 1/3 \\ -2/3 \end{pmatrix}.$$

$$= 1^n \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix} + \left(\frac{-1}{2}\right)^n \begin{pmatrix} 1/3 \\ -2/3 \end{pmatrix}.$$

$$= \begin{pmatrix} 2/3 + \left(-\frac{1}{2}\right)^n \frac{1}{3}. \\ 2/3 - \left(-\frac{1}{2}\right)^n \frac{2}{3} \end{pmatrix}$$

The nth Gibonacci number is

$$G_n = \frac{2}{3} + \frac{(-1)^{n-1}}{3 \cdot 2^{n-1}}$$

$$\rightarrow \frac{2}{3} \quad \text{as} \quad n \longrightarrow \infty.$$

A.2. Owls vs. Rats.

$$O_{k+1} = (.5)\,O_k + (.4)\,R_k$$

$$R_{k+1} = -p\,O_k + (1.1)\,R_k.$$

Recurrence:

$$\begin{pmatrix} O_{k+1} \\ R_{k+1} \end{pmatrix} = \begin{pmatrix} .5 & .4 \\ -p & 1.1 \end{pmatrix} \begin{pmatrix} O_k \\ R_k \end{pmatrix}.$$

Initial conditions NOT GIVEN.

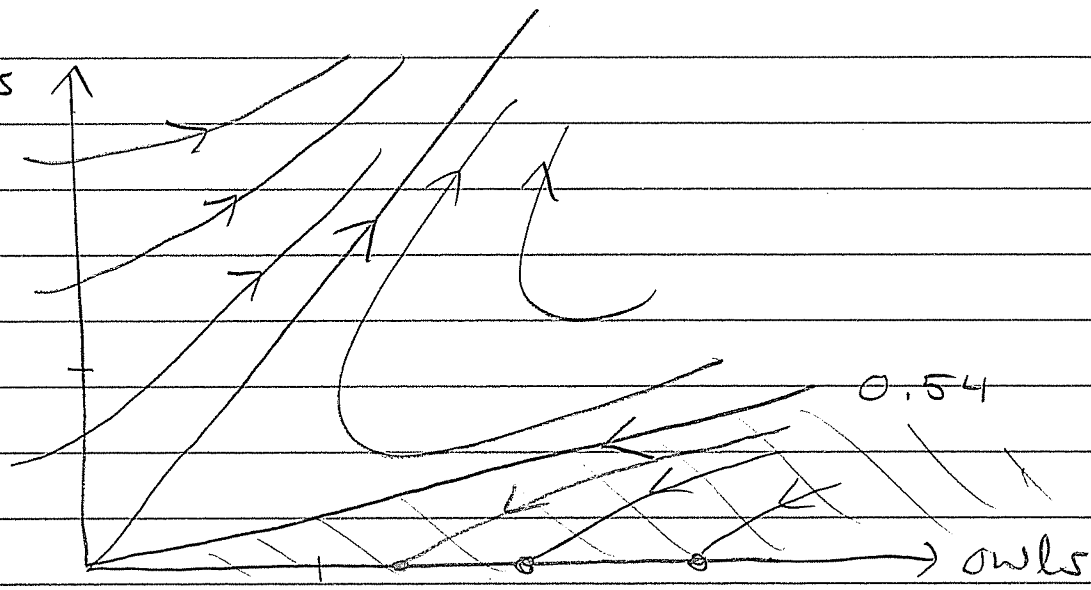
Draw the Phase Portrait for three values of p.

1) $\underline{p = 0.056}$. We have

$$\begin{pmatrix} .5 & .4 \\ -0.056 & 1.1 \end{pmatrix} \begin{pmatrix} 1 \\ 1.4 \end{pmatrix} = (1.06) \begin{pmatrix} 1 \\ 1.4 \end{pmatrix}$$

and $$\begin{pmatrix} .5 & .4 \\ -0.056 & 1.1 \end{pmatrix} \begin{pmatrix} 1 \\ .1 \end{pmatrix} = (0.54) \begin{pmatrix} 1 \\ .1 \end{pmatrix}$$

Phase Portrait:



If $O_0 > 10 R_0$ then everyone goes extinct.
Otherwise we have

$$\frac{R_n}{O_n} \to 1.4$$ (40% more rats than owls)

and both populations will grow at $\approx$ 6% per year.

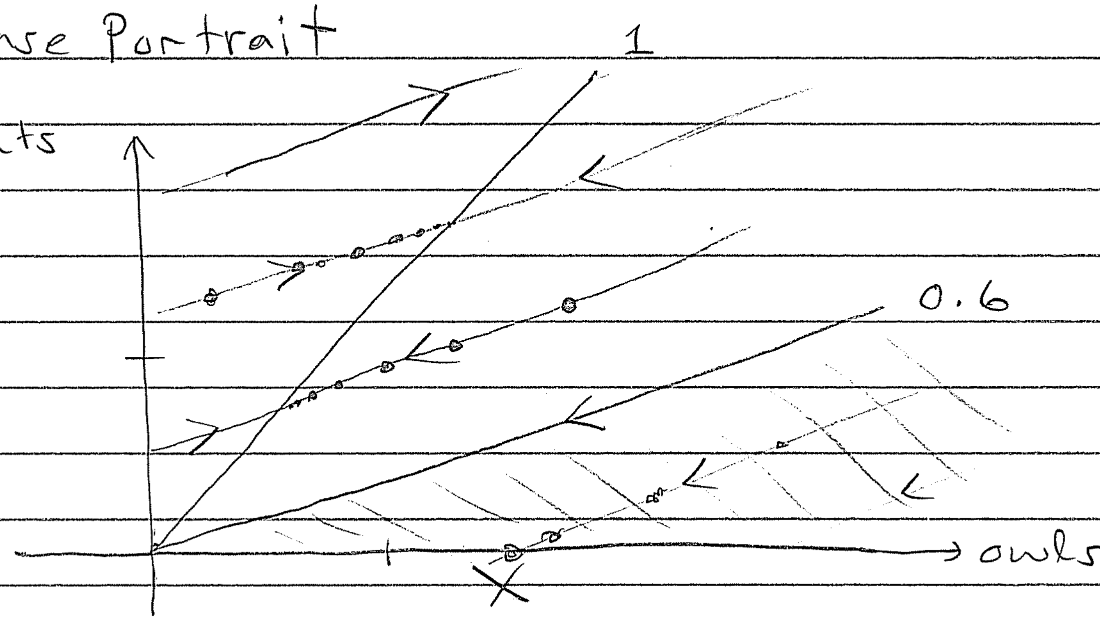
I guess we'll call that "good".

(2) $p = 0.125$. We have

$$\begin{pmatrix} .5 & .4 \\ -0.125 & 1.1 \end{pmatrix} \begin{pmatrix} 1 \\ 1.25 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1.25 \end{pmatrix}$$

and
$$\begin{pmatrix} .5 & .4 \\ -0.125 & 1.1 \end{pmatrix} \begin{pmatrix} 1 \\ 0.25 \end{pmatrix} = (0.6) \begin{pmatrix} 1 \\ 0.25 \end{pmatrix}$$

Phase Portrait



If $O_0 > 4 R_0$ then everyone goes extinct.
Otherwise the populations approach a steady state with

$-0.25\, O_0 + R_0$ owls and
$-0.3125\, O_0 + 1.25\, R_0$ rats

We'll call this "Okay".

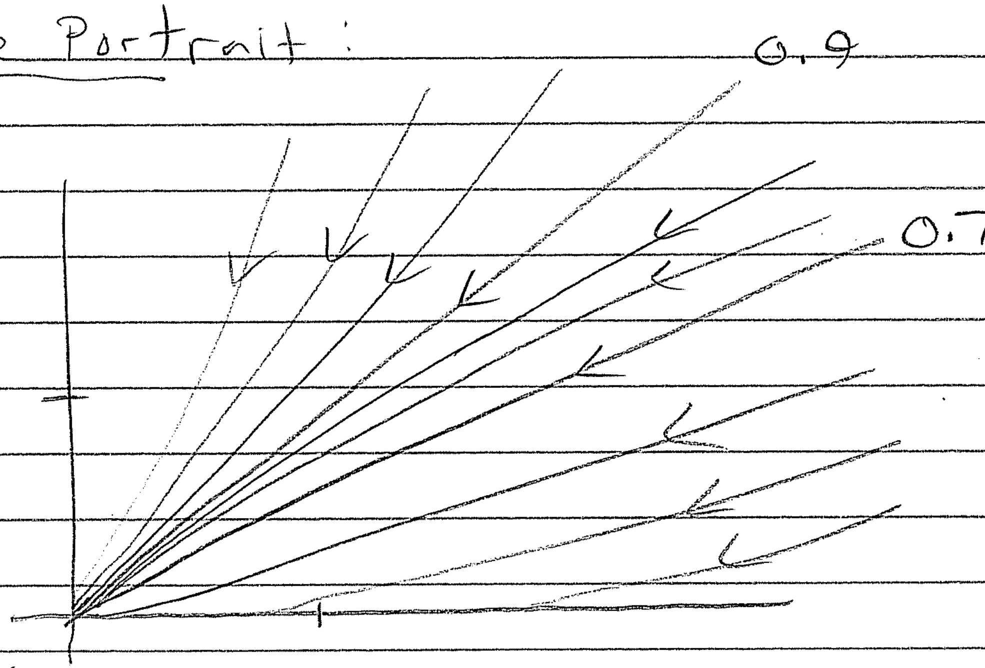
(3) $\underline{p = 0.2}$. We have

$$\begin{pmatrix} .5 & .4 \\ -0.2 & 1.1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (0.9) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} .5 & .4 \\ -0.2 & 1.1 \end{pmatrix} \begin{pmatrix} 1 \\ .5 \end{pmatrix} = (0.7) \begin{pmatrix} 1 \\ .5 \end{pmatrix}$$

Phase Portrait:



Everyone goes extinct no matter what O_0 and R_0 are

We'll call this "bad".

Memo to owls: control your appetite.