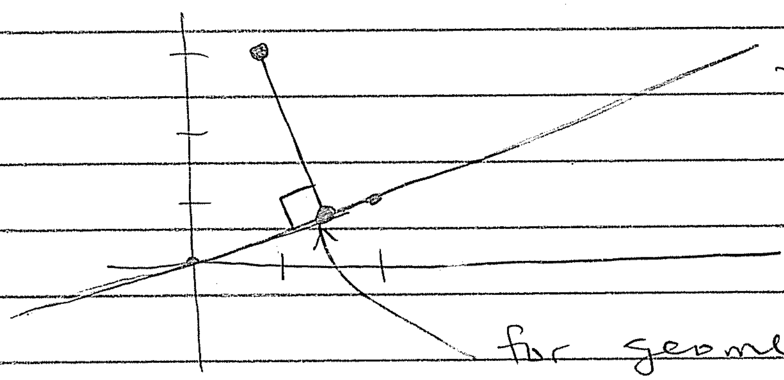
HW 5 Ave 27/30
Exam 1 Ave 17/30
A = 20 - 29, B = 13 - 19, C = 6 - 11

Mon Mar 18

HW 6 due Friday

Now : Orthogonal Projection.

Problem: Find the closest point to $\binom{1}{3}$ on the line $x - 2y = 0$.

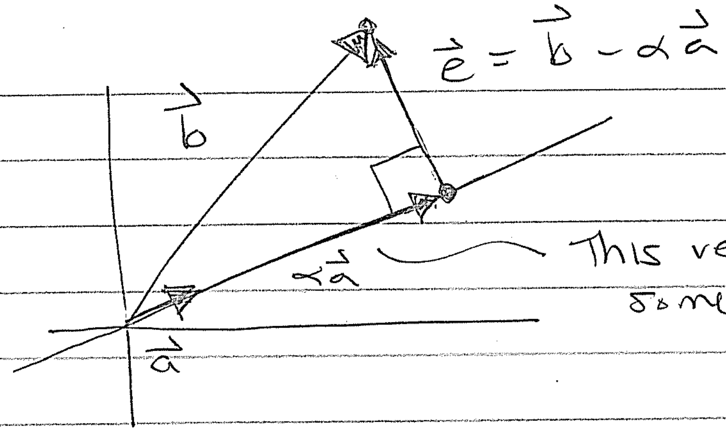


for geometric reasons the solution makes a right angle.

Call this the orthogonal projection of $\binom{1}{3}$ onto the line $x - 2y = 0$.

Compute It!

In general, to project $\vec{b}$ onto the line $t\vec{a}$ ...



We have $\vec{a} \perp \vec{e}$. That is.

$$\vec{a}^T(\vec{b} - \alpha\vec{a}) = 0.$$

$$\vec{a}^T\vec{b} - \alpha\,\vec{a}^T\vec{a} = 0$$

$$\vec{a}^T\vec{b} = \alpha\,\vec{a}^T\vec{a} \quad \left(= \alpha\|\vec{a}\|^2\right).$$

$$\Longrightarrow \alpha = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}. \quad \left(= \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\right).$$

Hence the projection is

$P_{\vec{a}}(\vec{b})$ = projection of $\vec{b}$ onto the line $t\vec{a}$.

$$= \left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)\vec{a}$$

vector

number

$$= \vec{a}\left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)$$

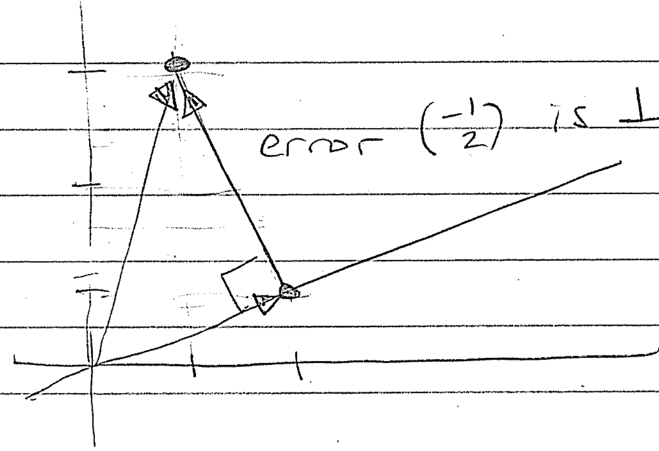
The line $x - 2y = 0$ is $t\begin{pmatrix}2\\1\end{pmatrix}$.

Project $\vec{b} = \begin{pmatrix}1\\3\end{pmatrix}$ onto $t\vec{a} = t\begin{pmatrix}2\\1\end{pmatrix}$ to get

$$P_{\binom{1}{2}}\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}\left[\frac{(2\ 1)\begin{pmatrix}1\\3\end{pmatrix}}{(2\ 1)\begin{pmatrix}2\\1\end{pmatrix}}\right]$$

$$= \begin{pmatrix}2\\1\end{pmatrix}\left[\frac{2+3}{4+1}\right] = 1\cdot\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

error $\begin{pmatrix}-1\\2\end{pmatrix}$ is $\perp$ to $\begin{pmatrix}2\\1\end{pmatrix}$



Now we can think of $P_{\vec{a}}$ as a function

$$\mathbb{R}^2 \xrightarrow{P_{\vec{a}}} \mathbb{R}^2$$

that takes any vector $\vec{b} \in \mathbb{R}^2$ and projects it onto line $t\vec{a}$.

Q: What is the matrix of $P_{\vec{a}}$?

The answer illustrates the power of abstract algebra. Watch:

For any $\vec{b} \in \mathbb{R}^2$ we have

$$P_{\vec{a}}(\vec{b}) = \underset{\text{vector}}{\vec{a}} \underset{\mathbb{R}\text{ number}}{\left(\frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}}\right)}$$

Now rearrange brackets . . .

$$= \underset{\text{matrix}}{\left(\frac{\vec{a}\,\vec{a}^T}{\vec{a}^T \vec{a}}\right)} \underset{\text{vector}}{\vec{b}}.$$

Hence the matrix of $P_{\vec{a}}$ is $\dfrac{\vec{a}\,\vec{a}^T}{\vec{a}^T \vec{a}}$.

Example: Projection onto $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$$P_{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} = \frac{\begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix}}{\begin{pmatrix} 2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}} = \frac{\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}}{2+1}$$

$$= \frac{1}{5} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}.$$

This is the matrix that projects onto the line $x - 2y = 0$.

Important Note: If $\vec{a}$ is $n\times 1$ col vector

Then $\underset{1\times n \;—\; n\times 1}{\vec{a}^T \vec{a}} = \underset{1\times 1}{\vec{a}\cdot\vec{a}}$ is a number

and $\underset{n\times \cancel{1} — 1\times n}{\vec{a}\,\vec{a}^T}$ is an $n\times n$ matrix.

In general, given a line $t\vec{a}$ in $\mathbb{R}^n$, the matrix that projects onto the line is

$$P_{\vec{a}} = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}} \quad \begin{matrix} \text{~~matrix~~} \\ \text{number} \end{matrix}$$

Q: Is $P_{\vec{a}}$ an invertible matrix /function?

A: Certainly NOT!

Reason: Let $\vec{b}$ be any vector $\perp \vec{a}$, i.e. with $\vec{a}^T\vec{b} = 0$.

Then $P_{\vec{a}}(\vec{b}) = \left(\frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}}\right)\vec{b}$

$$= \vec{a}\left(\frac{\cancelto{0}{\vec{a}^T\vec{b}}}{\vec{a}^T\vec{a}}\right) = \vec{0}.$$

But we also have $P_{\vec{a}}(\vec{0}) = \vec{0}$.

So if $P_{\vec{a}}^{-1}$ existed, how would we define

$$P_a^{-1}(\vec{0}) \overset{?}{=} \vec{b} \overset{?}{=} \vec{0}\ ?$$

WE ARE STUCK ☹

So P_a^{-1} does not exist.

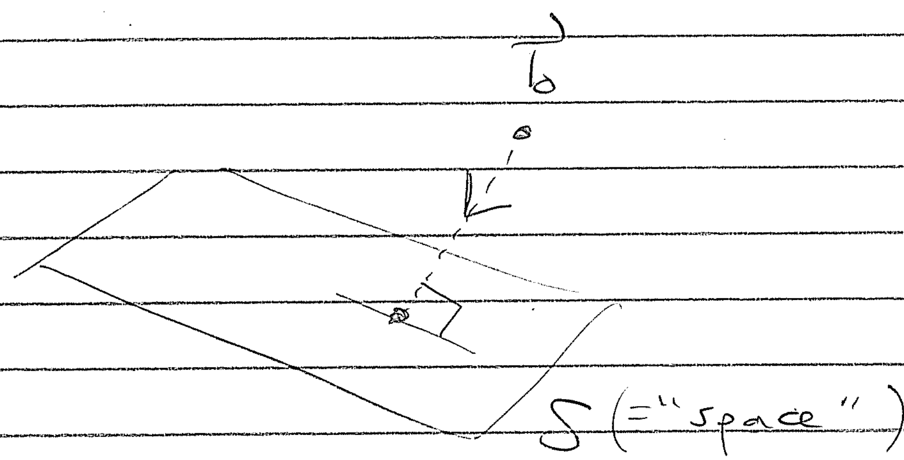
Slogan: When you project, you lose information. So you can't "un-project".

Wed Mar 20

HW 6 due Friday
Office Hours Today 3-4.

Today: Projecting onto a plane

Goal: Given a plane S and a point $\vec{b} \in \mathbb{R}^3$, find the closest point on S to $\vec{b}$.



The answer is the orthogonal projection of $\vec{b}$ onto S.

Actually, we'll solve the general case.

Consider n vectors in $\mathbb{R}^m$

$$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$$

They probably generate an n-dimensional subspace of $\mathbb{R}^m$ consisting of the "linear combinations"

$$S = \{ t_1 \vec{a}_1 + t_2 \vec{a}_2 + \cdots + t_m \vec{a}_m \}$$

parameters.

We can also think of this subspace as the "image" of a function:

$$\text{Let } A = m\left\{ \underbrace{\begin{pmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{pmatrix}}_{n} \right.$$

Then we have

$$A \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = t_1 \vec{a}_1 + \cdots + t_n \vec{a}_n$$

So the space S is just all vectors of the form

$$A\vec{t} \text{ for some } \vec{t} \in \mathbb{R}^n.$$

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m.$$

We say that the space S is the image of the function A

$$S = \text{im } A.$$

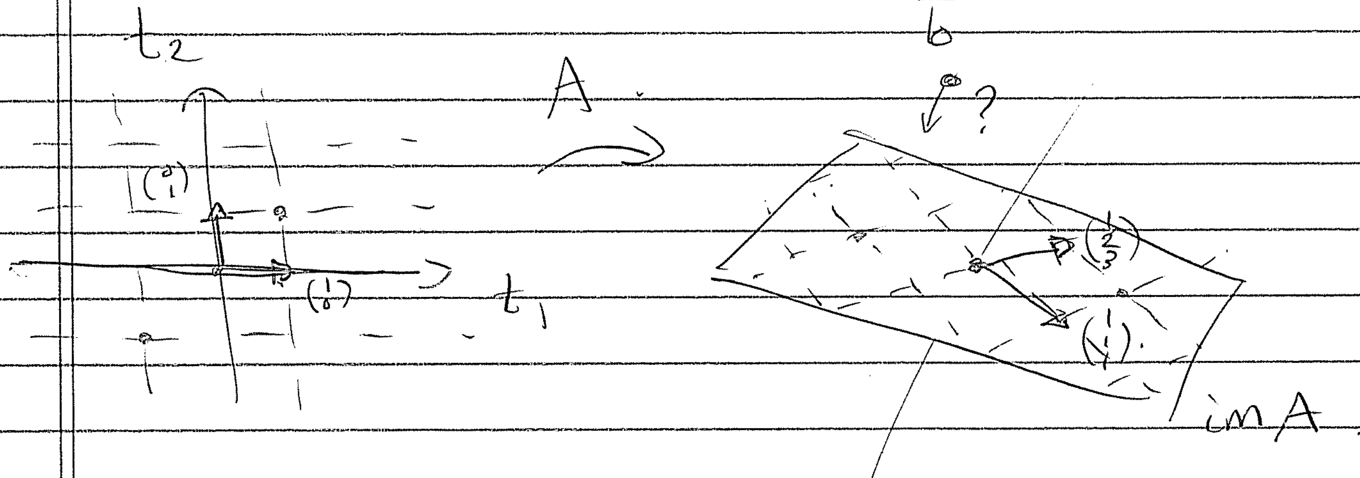
(A.K.A. the column space of A).

Example: Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$.

The image is a plane in $\mathbb{R}^3$.

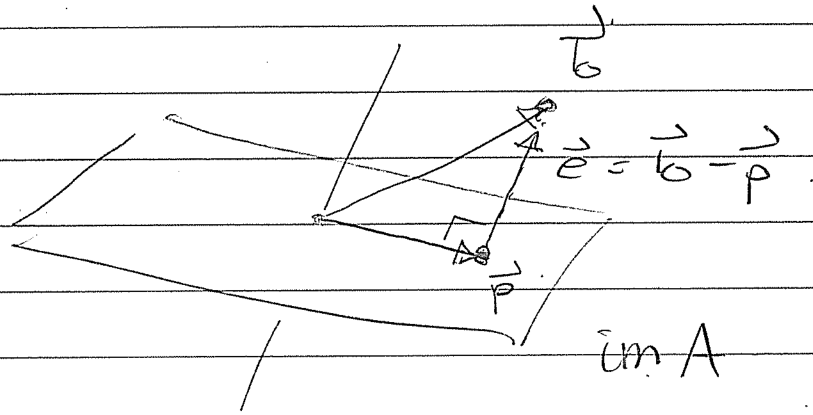
$$\mathbb{R}^2 \xrightarrow{A} \mathbb{R}^3.$$

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \longrightarrow A\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = t_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$



To Project onto im A :

Given any $\vec{b} \in \mathbb{R}^m$ try to find the point $\vec{p}$ in im A closest to $\vec{b}$



We Know:

(1) $\vec{p} = A\hat{x}$ for some $\hat{x}$ because it is in the image in A.

(2) Error $\vec{e} = \vec{b} - \vec{p} = \vec{b} - A\hat{x}$ is $\perp$ to every vector in im A.

i.e. we have
$$\left.\begin{aligned} \vec{a}_1^{\,t}\,\vec{e} &= 0 \\ \vec{a}_2^{\,t}\,\vec{e} &= 0 \\ &\vdots \\ \vec{a}_n^{\,t}\,\vec{e} &= 0. \end{aligned}\right\}$$

This can be written very nicely as a matrix equation

$$\begin{pmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \\ & \vdots & \\ - & \vec{a}_n^T & - \end{pmatrix} \vec{e} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$A^T \vec{e} = \vec{0}$$

$$A^T(\vec{b} - A\hat{x}) = \vec{0}$$

$$A^T\vec{b} - A^TA\hat{x} = \vec{0}$$

$$A^T\vec{b} = A^TA\hat{x}$$

Can we solve for $\hat{x}$, and hence $\vec{p} = A\hat{x}$?

Yes, <u>if</u> matrix A^TA is invertible.

Well $\underset{n\times m \;\; m\times n}{A^TA}$ has shape $n \times n$ (square)

so it could be invertible . . .

Say it is.

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

$$\Rightarrow \vec{p} = A\hat{x} = A(A^T A)^{-1} A^T \vec{b}$$

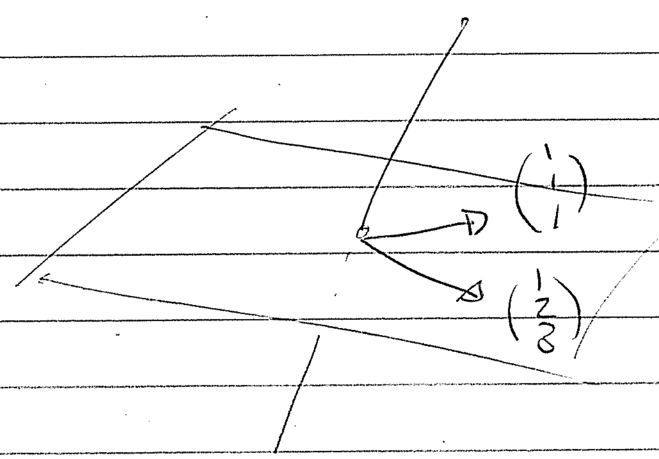
Conclusion: The projection of $\vec{b}$ onto the image (column space) of A is

$$A(A^T A)^{-1} A^T \vec{b}$$

The matrix of the projection function is

$$A(A^T A)^{-1} A^T$$

Example: Project onto the plane



Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$. Compute $A(A^TA)^{-1}A^T$.

First, $A^TA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$

Then $(A^TA)^{-1} = \dfrac{1}{14 \cdot 3 - 6 \cdot 6} \begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}$

$$= \frac{1}{6} \begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}$$

Finally, $A(A^TA)^{-1}A^T$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \frac{1}{6} \begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 8 & 2 & -4 \\ -3 & 0 & 3 \end{pmatrix}$$

$$= \frac{1}{6} \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$

Mon Mar 25

HW 7 due this Friday
Office Hours Today 2-3

Today: Fitting a line to data.

We measured a quantity at times $t_1, t_2, \ldots, t_m$. We got results $b_1, b_2, \ldots, b_m$. Theoretically, we expect a linear relationship.

$$C + Dt = b.$$

Use the data to estimate C and D. In other words, find the line $C+Dt=b$ that best fits the points

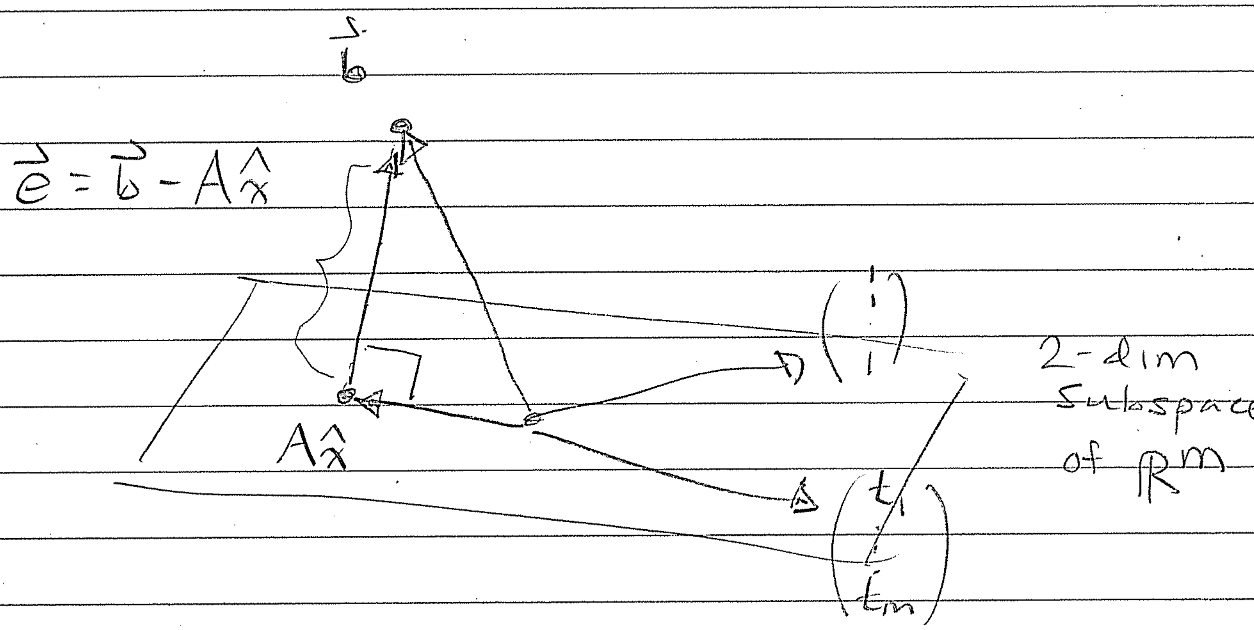
$$\begin{pmatrix} t_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} t_2 \\ b_2 \end{pmatrix}, \ldots, \begin{pmatrix} t_m \\ b_m \end{pmatrix}.$$

Matrix equation

$$A \vec{x} = \vec{b}$$

$$\begin{aligned} C + Dt_1 &= b_1 \\ C + Dt_2 &= b_2 \\ &\vdots \\ C + Dt_m &= b_m \end{aligned} \longrightarrow \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Probably has NO solution because $\vec{b}$ is probably NOT in the image of A:



Find the point $A\hat{x} = A\binom{C}{D}$ in the plane that minimizes the error $\|\vec{e}\|^2 = \|b - A\hat{x}\|^2$

Answer: The error vector should be orthogonal to the plane.

$$A^T\vec{e} = \vec{0}$$

$$A^T(\vec{b} - A\hat{x}) = \vec{0}$$

$$A^T\vec{b} - A^TA\hat{x} = \vec{0}$$

$$\boxed{A^TA\hat{x} = A^T\vec{b}}$$

Solve this for $\hat{x} = \binom{C}{D}$.

Note:

$$A^T A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_m \end{pmatrix} \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{pmatrix}$$

$$= \begin{pmatrix} m & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{pmatrix}$$

$$A^T \vec{b} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_m \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} \Sigma b_i \\ \Sigma t_i b_i \end{pmatrix}$$

So we want to solve

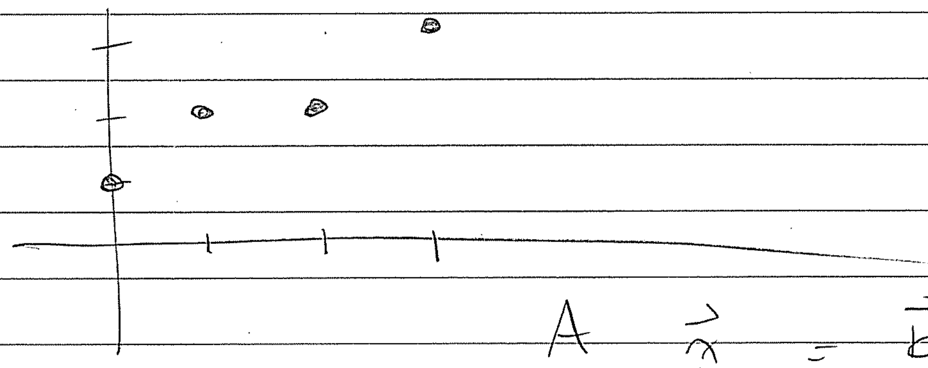
$$\boxed{\begin{pmatrix} m & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} \Sigma b_i \\ \Sigma t_i b_i \end{pmatrix}}$$

These are called the "normal equations" in statistics

"normal" = $\perp$

Example: Four data points

$$\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$



$$\begin{array}{ccc} A & \vec{x} & = \vec{b} \end{array}$$

Try to solve $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}$.

No solution. So instead try to solve

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 8 \\ 15 \end{pmatrix}$$

$$A^T A \hat{x} = A^T \vec{b}$$ Normal Equations

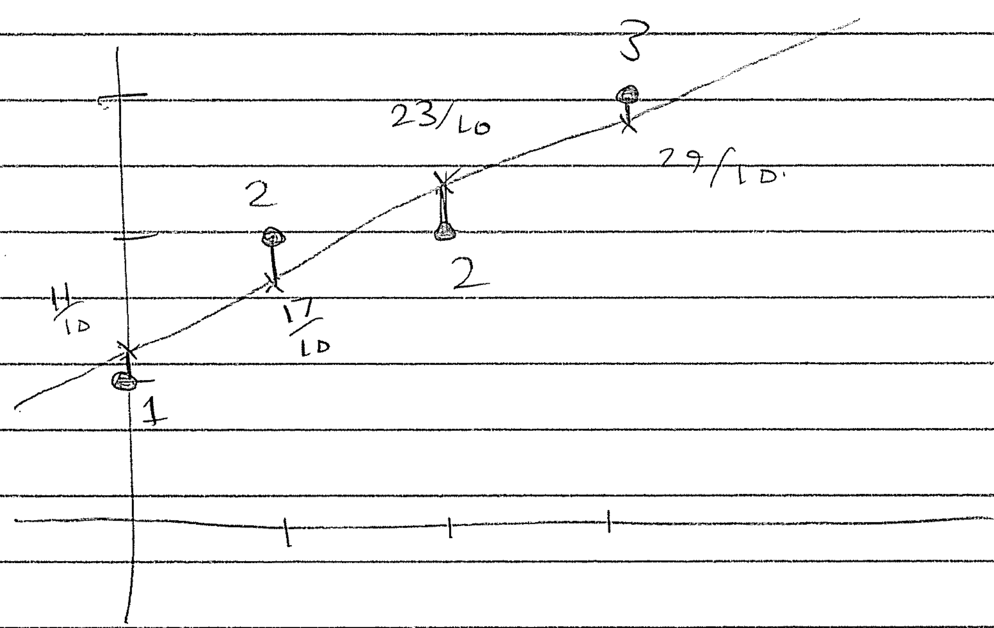
Solve for $\hat{x} = \binom{C}{D}$.

$$\left(\begin{array}{cc|c} 4 & 6 & 8 \\ 6 & 14 & 15 \end{array}\right) \to \left(\begin{array}{cc|c} 4 & 6 & 8 \\ 0 & 5 & 3 \end{array}\right) \to \left(\begin{array}{cc|c} 4 & 6 & 8 \\ 0 & 1 & 3/5 \end{array}\right)$$

$$\to \left(\begin{array}{ccc} 4 & 0 & 22/5 \\ 0 & 1 & 3/5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 11/10 \\ 0 & 1 & 6/10 \end{array}\right).$$

Least squares solution is

$$\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 11/10 \\ 6/10 \end{pmatrix} \to \boxed{\frac{11}{10} + \frac{6}{10} t = b}$$



Our data vector is $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}$

The projection is $A\hat{x} = \frac{1}{10}\begin{pmatrix} 11 \\ 17 \\ 23 \\ 29 \end{pmatrix}$

The length (squared) of the error is

$$\|\vec{e}\|^2 = \|\vec{b} - A\hat{x}\|^2$$

$$= \left(1 - \frac{11}{10}\right)^2 + \left(2 - \frac{17}{10}\right)^2 + \left(2 - \frac{23}{10}\right)^2 + \left(3 - \frac{29}{10}\right)^2$$

$$= \left(-\frac{1}{10}\right)^2 + \left(+\frac{3}{10}\right)^2 + \left(-\frac{3}{10}\right)^2 + \left(+\frac{1}{10}\right)^2$$

$$= \frac{20}{100} = 1/5.$$

= The distance squared from $\vec{b}$ to the plane im A.