Mon Feb 11

HW 4 due Fri
Office Hours Today 2-3
Reminder: Exam 1 is Fri May 1.

Point of View:

We think of 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$
as a function that sends vector $\begin{pmatrix} x \\ y \end{pmatrix}$

to vector $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$

$$= x\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + y\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

Eg Matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the function that rotates vectors by $90°$ c.c.w.

So what?

Functions can be composed.

Let $\mathbb{R}^2$ be the set of vectors $\begin{pmatrix} x \\ y \end{pmatrix}$.

$\mathbb{R}^2$ = the Cartesian plane.

Schematically, we can draw functions as arrows

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "A"', "\text{rotate } 90^\circ"] \arrow[rr, bend right, "\text{rotate } 180^\circ"'] & \mathbb{R}^2 \arrow[r, "A"', "\text{rotate } 90^\circ"] & \mathbb{R}
\end{tikzcd}$$

For general 2×2 matrices A & B,

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "B"] \arrow[rr, bend right, "A \circ B = \text{first } B \text{ then } A."'] & \mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2
\end{tikzcd}$$

$A \circ B$ is the <u>composition</u> of functions.

$A \circ B$ = "A follows B".

Question: Is $A \circ B$ also a matrix?

Answer: Yes! Let's compute what it is.

Say $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

Starting with $\begin{pmatrix} x \\ y \end{pmatrix}$, first do B

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_{11}x + b_{12}y \\ b_{21}x + b_{22}y \end{pmatrix}$$

then do A.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11}x + b_{12}y \\ b_{21}x + b_{22}y \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}(b_{11}x + b_{12}y) + a_{12}(b_{21}x + b_{22}y) \\ a_{21}(b_{11}x + b_{12}y) + a_{22}(b_{21}x + b_{22}y) \end{pmatrix}$$

(Okay. Factor out x and y.)

$$= \begin{pmatrix} (a_{11}b_{11} + a_{12}b_{21})x + (a_{11}b_{12} + a_{12}b_{22})y \\ (a_{21}b_{11} + a_{22}b_{21})x + (a_{21}b_{12} + a_{22}b_{22})y \end{pmatrix}$$

(Can we make it look nicer?)

$$= x\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} \\ a_{21}b_{11}+a_{22}b_{21}\end{pmatrix} + y\begin{pmatrix} a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22}\end{pmatrix}}\begin{pmatrix} x \\ y \end{pmatrix}.$$

this matrix does "B and then A"

what should we call it?

We can call it $A \circ B$.

But usually we shorten it to just AB.

The matrix product.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix} = \begin{pmatrix} (a_{11}, a_{12})\circ\begin{pmatrix} b_{11} \\ b_{21}\end{pmatrix} & (a_{11}, a_{12})\circ\begin{pmatrix} b_{12} \\ b_{22}\end{pmatrix} \\ (a_{21}, a_{22})\circ\begin{pmatrix} b_{11} \\ b_{21}\end{pmatrix} & (a_{21}, a_{22})\circ\begin{pmatrix} b_{12} \\ b_{22}\end{pmatrix}\end{pmatrix}$$

rows of A
dot product with
columns of B.

$$\begin{pmatrix} - \vec{a}_{1*} - \\ - \vec{a}_{2*} - \end{pmatrix} \begin{pmatrix} | & | \\ \vec{b}_{*1} & \vec{b}_{*2} \\ | & | \end{pmatrix} = \begin{pmatrix} \vec{a}_{1*} \circ \vec{b}_{*1} & \vec{a}_{1*} \circ \vec{b}_{*2} \\ \vec{a}_{2*} \circ \vec{b}_{*1} & \vec{a}_{2*} \circ \vec{b}_{*2} \end{pmatrix}$$

This suggests how to define the product of general matrices.

Suppose $A = m\left\{\underbrace{\begin{pmatrix} - & \vec{a}_{1*} & - \\ & \vdots & \\ - & \vec{a}_{m*} & - \end{pmatrix}}_{n}\right.$

$B = n\left\{\underbrace{\begin{pmatrix} | & & | \\ \vec{b}_{*1} & \cdots & \vec{b}_{*p} \\ | & & | \end{pmatrix}}_{p}\right.$

Then we define

$$\underset{m\times n}{A} \cdot \underset{n\times p}{B} = m\left\{\overbrace{\begin{pmatrix} \vec{a}_{1*} \circ \vec{b}_{*1} & \cdots & \vec{a}_{1*} \circ \vec{b}_{*p} \\ \vdots & & \vdots \\ \vec{a}_{m*} \circ \vec{b}_{*1} & & \vec{a}_{m*} \circ \vec{b}_{*p} \end{pmatrix}}^{p}\right.$$

cancel

$m \times p$

In short: If

\# columns of A = \# rows of B,

we can define matrix AB where

entry of AB in row i & col j = (row i of A) $\cdot$ (col j of B)

otherwise, AB is NOT DEFINED.

Examples:

(1)
$$\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \left(1\cdot 1 + 2\cdot 3 + 1\cdot(-1)\right) = (7).$$

$1\times 3 \quad 3\times 1 \qquad 1\times 1$

n-row $\cdot$ n-col = 1×1 matrix (number)

[The dot product is a special example of the matrix product.]

(2) Square Matrices can always multiply

$$\underset{n\times n}{A}\cdot\underset{n\times n}{B} = \underset{n\times n}{AB}$$

$$\underset{n\times n}{B}\cdot\underset{n\times n}{A} = \underset{n\times n}{BA}$$

But, watch out!

Usually we have $AB \neq BA$
(not commutative)

eg. let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\not\parallel$$

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Wed Feb 13

HW 4 due Fri
Office Hours Today 3–4
Exam 1 Fri Mar 1.

Today: Matrix Algebra

The matrix product is a simultaneous generalization of

(1) Multiplication of numbers

$$(1\times 1)\cdot(1\times 1) = (1\times 1)$$

(2) Dot product of vectors

$$\text{row} \cdot \text{column} = \text{number}$$
$$(1\times n)\cdot(n\times 1) = (1\times 1).$$

(3) Composition of functions

$$(\text{Rotate by } \alpha)\circ(\text{Rotate by } \beta) = \text{Rotate by } \alpha+\beta.$$

The definition has a few forms.

Given $m \times n$ matrix A and

$n \times p$ matrix B

we define the $m \times p$ matrix AB by

- $\underbrace{i,j \text{ entry of } AB}_{1 \times 1} = \underbrace{(i\text{th row } A)}_{1 \times n} \underbrace{(j\text{th col } B)}_{n \times 1}$

- $\underbrace{i\text{th row of } AB}_{1 \times p} = \underbrace{(i\text{th row } A)}_{1 \times n} \cdot \underbrace{B}_{n \times p}$.

- $\underbrace{j\text{th col of } AB}_{m \times 1} = \underbrace{A}_{m \times n} \cdot \underbrace{(j\text{th col } B)}_{n \times 1}$

Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}$.

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 0 \cdot 0 & 1(-1) + 0 \cdot 3 \\ 0 \cdot 1 + 1 \cdot 0 & 0(-1) + 1 \cdot 3 \\ 2 \cdot 1 + 1 \cdot 0 & 2(-1) + 1 \cdot 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 1 \end{pmatrix}$$

OR $AB = A\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} = \left(A\begin{pmatrix} 1 \\ 0 \end{pmatrix} \ A\begin{pmatrix} -1 \\ 3 \end{pmatrix} \right)$

$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 3 \end{pmatrix} \right)$$

$$= \left(1\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \ -1\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 3 \\ 2 & 1 \end{pmatrix}$$ as expected ✓

OR $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} B = \begin{pmatrix} (1\ 0)\, B \\ (0\ 1)\, B \\ (2\ 1)\, B \end{pmatrix} =$ etc.

Note BA is NOT DEFINED.

Matrix Algebra behaves mostly as you expect.

Let A, B, C be matrices
Let α be a number.
If the sums and products are defined, then we have.

$$\left.\begin{aligned} A+B &= B+A \\ \alpha(A+B) &= \alpha A+\alpha B. \\ A+(B+C) &= (A+B)+C. \end{aligned}\right\} \text{addition laws.}$$

$$\left.\begin{aligned} A(\alpha B) &= (\alpha A)B = \alpha(AB). \\ C(A+B) &= CA+CB \\ (A+B)C &= AC+BC \\ A(BC) &= (AB)C \end{aligned}\right\} \text{product laws.}$$

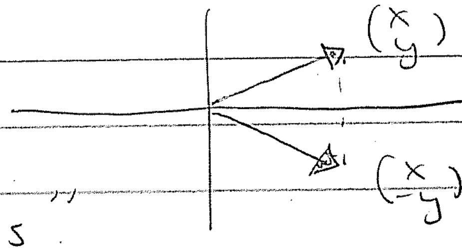
However, BE CAREFUL.
We generally have

$$AB \neq BA$$

even when both are defined.

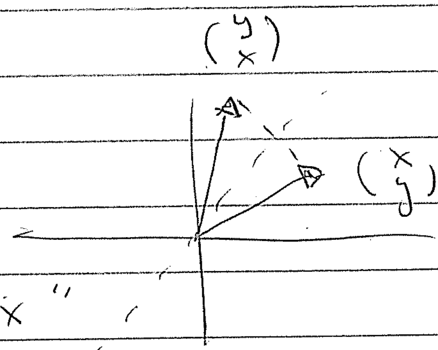
Example:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



"Reflect across x-axis"

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



"Reflect across line $y=x$"

Then.

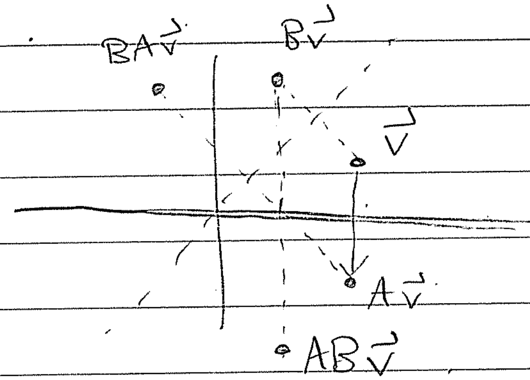
$$AB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

"Rotate 90° c.w."

$$BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

"Rotate 90° c.c.w."

So $AB \neq BA$.



Application of Matrix Algebra:

Problem 2.2.11 from HW 3.
A system of linear equations can't have just 2 solutions. Why?

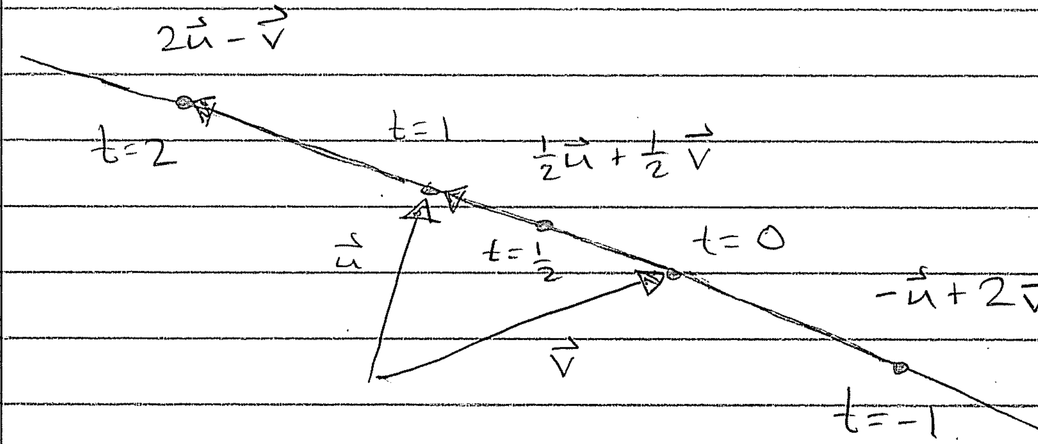
Consider the linear system

$$A\vec{x} = \vec{b}$$

and suppose we have 2 solutions, say

$$A\vec{u} = \vec{b} \quad \& \quad A\vec{v} = \vec{b}.$$

Thinking of $\vec{u}, \vec{v}$ as points in some n-dimensional space, what is the line that connects them?



The line connecting $\vec{u}$ and $\vec{v}$ has parametric equation

$$t\vec{u} + (1-t)\vec{v} \quad \text{for all } t.$$

Claim: Then every point on the line satisfies $A\vec{x} = \vec{b}$.

Indeed, let $\vec{x} = t\vec{u} + (1-t)\vec{v}$. Then

$$A\vec{x} = A(t\vec{u} + (1-t)\vec{v})$$

$$= tA\vec{u} + (1-t)A\vec{v}$$

$$= t\vec{b} + (1-t)\vec{b}$$

$$= \cancel{t\vec{b}} + \vec{b} - \cancel{t\vec{b}}$$

$$= \vec{b}.$$

///.

Hence $A\vec{x} = \vec{b}$ must have ∞ many solutions, including a whole line.

HW 3 total 30
Ave 28.

Fri Feb 15

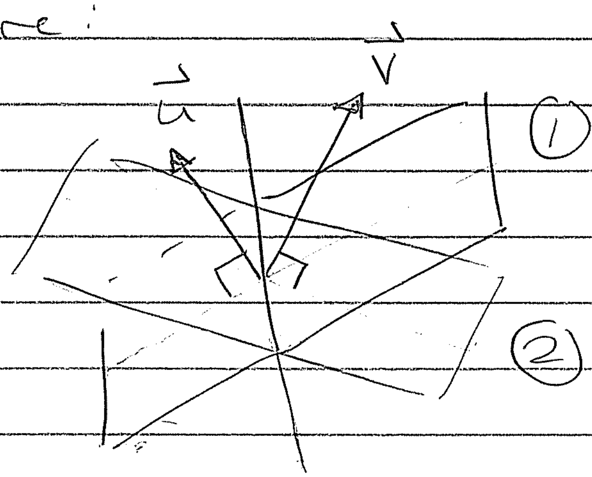
HW 4 due NOW
HW 5 due next Fri Feb 22
Exam 1 Fri Mar 1

Today: HW 4 discussion.

Solve $\begin{cases} u_1 x + u_2 y + u_3 z = 0 & (1) \\ v_1 x + v_2 y + v_3 z = 0. & (2) \end{cases}$

In vector form $\begin{cases} \vec{u} \cdot \vec{x} = 0 & (1) \\ \vec{v} \cdot \vec{x} = 0. & (2) \end{cases}$

Picture:



Solutions $\vec{x}$ are $\perp$ to both $\vec{u}$ & $\vec{v}$.

We expect a line of solutions
unless $\vec{u} = k\vec{v}$
$(1) = k(2)$

In this case we have

$u_1 = kv_1 \quad , \quad u_2 = kv_2 \quad , \quad u_3 = kv_3$

$$\boxed{\frac{u_1}{v_1} = \frac{u_2}{v_2} = \frac{u_3}{v_3}} = k$$

BAD.

So let's assume either $\frac{u_1}{v_1} \neq \frac{u_2}{v_2}$ ✓

or $\frac{u_2}{v_2} \neq \frac{u_3}{v_3}$, or both.

Without loss of generality, say.

$$\frac{u_1}{v_1} \neq \frac{u_2}{v_2} \implies u_1 v_2 \neq u_2 v_1$$

$$\implies u_1 v_2 - u_2 v_1 \neq 0.$$

Also assume without loss that $u_1 \neq 0$.

Now let's solve:

$$\left(\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \end{array}\right) \quad \begin{array}{c} (1) \\ (2) \end{array}$$

Replace ② by ② $- \frac{v_1}{u_1}$ ①.

$$\left(\begin{array}{ccc|c} 0 & v_2 - \frac{v_1}{u_1} u_2 & v_3 - \frac{v_1}{u_1} u_3 & 0 \end{array}\right).$$

$$\left(\begin{array}{ccc|c} 0 & \frac{u_1 v_2 - u_2 v_1}{u_1} & \frac{u_1 v_3 - u_3 v_1}{u_1} & 0 \end{array}\right)$$

Multiply by u_1 to get.

$$\begin{array}{c} ①' \\ ②' \end{array} \left(\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ 0 & u_1 v_2 - u_2 v_1 & u_1 v_3 - u_3 v_1 & 0 \end{array}\right)$$

Next replace ①' by ①' $- \left(\frac{u_2}{u_1 v_2 - u_2 v_1}\right)$ ②' :

$$\left(\begin{array}{ccc|c} u_1 & 0 & u_3 - \left(\frac{u_2}{u_1 v_2 - u_2 v_1}\right)(u_1 v_3 - u_3 v_1) & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} u_1 & 0 & \frac{u_3(u_1 v_2 - u_2 v_1) - u_2(u_1 v_3 - u_3 v_1)}{u_1 v_2 - u_2 v_1} & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} u_1 & 0 & \frac{u_3 u_1 v_2 - \cancel{u_3 u_2} v_1 - u_2 u_1 v_3 + \cancel{u_2 u_3} v_1}{u_1 v_2 - u_2 v_1} & 0 \end{array}\right)$$

Multiply through by $u_1 v_2 - u_2 v_1$:

$$\left(u_1(u_1v_2-u_2v_1) \quad 0 \quad u_1(u_3v_2-u_2v_3) \;\middle|\; 0\right)$$

Divide by u_1 to get

$$\begin{array}{c} ①'' \\ ②'' \end{array} \left(\begin{array}{ccc|c} u_1v_2-u_2v_1 & 0 & u_3v_2-u_2v_3 & 0 \\ 0 & u_1v_2-u_2v_1 & u_1v_3-u_3v_1 & 0 \end{array}\right)$$

pivot variables x, y free variable z.

Let $z = t$. Then

$$x = -\left(\frac{u_3v_2-u_2v_3}{u_1v_2-u_2v_1}\right) t$$

$$y = -\left(\frac{u_1v_3-u_3v_1}{u_1v_2-u_2v_1}\right) t.$$

$$z = t.$$

I don't like that. Reparametrize:

Let $t = (u_1v_2-u_2v_1)\, s$

Finally we have

$$\left.\begin{aligned} x &= (u_2 v_3 - u_3 v_2)\, s \\ y &= (u_3 v_1 - u_1 v_3)\, s \\ z &= (u_1 v_2 - u_2 v_1)\, s \end{aligned}\right\}$$

So we never have to do this again, let's give it a name.

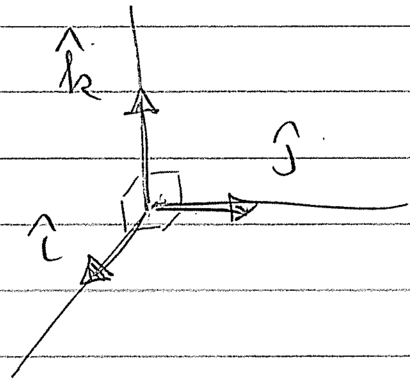
Define the cross product

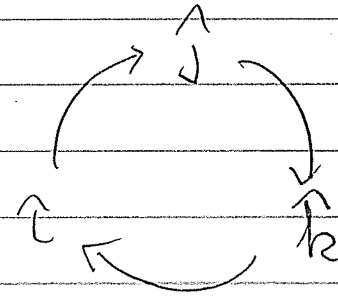
$$\vec{u} \times \vec{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

Very imporant in physics/chemistry.

Define the standard basis vectors

$$\hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$





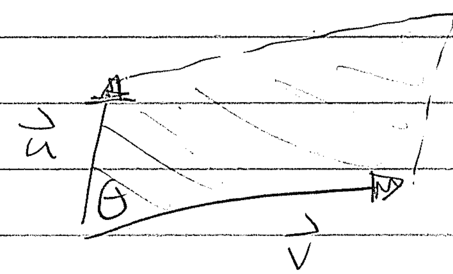
Then
$$\left.\begin{aligned} \hat{i}\times\hat{j} &= \hat{k} \\ \hat{j}\times\hat{k} &= \hat{i} \\ \hat{k}\times\hat{i} &= \hat{j} \end{aligned}\right\} \text{"Right-Handed" System.}$$

In general, note.

$$\vec{u}\times\vec{v} = -(\vec{v}\times\vec{u})$$

anti-commutative

Fun Fact: Given



$$\vec{u}\circ\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\theta.$$

$$\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\sin\theta$$
$$= \text{area of parallelogram.}$$

HW 5 due Fri
Office Hours Today 2-3
Exam 1 Fri Mar 1.

When is a function invertible?

Let X and Y we sets. We say a function $f: X \to Y$ is invertible if there exists a function $g: Y \to X$

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

such that

- $g \circ f$ = "do nothing" function on X. ("identity")
- $f \circ g$ = "do nothing" function on Y

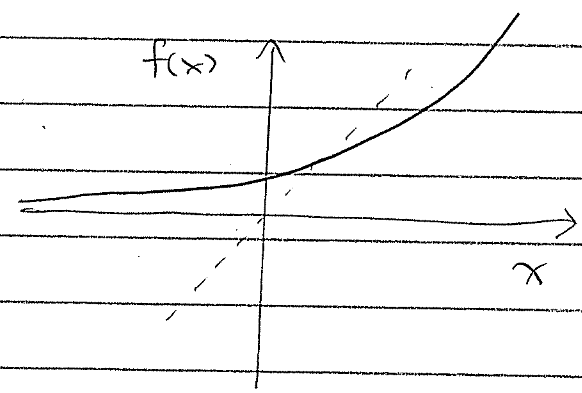
Notation: In this case we say

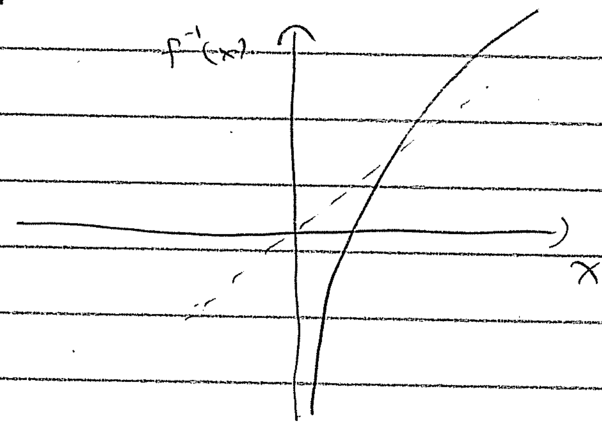
$$g = f^{-1} = \text{the inverse of } f$$

Examples from Calculus:

(1) The function $f(x) = e^x$

f : all numbers $\longrightarrow$ positive numbers



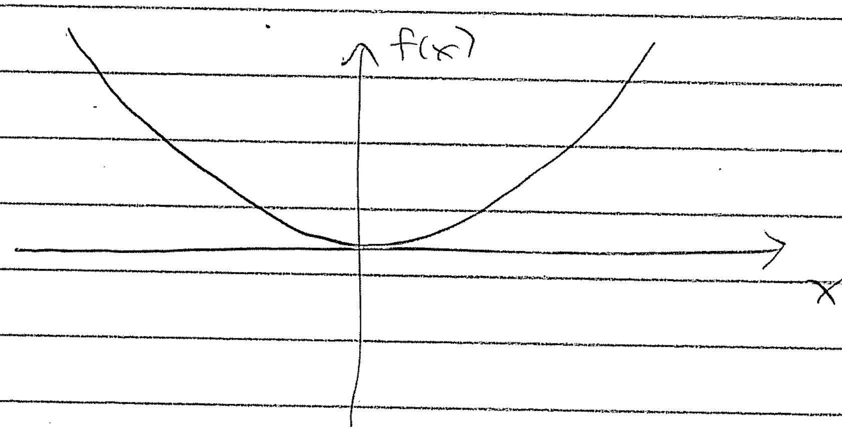


is invertible with inverse $f^{-1}(x) = \log(x)$.

Check: $e^{\log(x)} = \log(e^x) = x$ ✓

(2) The function $f(x) = x^2$

f : all numbers $\longrightarrow$ positive numbers



is NOT invertible! Why not?

$\mathbb{R}_{\geq 0}$ $\mathbb{R}$

Suppose we had f^{-1}: pos. numbers $\to$ all numbers

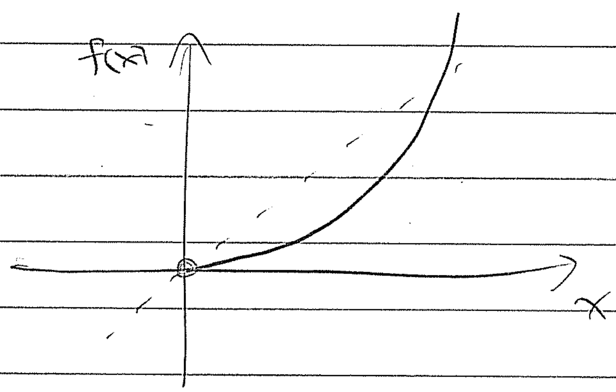
Then $f^{-1}(4) = 2$, or -2 ?

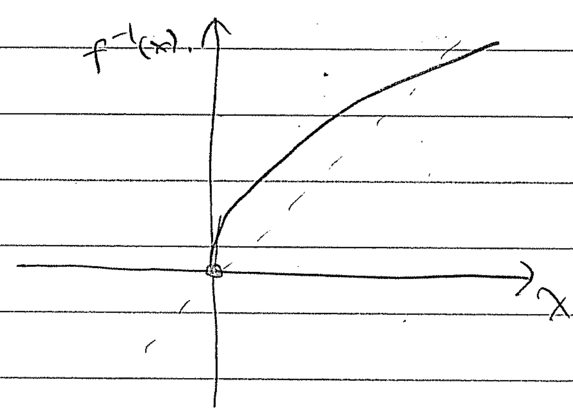
we must make a choice.

So we change the domain:

$\mathbb{R}_{\geq 0}$ $\mathbb{R}_{\geq 0}$

$f(x) = x^2$ from pos. numbers $\to$ pos. numbers





NOW it's invertible with $f^{-1}(x) = \sqrt{x}$.

Check
$$\begin{aligned} f \circ f^{-1}(x) &= f(f^{-1}(x)) \\ &= f(\sqrt{x}) \\ &= (\sqrt{x})^2 = x \end{aligned}$$

and
$$\begin{aligned} f^{-1} \circ f(x) &= f^{-1}(f(x)) \\ &= f^{-1}(x^2) \\ &= \sqrt{x^2} = x \end{aligned}$$

When is a matrix function invertible?

Say $A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by

$$\vec{x} \longrightarrow A\vec{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Can we find another matrix that undoes this?

We want matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I$$

Suppose $A^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

By definition of multiplication we have

$$AA^{-1} = A\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \left(A\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \ A\begin{pmatrix} \beta \\ \delta \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which gives two matrix equations

$$A\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad A\begin{pmatrix} \beta \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

SOLVE for $\alpha, \beta, \gamma, \delta$!

Example: let $A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$, $A^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$,

so $\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \beta \\ \delta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Solve: $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & -1 \end{array}\right)$

$\longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1/2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \end{array}\right)$

We conclude $\alpha = 3/2$, $\gamma = -1/2$.

Next, $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 3 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 2 & 1 \end{array}\right)$

$\longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1/2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \end{array}\right)$

We conclude $\beta = -1/2$, $\delta = 1/2$.

$$\Longrightarrow A^{-1} = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

Let's Check:

$$A A^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$A^{-1} A = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

We are happy.

TRICK (Gauss Jordan Method):
We could have solved both systems at once:

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 3/2 & -1/2 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right)$$

$$\boxed{(A \mid I) \xrightarrow[\text{RREF}]{} (I \mid A^{-1}).}$$

Try to invert $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right)$$

Oops. No pivot!

This means A^{-1} does not exist

Say "A is not invertible"
or "A is a singular matrix"

Fun Fact :

Matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

Wed Feb 20

HW 5 due Friday
Office Hours Today 3–4

Recall the matrix product
Given $m \times n$ matrix A and $n \times p$ matrix B
define the $m \times p$ matrix AB by

i,j entry AB = (ith row A)(jth col B)
OR ith row AB = (ith row A) B
OR jth col. AB = A(jth col B)

It has the <u>defining</u> <u>property</u>:

$$A(B\vec{x}) = (AB)\vec{x}$$

for all $p \times 1$ column vectors $\vec{x}$.
This is <u>composition</u> <u>of</u> <u>functions</u>.

$$\begin{tikzcd} \mathbb{R}^p \arrow[r, "B"] \arrow[rr, bend right, "AB = A \circ B"'] & \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m \end{tikzcd}$$

If A is $n \times n$ square, then it might have an inverse

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend left, "A"] & \mathbb{R}^n \arrow[l, bend left, "A^{-1}"] \end{tikzcd}$$

i.e. $AA^{-1} = A^{-1}A = I = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}$

But it might not.

To compute the inverse, apply row operations to the augmented matrix

$$(A \mid I) \xrightarrow{RREF} (I \mid A^{-1})$$

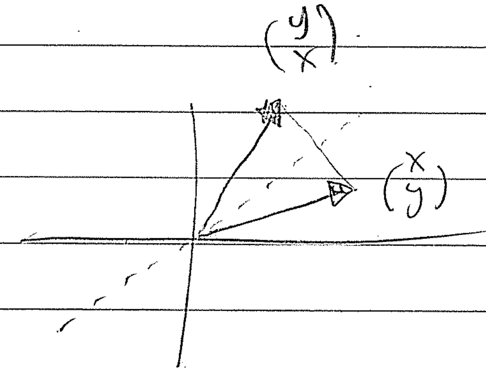
"Gauss - Jordan method"

If elimination fails it means A^{-1} does not exist.

i.e. "A is not invertible"

OR "A is singular"

Sometimes we can guess the inverse without computation.



eg $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$

"Reflect across line $y=x$".

Reflect twice = Do nothing.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

---

eg. Let $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

= rotate by θ c.c.w.

Rotate by 0 = Do nothing.

$$R_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(Rotate by θ) $\circ$ (Rotate by $-\theta$) = Do nothing.

$$R_\theta R_{-\theta} = I.$$

Hence

$$R_\theta^{-1} = R_{-\theta} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

eg. "Diagonal" matrices are easy.

If $a, b, c \neq 0$ then

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}.$$

Otherwise the inverse doesn't exist.

eg. $\underline{E} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\underline{E}\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 - 2x_1 \\ x_3 \end{pmatrix}$$

We can undo this by adding 2 times the 1st coord. to the 2nd coord.

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 - 2x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ (x_2 - \cancel{2x_1}) + \cancel{2x_1} \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

We conclude

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These E, E^{-1} are called "elementary matrices" or "elimination matrices".
Why?

Consider a 3×3 matrix $A = \begin{array}{c} \begin{pmatrix} - \vec{a}_{1*} - \\ - \vec{a}_{2*} - \\ - \vec{a}_{3*} - \end{pmatrix} \end{array} \begin{array}{c} ① \\ ② \\ ③ \end{array}$.

Then $\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{a}_{1*} \\ \vec{a}_{2*} \\ \vec{a}_{3*} \end{pmatrix} = \begin{pmatrix} \vec{a}_{1*} \\ \vec{a}_{2*} - 2\vec{a}_{1*} \\ \vec{a}_{3*} \end{pmatrix} \begin{array}{l} ① \\ ② - 2① \\ ③ \end{array}$.

We subtracted 2① from ②

We can do the whole elimination process with "elimination matrices".

$$E_k \cdots E_3 E_2 E_1 A = I$$

So what? Then we have

$$A^{-1} = E_k E_{k-1} \cdots E_3 E_2 E_1$$

Example: $A = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$.

Elimination

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{matrix} (1) \\ (2) \end{matrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{matrix} (1)' = (1) \\ (2)' = (2) - (1). \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{matrix} (1)'' = (1)' \\ (2)'' = \frac{1}{2}(2)' \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{matrix} (1)''' = (1)'' - (2)'' \\ (2)''' = (2)''. \end{matrix}$$

We did it in 3 steps.

Matrix form :

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \underbrace{\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{\text{elementary matrices}}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = I.$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Fri Feb 22

HW5 due NOW

Exam 1 next Friday in class
- closed book
- no cheating

Today: HW5 discussion

Recall, we can do elimination with "elimination matrices" E. If

$$E = \begin{array}{cc} & \begin{array}{cccc} & j & & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ \end{array} & \begin{pmatrix} 1 & & & \\ & 1 & & \\ & \ell & 1 & \\ & & & 1 \end{pmatrix} \end{array} \quad \text{for } i \neq j$$

Then EA = (ith row A) by (ith row A) + ℓ(jth row A).

Eg

$$\begin{array}{cc} & \begin{array}{ccc} & & 3 \end{array} \\ \begin{array}{c} 1 \\ \\ \\ \end{array} & \begin{pmatrix} 1 & 0 & \ell \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a+\ell g & b+\ell h & c+\ell i \\ d & c & f \\ g & h & i \end{pmatrix}$$

Replace $(a\ b\ c)$ by $(a\ b\ c) + \ell(g\ h\ i)$

1 $\qquad$ 1 + $\ell \cdot$ 3.

If $E = \begin{array}{cc} & \begin{array}{ccccccc} & & i & & j & & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & 0 & \cdots & 1 & \\ & & & \vdots & \ddots & \vdots & \\ & & & 1 & \cdots & 0 & \\ & & & & & & 1 \ddots 1 \end{pmatrix} \end{array}$

Then EA = swap rows i and j of A.

$$\begin{array}{cc} & \begin{array}{ccc} & 2 & 3 \end{array} \\ \begin{array}{c} \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix}$$

If $E = \begin{array}{cc} & \begin{array}{ccccc} & & i & & \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ \\ \end{array} & \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & l & \\ & & & & 1 \ddots 1 \end{pmatrix} \end{array}$

Then EA = multiply ith row A by l

$$\begin{pmatrix} 1 & & \\ & l & \\ & & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & b & c \\ ld & le & lf \\ g & h & i \end{pmatrix}$$

[Remark: Elementary matrices are easily invertible. No calculation needed.]

Goal: Apply elementary matrices to convert A into I. If possible, we get

$$\underbrace{E_k \cdots E_2 E_1}_{\text{the inverse of } A} A = I.$$

$$A^{-1} = E_k E_{k-1} \cdots E_2 E_1$$

This explains the Gauss-Jordan method.

$$\left(A \mid I \right)$$

$$\to \left(E_1 A \mid E_1 I \right)$$

$$\to \left(E_2 E_1 A \mid E_2 E_1 I \right)$$

$$\vdots$$

$$\to \left(E_k \cdots E_1 A \mid E_k \cdots E_1 I \right)$$

$$= \left(I \mid A^{-1} \right)$$

//

Example Computation: $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}^{-1}$.

$$\left(\begin{array}{ccc|ccc} 2 & 1 & & 1 & & \\ 1 & 2 & 1 & & 1 & \\ & 1 & 2 & & & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1/2 & & 1/2 & & \\ 1 & 2 & 1 & & 1 & \\ & 1 & 2 & & & 1 \end{array}\right) \tfrac{1}{2}①$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1/2 & & 1/2 & & \\ & 3/2 & 1 & -1/2 & 1 & \\ & 1 & 2 & & & 1 \end{array}\right) ② - \tfrac{1}{2}①$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1/2 & & 1/2 & & \\ & 1 & 2/3 & -1/3 & 2/3 & \\ & 1 & 2 & & & 1 \end{array}\right) 2/3\ ②$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1/2 & & 1/2 & & \\ & 1 & 2/3 & -1/3 & 2/3 & \\ & & 4/3 & 1/3 & -2/3 & 1 \end{array}\right) ③ - ②$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1/2 & & 1/2 & & \\ & 1 & 2/3 & -1/3 & 2/3 & \\ & & 1 & 1/4 & -2/4 & 3/4 \end{array}\right) 3/4\ ③$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1/2 & & 1/2 & & \\ & 1 & & -1/2 & 1 & -1/2 \\ & & 1 & 1/4 & -2/4 & 3/4 \end{array}\right) ② - \tfrac{2}{3}③$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & & & 3/4 & -1/2 & 1/4 \\ & 1 & & -1/2 & 1 & -1/2 \\ & & 1 & 1/4 & -1/2 & 3/4 \end{array}\right) \quad ① - \tfrac{1}{2}②$$

Conclusion.

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 3/4 & -1/2 & 1/4 \\ -1/2 & 1 & -1/2 \\ 1/4 & -1/2 & 3/4 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{pmatrix}$$

Q: When does Gauss-Jordon fail?

A: When elimination on A creates a row of zeroes.

If this happens it means we have

$$c_1(\text{row } 1) + c_2(\text{row } 2) + \cdots + c_n(\text{row } n) = \vec{0}$$

For some nontrivial coefficients c_i. (i.e. not all zero) "redundancy"

We call this a "dependence" among the rows.

Eg. The vectors $(1,1,2)$, $(2,-1,-1)$ and $(-2,4,6)$ are dependent

$$\boxed{1(1,1,2) - 1(2,-1,-1) + \frac{1}{2}(-2,4,6) = (0,0,0)}$$

This means that

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & -1 \\ -2 & 4 & 6 \end{pmatrix}$$ has no inverse

In fact, we can say:

For a square matrix A.

$$\boxed{A \text{ is invertible} \iff \text{Rows of } A \text{ are independent}}$$

HW 4 Average 28/30
Exam 1 this Friday
- absolutely NO CHEATING.
Office Hours Today 2-3

Now: Discussion of inverses.

True or False: Invertible matrices are square.

TRUE!

One can imagine $m \times n$ matrix A and $n \times m$ matrix B such that

$$\underset{m\times n}{A} \; \underset{n\times m}{B} = \underset{m\times m \text{ identity}}{I_m}$$

and

$$\underset{n\times m}{B} \cdot \underset{m\times n}{A} = \underset{n\times n \text{ identity}}{I_n}$$

But it turns out this is impossible unless $m = n$.

Example: Try to find the "inverse" of 2×3 matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$

Say $B = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$. We want

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \begin{pmatrix} a+2b+3c & d+2e+3f \\ a+b+c & d+e+f \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Get 4 linear equations:

$$a + 2b + 3c = 1 \quad (1)$$
$$a + b + c = 0 \quad (2)$$
$$d + 2e + 3f = 0 \quad (3)$$
$$d + e + f = 1. \quad (4)$$

... some work .... (omitted)

pivot variables a, b, d, e.

free parameters $c = s$, $f = t$.

Solution :

$$B = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \underbrace{\begin{pmatrix} -1 & 2 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} + s\begin{pmatrix} 1 & 0 \\ -2 & 0 \\ 1 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & 1 \\ 0 & -2 \\ 0 & 1 \end{pmatrix}}_{\text{Lots of ``right inverses''}}$$

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We also want

$$BA = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a+d & 2a+d & 3a+d \\ b+e & 2b+e & 3b+e \\ c+f & 2c+f & 3c+f \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Get 9 linear equations

$a + d = 1$ ①
$b + e = 0$ ②
$c + f = 0$ ③
$2a + d = 0$ ④
$2b + e = 1$ ⑤
$2c + f = 0$ ⑥
$3a + d = 0$ ⑦
$3b + e = 0$ ⑧
$3c + f = 1$ ⑨

9 linear equation in 6 unknowns probably HAS NO SOLUTION.

I'll show this with less work.

Observe that

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now suppose $B\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then we must have

$$\underbrace{B\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$$B\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

Is this possible? NO.

In general, if A sends some nonzero $\vec{x}$ to $\vec{0}$, then A HAS NO LEFT INVERSE

Otherwise: $BA = I$

Then $B\underbrace{A\vec{x}} = I\vec{x}$

$$B\vec{0} = \vec{x} \neq \vec{0}. \text{ IMPOSSIBLE}$$

In general, consider $m \times n$ matrix A

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$$

Consider the system

$A\vec{x} = \vec{0}$ m linear equations in n unknowns

If $m < n$ there will be ∞ many solutions.

Choose $\vec{v} \neq \vec{0}$ with $A\vec{v} = \vec{0}$.

It follows that A has NO LEFT INVERSE,

So A is not "invertible"