Mon Feb 4.

HW 3 due Friday
Office Hours Today 2–3.

We discussed the central PROBLEM of Linear Algebra; i.e. to solve a system of m linear equations in n unknowns.

Today we will discuss the SOLUTION.

First by Example:

$$\begin{cases} 2x + 4y - 2z = 2 & (1) \\ 4x + 9y - 3z = 8 & (2) \\ -2x - 3y + 7z = 10 & (3) \end{cases}$$

First "pivot" is $2x$. Eliminate everything below.

$$\begin{cases} 2x + 4y - 2z = 2 & (1)' = (1) \\ 0 + y + z = 4 & (2)' = (2) - 2(1) \\ 0 + y + 5z = 12 & (3)' = (3) + 1(1) \end{cases}$$

New system ①', ②', ③' has same solution as the old system, but it's simpler.

Next "pivot" is y. Eliminate everything below.

$$\begin{aligned} 2x + 4y - 2z &= 2 \qquad ①'' = ①' \\ 0 + y + z &= 4 \qquad ②'' = ②' \\ 0 + 0 + 4z &= 8 \qquad ③'' = ③' - ②' \end{aligned}$$

Now we have an "upper triangular" system :)

This is called "Echelon form".

We can "Back-Substitute" to get the solution:

③'' $4z = 8 \implies z = 2$

②'' $y + z = 4 \implies y + 2 = 4 \implies y = 2$

①'' $2x + 4y - 2z = 2$

$$\begin{aligned} \implies 2x + 4 \cdot 2 - 2 \cdot 2 &= 2 \\ 2x + 8 - 4 &= 2 \\ 2x + 4 &= 2 \\ 2x &= -2 \implies x = -1 \end{aligned}$$

Solution is :

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} \quad \checkmark$$

OR we can continue to eliminate above the pivots.

Scale the pivots to 1 :

$$\begin{cases} x + 2y - z = 1 & (1)''' = \frac{1}{2}(1)'' \\ \quad\; y + z = 4 & (2)''' = (2)'' \\ \qquad\quad z = 2 & (3)''' = \frac{1}{4}(3)'' \end{cases}$$

Eliminate above pivot z :

$$\begin{cases} x + 2y + 0 = 3 & (1)^{(4)} = (1)^{(3)} + (3)^{(3)} \\ \quad\; y + 0 = 2 & (2)^{(4)} = (2)^{(3)} - (3)^{(3)} \\ \qquad\quad z = 2 & (3)^{(4)} = (3)^{(3)} \end{cases}$$

Finally, Eliminate above pivot y :

$$\begin{cases} x + 0 + 0 = -1 & ①^{(5)} = ①^{(4)} - 2②^{(4)} \\ \quad y + 0 = 2 & ②^{(5)} = ②^{(4)} \\ \quad\quad z = 2 & ③^{(5)} = ③^{(4)} \end{cases}$$

We're Done.

This is called the

Reduced Row-Echelon Form
(RREF).

$$\begin{aligned} x \quad\quad &= -1 \\ y \quad &= 2 \\ z &= 2 \end{aligned}$$

Another Example:

$$\begin{cases} x + y + z = 2 & ① \\ x + 2y + z = 3 & ② \\ 2x + 3y + 2z = 5 & ③ \end{cases}$$

$$\begin{cases} x + y + z = 2 & ①' = ① \\ 0 + y + 0 = 1 & ②' = ② - ① \\ 0 + y + 0 = 1 & ③' = ③ - 2① \end{cases}$$

$$\begin{cases} x + y + z = 2 & ①'' = ①' \\ 0 + y + 0 = 1 & ②'' = ②' \\ 0 + 0 + 0 = 0 & ③'' = ③' - ②' \end{cases}$$

OOPS! There is no 3rd pivot ...
oh well.

Finally: Up - Elimination

RREF $$\begin{cases} x + 0 + z = 1 & ①''' = ①'' - ②'' \\ 0 + y + 0 = 1 & ②''' = ②'' \\ 0 + 0 + 0 = 0 & ③''' = ③'' \end{cases}$$

pivot columns

Free column (no pivot).

Free columns $\Longrightarrow$ parameters.
so let $z = t$. Then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-t \\ 1 \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

as we know.

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Wed Feb 6

HW 3 due Friday
Office Hours Today 3 – 4
MATH CLUB TODAY 5:30 pm Ungar 402

Today: Gaussian Elimination
(in general)

Idea: Given a system of equations we can do three basic "row operations" that leave the solution the same.

A) Exchange two equations

B) Scale an equation: $(i) \mapsto c(i)$

C) Replace equation (i) by $(i) - c(j)$ for some number c.

These are called Elementary Row Operations (ERO).

Goal: Perform a sequence of ERO's to put your system in a very nice form

Reduced Row Echelon Form (RREF).

Example: Solve the system

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 & (1) \\ 0 + 0 + x_3 + 4x_4 = 6 & (2) \\ x_1 + 3x_2 + x_3 + 6x_4 = 7 & (3) \end{cases}$$

Eliminate below pivot x_1 in (1)

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 & (1)' = (1) \\ 0 + 0 + x_3 + 4x_4 = 6 & (2)' = (2) \\ 0 + 0 + x_3 + 4x_4 = 6 & (3)' = (3) - (1). \end{cases}$$

We accidentally eliminated x_2 also.
That's okay.

Next pivot is x_3 in (2)'

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 & (1)'' = (1)' \\ 0 + 0 + x_3 + 4x_4 = 6 & (2)'' = (2)' \\ 0 + 0 + 0 + 0 = 0. & (3)'' = (3)' - (2)'. \end{cases}$$

Finally, eliminate above pivot x_3 in (2)''
(It's already done)

Conclusion: x_1, x_3 pivot variables
x_2, x_4 free variables.

Write down the solution in terms of the free variables

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1-3x_2 \\ x_2 \\ 6-4x_4 \\ x_4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 6 \\ 0 \end{pmatrix} + \underbrace{x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -4 \\ 1 \end{pmatrix}}_{\text{a 2-dimensional space of solutions.}}$$

Our 3 hyperplanes in 4D meet along a 2-dimensional space.

Example : Change the constant terms

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 \\ 0 + 0 + x_3 + 4x_4 = 6 \\ x_1 + 3x_2 + x_3 + 6x_4 = 8 \end{cases}$$ changed.

$$\Big\downarrow \text{RREF}$$

$$\begin{cases} x_1 + 3x_2 + 0 + 2x_4 = 1 \\ 0 + 0 + x_3 + 4x_4 = 6 \\ 0 + 0 + 0 + 0 = 1 \end{cases}$$ changed

What?

Equation $0 = 1$ means
no solution exists

Fact : A linear system has

0, 1, or ∞

solutions. E.g. 2 is not possible
(See HW2 Problem 2.2.11).

Notation: We prefer to do Gaussian Elimination (or "Row-Reduction") on matrices

Write $A\vec{x} = \vec{b}$ as an augmented matrix $(A \mid \vec{b})$

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5 \end{aligned} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{array}\right)$$

Now push the RREF button:

$$\left(\begin{array}{ccc|c} 1 & 3 & 3 & 1 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 5 \end{array}\right) \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Pivot variables : x, z

Free Variables : y.

Solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2-3y \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + y \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

Fri Feb 8

HW 3 due NOW

HW 4 due next Friday

HW 2 Total 30
Ave 25

Today: Discuss HW 3

Recall the matrix notation for a linear system

Eg

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_1$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$A\vec{x} = \vec{b}$$

This is more than just a notation, it is a new point of view.

By Definition:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

$$= x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix}$$

I can think of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a "function", or a machine that eats the vector

$\begin{pmatrix} x \\ y \end{pmatrix}$ and spits out vector $\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$.

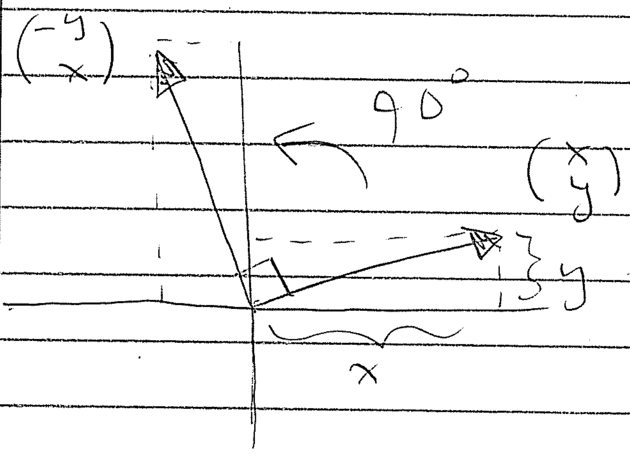
Q: which matrix/machine "does nothing"?

Answer: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1x+0y \\ 0x+1y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

Call it $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the "identity matrix"

We have $I\vec{x} = \vec{x}$ for all $\vec{x}$.

Q: Which matrix rotates $\begin{pmatrix} x \\ y \end{pmatrix}$ by 90° counterclockwise?



Should send

$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -y \\ x \end{pmatrix}$

We want $R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$. What is R?

Must be 2×2, so $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Answer: $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Q: Which matrix rotates $180°$?
(clockwise OR counterclockwise)

Point of view:
Rotate by $180°$ = Rotate by $90°$ twice.

$$R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}.$$

Do it again.

$$R\left[R\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0(-y) + (-1)x \\ 1(-y) + 0x \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}.$$

180° rotation sends $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} -x \\ -y \end{pmatrix}$

Matrix: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$

Look at this:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Point of View: can I move the brackets?

$$\underbrace{\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right]}_{\text{"multiply" matrices?}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \; ?$$

Sure, why not? Let's say

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$R^2 = -I.$$

(R is a square root of $-I$?)

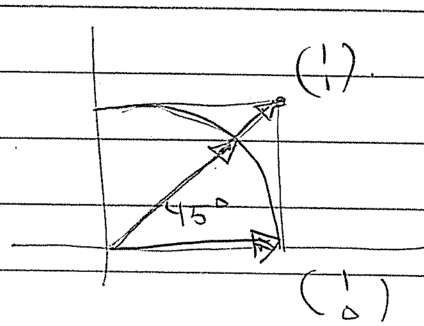
Thinking Homework:

Compute $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = ?$

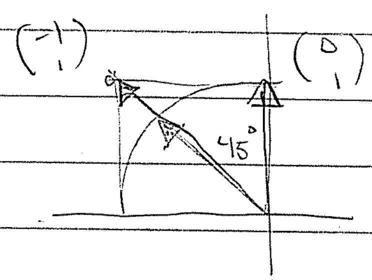
Q: Which matrix M rotates by $45°$ counterclockwise?

Say $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

What do we know?



$$M\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



$$M\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

And that's enough.

Because

$$M\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a\\c\end{pmatrix}$$

$$M\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}b\\d\end{pmatrix}.$$

$$\Longrightarrow \quad M = \begin{pmatrix}a & b\\c & d\end{pmatrix} = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}.$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}.$$

Question: $M^2 = ?$

Answer: $M^2 = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ = rotate by $90°$ c.c.w.

In general, which matrix rotates by angle θ c.c.w.?

Answer: $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$