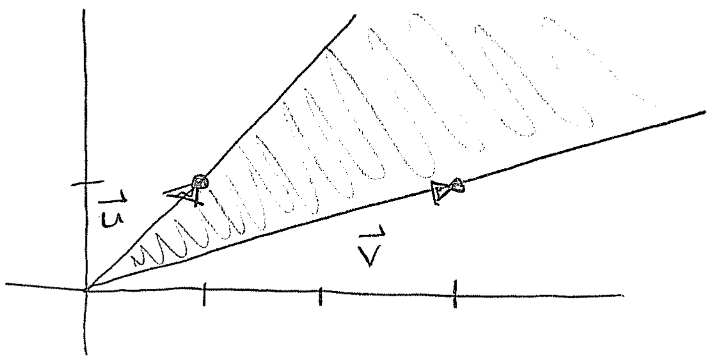
Math 210 E
Spring 2013

Exam 1
Friday March 1

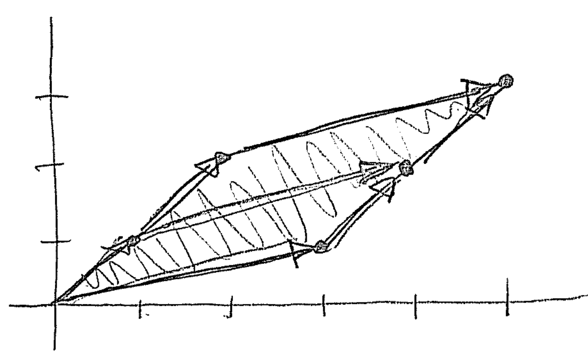
This is a closed book test. No electronic devices are allowed. If two students submit exams in which any solution has been copied, **both students will receive a score of zero.** There are 5 pages and 5 problems, each worth 6 points.

Problem 1. Consider the vectors $\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ in the Cartesian plane.

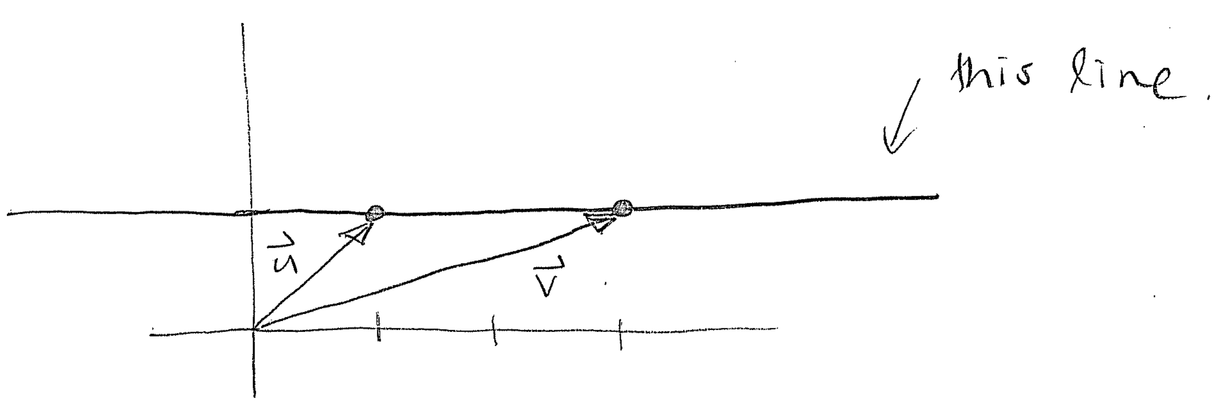
(a) **Draw** the set of vectors $s\vec{u} + t\vec{v}$, for all $s \geq 0$ and $t \geq 0$.



(b) **Draw** the set of vectors $s\vec{u} + t\vec{v}$, for all $0 \leq s \leq 2$ and $0 \leq t \leq 1$.



(c) **Draw** the set of vectors $s\vec{u} + t\vec{v}$, for all $s + t = 1$.



Problem 2. Consider the vectors $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ in Cartesian space.

(a) Let $\vec{x}$ be a vector perpendicular to **both** $\vec{u}$ and $\vec{v}$. That is, $\vec{x} \bullet \vec{u} = 0$ and $\vec{x} \bullet \vec{v} = 0$. Write down the matrix equation $A\vec{x} = \vec{b}$ that $\vec{x}$ must satisfy.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(b) **Solve** the system $A\vec{x} = \vec{b}$ from part (a) to find **all** such vectors $\vec{x}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \longrightarrow \begin{array}{cccc} x & y & z & \\ \hline \end{array} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

Pivot: x, y

Free: z.

$$\Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ z \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

it's a line

(c) Tell me the **equation** of the plane described by $s\vec{u} + t\vec{v}$ for all s, t.

The plane with normal line $z \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is

$$-x + y + z = 0.$$

(d) Tell me the equation of a **different** plane, parallel to the plane from part (c).

$$-x + y + z = k$$

for some $k \neq 0$.

Problem 3.

(a) Let A and B be matrices such that the product AB exists. Finish the sentence: The j**-th column** of AB is...

$A \cdot (j\text{th column of } B).$

(b) Now let A be a 3×3 invertible matrix. Finish the sentence: The **1st column** of AA^{-1} is...

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

(c) Now let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ and let $\vec{x}$ be the **1st column** of A^{-1}. Write a matrix equation that $\vec{x}$ must satisfy.

From (a) and (b):

$$A(1\text{st col of } A^{-1}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

(d) **Solve** your equation to find $\vec{x}$. (I don't want to know the whole matrix A^{-1}. I just want to know the 1st column.)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Hence $\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

Problem 4. Consider two points (i.e. column vectors) $\vec{u}$ and $\vec{v}$ in n-dimensional space.

(a) Consider the set of points $t\vec{u} + (1-t)\vec{v}$ for all t. **Describe** this set geometrically.

The line going through points $\vec{u}$ and $\vec{v}$.

(b) Suppose that $A\vec{u} = \vec{b}$ and $A\vec{v} = \vec{b}$ for some matrix A and vector $\vec{b}$. Show that the **midpoint** also solves this equation: i.e. show that $A(\frac{1}{2}(\vec{u}+\vec{v})) = \vec{b}$.

$$\begin{aligned} A\left(\tfrac{1}{2}(\vec{u}+\vec{v})\right) &= \tfrac{1}{2}A(\vec{u}+\vec{v}) \\ &= \tfrac{1}{2}A\vec{u} + \tfrac{1}{2}A\vec{v} \\ &= \tfrac{1}{2}\vec{b} + \tfrac{1}{2}\vec{b} = \vec{b}. \end{aligned}$$

(c) If $A\vec{u} = \vec{b}$ and $A\vec{v} = \vec{b}$, show that we have $A(t\vec{u} + (1-t)\vec{v}) = \vec{b}$ for all t.

$$\begin{aligned} A(t\vec{u} + (1-t)\vec{v}) &= tA\vec{u} + (1-t)A\vec{v} \\ &= t\vec{b} + (1-t)\vec{b} \\ &= \cancel{t\vec{b}} + \vec{b} - \cancel{t\vec{b}} \\ &= \vec{b}. \end{aligned}$$

(d) Finish the sentence: If a system of linear equations has at least **two** solutions, then it must have...

∞ many solutions
(in fact, a whole line)

Problem 5.

(a) Fill in the blanks: Any matrix A with m rows and n columns can be though of as a function from __n__-dimensional space to __m__-dimensional space.

$$m\left\{\underbrace{\left(\qquad\right)}_{n}\right.\left(\quad\right)\left.\right\}n = \left(\quad\right)\left.\right\}m$$

(b) Finish the sentence: If the matrix A has m rows and n columns, then the matrix equation $A\vec{x} = \vec{b}$ can be thought of **geometrically** as the intersection of...

m hyperplanes in

n-dimensional space

(c) Fill in the blank: In general (for a **typical** matrix A), the solution of the equation $A\vec{x} = \vec{b}$ from part (b) can be described using __n − m__ free parameters.

(d) Finish the sentence: A collection of 5 hyperplanes in 5-dimensional space **most likely** intersect in...

a point.

5 − 5 = 0 dimensional.