HW4 notes

Our discussion of Gaussian elimination is done and now we will move on to a new topic. Actually, it's the same topic but written is a new language:

the language of "matrix algebra".

Recall that the central problem of linear algebra is to solve a system of m linear equations in n unknowns, which we can write as follows:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

However, this notation is quite cumbersome. So far we have seen two ways to simplify it.

1. Row Picture.

When thinking of the system as an intersection of m hyperplanes in n-dimensional space, we can rewrite the i^{th} equation as

$$\boxed{\vec{a}_{i*} \cdot \vec{x} = b_i}$$

where $\vec{a}_{i*}$ is the i^{th} row vector of the system and $\vec{x}$ is the vector of variables:

$$\vec{a}_{i*} = \begin{pmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{pmatrix} \quad \& \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

We have seen that this is the unique hyperplane in n-dimensional space that

- is perpendicular to $\vec{a}_{i*}$
- has distance $b_i / \|\vec{a}_{i*}\|$ from $\vec{0}$.

Then we can express the system of hyperplanes as

$$\begin{cases} \vec{a}_{1*} \cdot \vec{x} = b_1 \\ \vec{a}_{2*} \cdot \vec{x} = b_2 \\ \quad\vdots \\ \vec{a}_{m*} \cdot \vec{x} = b_m \end{cases}$$

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2. Column Picture.

When thinking of the system as a linear combination of n vectors in m-dimensional space, we can rewrite it as a single vector equation

$$\boxed{x_1 \vec{a}_{*1} + x_2 \vec{a}_{*2} + \cdots + x_n \vec{a}_{*n} = \vec{b}}$$

where $\vec{a}_{*j}$ is the j^{th} column vector of the system and $\vec{b}$ is the vector of constants:

$$\vec{a}_{*j} = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \quad \& \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \\ b_m \end{pmatrix}.$$

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But both of these notations still use a lot of symbols. Wouldn't it be nice if we could

- simplify the notation even further,
- express the row & column pictures simultaneously?

This is exactly what the language of matrix algebra does for us. In this language we will express the system simply as

$$\boxed{A\vec{x} = \vec{b}}$$

Where A is a rectangular array called the matrix of coefficients:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

To write it out fully, we have

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_m \end{pmatrix}$$

To emphasize that m & n may be different, here is a concrete example from HW3 Problem 4. We can rewrite the system of 3 equations in 6 unknowns

$$\begin{cases} 0 + x_2 + 0 + x_4 - x_5 - 4x_6 = -1 \\ x_1 + 2x_2 - x_3 + 4x_4 - x_5 - 4x_6 = 3 \\ x_1 + 2x_2 - x_3 + 4x_4 + 0 - x_6 = 5 \end{cases}$$

as a single "matrix equation":

$$\begin{pmatrix} 0 & 1 & 0 & 4 & -1 & -4 \\ 1 & 2 & -1 & 4 & -1 & -4 \\ 1 & 2 & -1 & 4 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}.$$

So what, you may ask. So we can express a linear system with a small number of symbols:

$$A\vec{x} = \vec{b}.$$

But does this actually help us to <u>solve</u> linear systems?

Well, yes it does. Like any good notation, this one gives us a new point of view and suggests new questions we can ask. For example:

The expression "$A\vec{x}$" on the left looks kind of like "multiplation". This suggests that maybe we could also "divide" to get

$$A\vec{x} = \vec{b}$$

$$\Longrightarrow \quad \vec{x} = \text{"}\frac{1}{A}\vec{b}\text{"}$$

and that would be pretty cool....

In fact we <u>will</u> learn how to do something like this but it will take us several weeks to make sense of it.

But don't feel bad. It took the human race thousands of years to take this step, so several weeks is actually pretty fast.

The whole key to the language of matrix algebra is the concept of

"matrix multiplication".

Stay tuned.

I

Last time I introduced the "matrix" notation, in which we rewrite a system of linear equations

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

as a single "matrix equation"

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

and then we replace each matrix and vector by a single symbol to write

$$\boxed{A\vec{x} = \vec{b}}$$

///

☆ Definition: A matrix is just a rectangular (or square) array of numbers. If the matrix has m rows and n columns we say it has shape $m \times n$.

We can also think of a vector with n components as a matrix of shape $n \times 1$. (By convention we always think of vectors as column matrices.)

[Remark: The word "matrix" (Latin for "womb") was first applied to a rectangle of numbers by James Joseph Sylvester in 1850. Pretty recently!]

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Thus the matrix notation is really just a rule for "multiplying" an $m \times n$ matrix A by an $n \times 1$ matrix/vector $\vec{x}$ to obtain an $m \times 1$ matrix/vector that we call "$A\vec{x}$".

Example: Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{pmatrix}$ & $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

2×3 $\qquad$ 3×1.

Then by definition we have

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 0 - y \\ 2x + 3y + 4z \end{pmatrix}.$$

$$2 \times 3 \quad 3 \times 1 \qquad\qquad 2 \times 1$$

match

As a special case we can multiply a $1 \times n$ matrix/row by an $n \times 1$ matrix/column to obtain a 1×1 matrix/number.

Example:

$$\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a_1 x_1 + a_2 x_2 + a_3 x_3 \end{pmatrix}.$$

Yes, you are right. This is just a fancy way to write the dot product of vectors.

To be explicit, we will define the concept of the "transpose" of a matrix.

☆ Definition: Let A be an $m \times n$ matrix with number a_{ij} in the i^{th} row & j^{th} column (we call this the (i,j) entry of the matrix):

$$A = \begin{array}{cc} & \begin{matrix} \phantom{a_{11}} & & j & & \phantom{a_{1n}} \end{matrix} \\ i & \underbrace{\left(\begin{matrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{matrix}\right)}_{n \text{ columns}} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \\ a \end{matrix}}\right\} m \text{ rows} \end{array}$$

Then we define the transpose matrix A^T as the matrix of shape $n \times m$ with entry a_{ij} in the j^{th} row & i^{th} column:

$$A^T = \begin{array}{cc} & \begin{matrix} \phantom{a_{11}} & & i & & \phantom{a_{m1}} \end{matrix} \\ j & \underbrace{\left(\begin{matrix} a_{11} & \cdots & a_{i1} & \cdots & a_{m1} \\ \vdots & & \vdots & & \vdots \\ a_{1j} & \cdots & a_{ij} & \cdots & a_{mj} \\ \vdots & & \vdots & & \vdots \\ a_{1n} & \cdots & a_{in} & \cdots & a_{mn} \end{matrix}\right)}_{m \text{ columns}} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \\ a \end{matrix}}\right\} n \text{ rows} \end{array}$$

Pretty basic idea, but it's useful because it allows us to turn the dot product of vectors into a product of matrices.

Consider two vectors (i.e. column matrices)

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad \& \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Then if we think of a 1×1 matrix as just a number, we have

$$\vec{a} \cdot \vec{x} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

$$= \left(a_1 x_1 + \cdots + a_n x_n \right)$$

$$= \left(a_1 \cdots a_n \right) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \vec{a}^T \vec{x}.$$

In summary,

$$\boxed{\vec{a} \cdot \vec{x} = \vec{a}^T \vec{x}}$$

dot product   matrix product.

Now we have an effective way to describe the two pictures of matrix notation.

Let A be an $m \times n$ and let $\vec{x}$ be an $n \times 1$ matrix / column / vector.

1. Column Picture.

Let $\vec{a_{*j}}$ be the j^{th} column vector of A (which has shape $m \times 1$). Then we have

$$A\vec{x} = \underbrace{x_1 \vec{a_{*1}} + x_2 \vec{a_{*2}} + \cdots + x_n \vec{a_{*n}}}_{\text{a "linear combination" of the columns of } A.}$$

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2. Row Picture.

Let $\vec{a}_{i*}$ be the i^{th} row vector (which we think of as an $n \times 1$ column, because we think of <u>every</u> vector as a column). Then

$$A\vec{x} = \begin{pmatrix} \vec{a}_{1*}^{\,T} \vec{x} \\ \vec{a}_{2*}^{\,T} \vec{x} \\ \vdots \\ \vec{a}_{m*}^{\,T} \vec{x} \end{pmatrix}.$$

///

It's very important that you understand these two different ways to compute $A\vec{x}$.

Example: Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & -2 \end{pmatrix}$ and

$$\vec{x} = (1,2,3,4) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^T.$$

Compute the product $A\vec{x}$ in two ways.

1. Column Picture.

$$A\vec{x} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$= 1 \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -6 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ -8 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}.$$

2. Row Picture.

$$A\vec{x} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 \\ 3 & 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} (1\ 0\ 1\ 0) \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \\ (0\ 2\ -2\ 1) \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \\ (3\ 1\ 0\ -2) \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1+0+3+0 \\ 0+4-6+4 \\ 3+2+0-8 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$$

same answer ✓

So far we have only defined the product

$$A\vec{x}$$

when A is an $m \times n$ matrix and $\vec{x}$ is an $n \times 1$ column. Next time we will think about how to define the product

$$\text{"}AB\text{"}$$

when B is a more general kind of matrix. Buckle your seatbelts; it will involve a radically new point of view.

We have decided two write a system of m linear equations in n unknowns as a single matrix equation

$$A\vec{x} = \vec{b}$$

where

- A is an $m \times n$ matrix of coefficients
- $\vec{x}$ is an $n \times 1$ matrix of variables
- $\vec{b}$ is an $m \times 1$ matrix of constants

Accordingly, we have two different ways to view the matrix product "$A\vec{x}$", coming from the row & column pictures of the linear system. To be specific, let

$$\vec{a}_{i*} = i^{th} \text{ row vector of } A,$$
$$\vec{a}_{*j} = j^{th} \text{ column vector of } A.$$

1. Row Picture.

The ith entry of the vector $A\vec{x}$ is

$$\vec{a_{i*}} \cdot \vec{x}.$$

2. Column Picture

The vector $A\vec{x}$ is a linear combination of the column vectors of A,

$$A\vec{x} = x_1 \vec{a_{*1}} + x_2 \vec{a_{*2}} + \cdots + x_n \vec{a_{*n}}.$$

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Today we will consider the following question:

☆ Is it possible to define ~~a~~ the "product" of two matrices

"AB"

when B is not just a single column?

A quick glance at Wikipedia (or a textbook) shows that the answer is <u>yes</u>.

You will also see that the product of matrices looks like a mess of symbols. So here's a better question:

☆ Why would we *want* to multiply matrices, and how *should* we think about the definition of matrix multiplication?

The reason we *want* to multiply matrices is to give us new tools for *solving* systems of equations:

$$\text{“}A\vec{x}\text{''} = \vec{b} \implies \vec{x} = \text{“}\frac{1}{A}\vec{b}\text{''} \;??$$

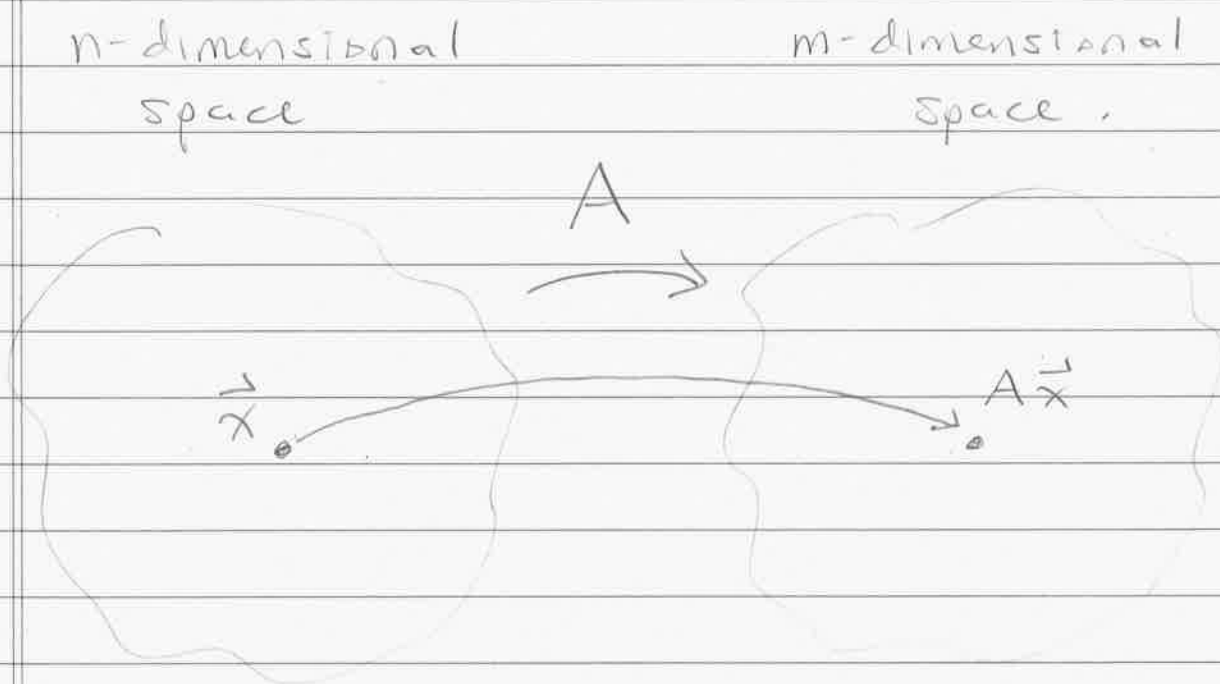
But to understand what matrix multiplication *should* be requires a radically new and modern point of view (first stated by Arthur Cayley in 1858).

[Remark: Is 1858 modern? Yes, by mathematical standards. The Calculus was invented in the 1660s.]

☆ Modern Point of View:

We will think of an $m \times n$ matrix A as a <u>function</u> that accepts a vector $\vec{x}$ with n coordinates and spits out a vector $A\vec{x}$ with m coordinates.

Picture:



Thus if A is <u>square</u> $(m=n)$, we can think of A as a function sending vectors to vectors in the <u>same space</u>

Let's try some examples.

Example: The 2×2 matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

sends points in the plane to points in the plane. What does it <u>do</u> to the points?

Given a point $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, the function I sends it to the point

$$I\vec{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Thus I sends every point to itself! We call this the "identity function" (or the "do-nothing function" on the Cartesian plane.

Example: How about the function

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ ?$$

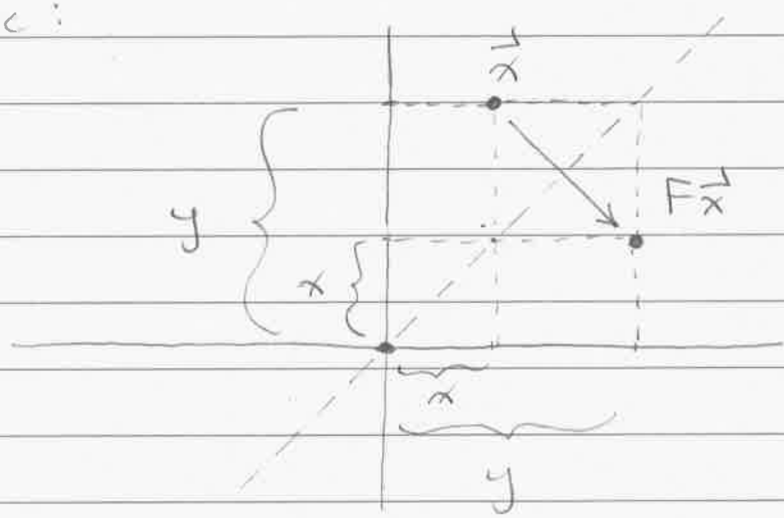
The Function F sends the point $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ to

$$F\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= x\begin{pmatrix} 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ x \end{pmatrix} + \begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}.$$

It switches the two coordinates.
Geometrically, we can think of this as a <u>reflection</u> across the line of slope 1.

Picture:



[Remark: F is for "Flip" or "reFlection".]

And what happens if we do F twice in succession?

Let's check:

$$F(F\vec{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix}$$

$$= y\begin{pmatrix} 0 \\ 1 \end{pmatrix} + x\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ y \end{pmatrix} + \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \vec{x}.$$

Does that surprise you? NO. If we reflect across the line and <u>then</u> reflect <u>again</u> we get back where we started. ///

To summarize these examples: For any point $\vec{x}$ in the plane we have

- $I\vec{x} = \vec{x}$
- $F(F\vec{x}) = \vec{x}$,

and hence

$$F(F\vec{x}) = I\vec{x}.$$

Now I'm really tempted to rearrange the parentheses and write

(*) $$(FF)\vec{x} = I\vec{x},$$

but is that allowed? Well, no because the expression "FF" is not defined.

OK, no problem. Let's just <u>define</u> $FF := I$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

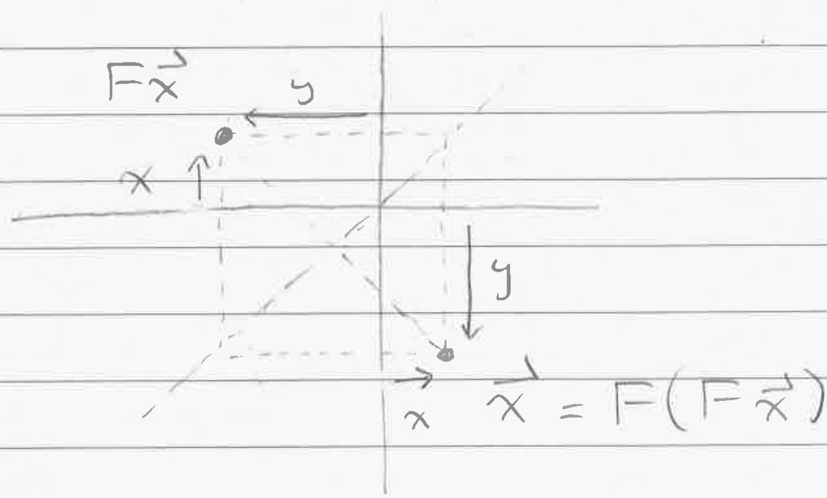
And now the equation (*) is perfectly true.

Congratulations: We just "multiplied" two 2×2 matrices to obtain another 2×2 matrix. <u>And</u> we understand what it <u>means</u>.

"Reflecting across the same line twice is the same as doing nothing once."

Last time we saw our first example of a matrix product "AB" when B is not a single column.

Recall: The matrix $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ sends the point $\vec{x} = (x, y)$ to the point $F\vec{x} = (y, x)$, which geometrically is a reflection across the line of slope 1:



If we perform the reflection again then we arrive back where we started:

$$F(F\vec{x}) = \vec{x} .$$

Thus "doing F twice" is the same as "doing nothing once" and we know that the "doing nothing" function is represented by the identity matrix

$$I\vec{x} = x$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

So we conclude that for all points $\vec{x}$ in the plane we have

$$F(F\vec{x}) = I\vec{x}.$$

and this makes it perfectly clear how we should define the matrix $F^2 = FF$: we just rearrange the parentheses,

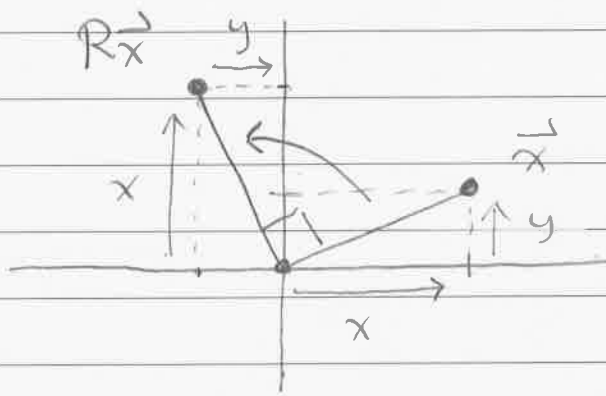
$$(FF)\vec{x} = I\vec{x}$$

and declare that $FF = I$, i.e.,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Another Example: Let R be the function that rotates each vector in the plane counterclockwise by $90°$. Can we represent R as a matrix?

Note that for all points $\vec{x} = (x, y)$ we must have $R\vec{x} = (-y, x)$ because of the following picture:



That is, for all $\vec{x} = (x, y)$ we must have

$$R\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0x - 1y \\ 1x + 0y \end{pmatrix}$$

and there is a unique matrix that does this:

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Can we compute the matrix $R^2 = RR$?

Sure: For all $\vec{x} = (x, y)$ we have

$$R\left(R\begin{pmatrix} x \\ y \end{pmatrix}\right) = R\begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} -(x) \\ (-y) \end{pmatrix}.$$

So we must have

$$(RR)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = \begin{pmatrix} -1x + 0y \\ 0x - 1y \end{pmatrix}$$

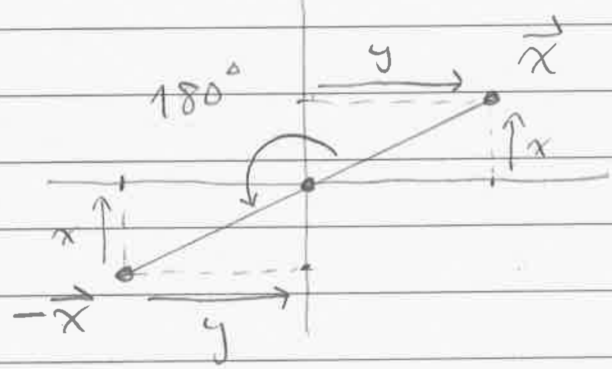
The unique solution is

$$R^2 = RR = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -I.$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Does this make sense? We know that rotating c.c.w. <u>twice by 90°</u> is the same as rotating c.c.w. <u>once by 180°</u>.

And <u>does</u> the matrix $-I$ rotate the plane c.c.w. by 180° ?



$-I\vec{x} = -\vec{x}$

Yes it does!

[Remark: We are starting to see some kind of "matrix algebra" emerging. Can you predict what the matrix R^3 is without doing any more computations?]

Ok, now let's try to compute the product of two general 2×2 matrices:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \& \quad B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

For all points $\vec{x} = (x, y)$ we have

$$A(B\vec{x}) = A \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= A \begin{pmatrix} a'x + b'y \\ c'x + d'y \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a'x + b'y \\ c'x + d'y \end{pmatrix}$$

$$= \begin{pmatrix} a(a'x + b'y) + b(c'x + d'y) \\ c(a'x + b'y) + d(c'x + d'y) \end{pmatrix}$$

$$= \begin{pmatrix} (aa' + bc')x + (ab' + bd')y \\ (ca' + dc')x + (cb' + dd')y \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}}_{\text{call this } C} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= C\vec{x}.$$

[No thinking was involved here; all of this was forced on me by the original definition of matrix × vector.]

We conclude that for all $\vec{x}$ we have

$$A(B\vec{x}) = C\vec{x},$$

thus we should define the matrix AB so that the equation

$$(AB)\vec{x} = C\vec{x}$$

is true for all $\vec{x}$. In other words, we <u>should</u> define

$$AB := C.$$

$$\star \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := \begin{pmatrix} aa' + bd' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}$$

and this is how we <u>will</u> define it.

$$(AB)\vec{x} = C\vec{x}$$

is true for all $\vec{x}$. In other words, we <u>should</u> define

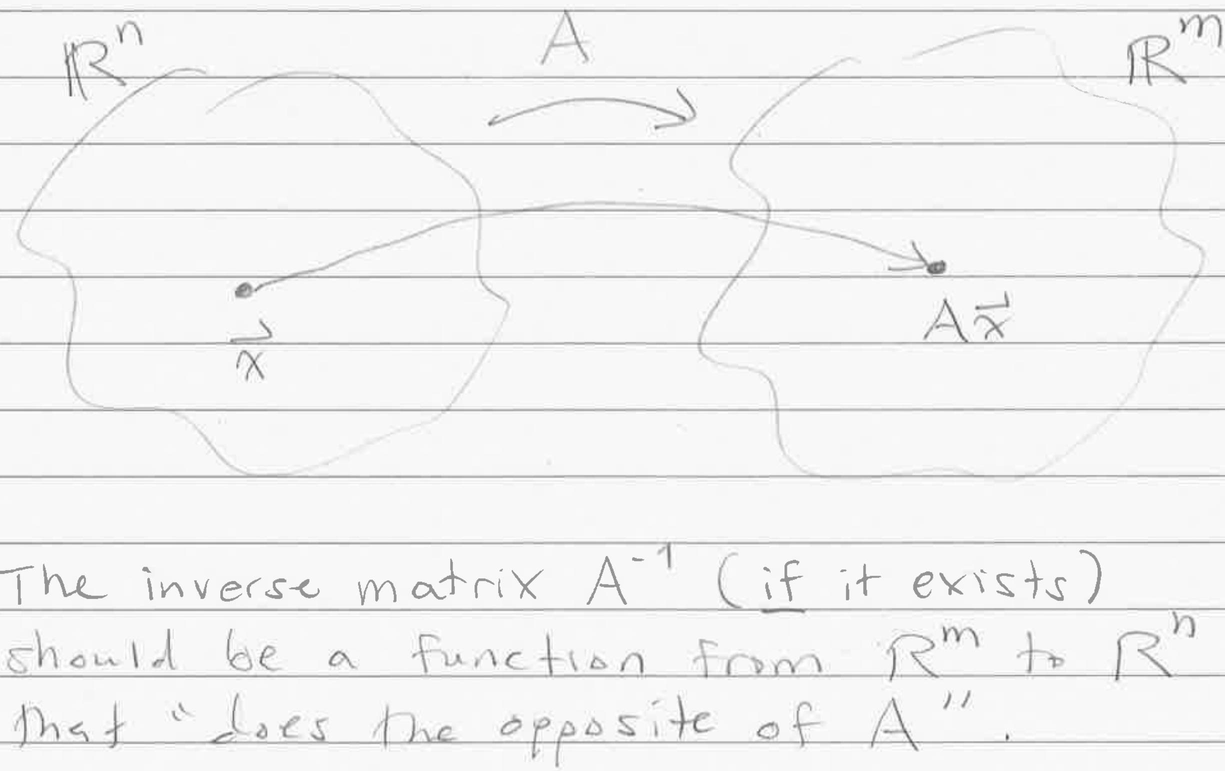
$$AB := C.$$

$$\star \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := \begin{pmatrix} aa' + bd' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}$$

and this is how we <u>will</u> define it.

Today: The Inverse of a Matrix.

Let A be an $m \times n$ matrix. Recall that we can think of this as a <u>function</u> from $\mathbb{R}^n$ to $\mathbb{R}^m$:



The inverse matrix A^{-1} (<u>if</u> it exists) should be a function from $\mathbb{R}^m$ to $\mathbb{R}^n$ that "does the opposite of A".

In other words, we should have

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[loop left, "A^{-1}A"] \arrow[r, bend left, "A"] & \mathbb{R}^m \arrow[l, bend left, "A^{-1}"] \arrow[loop right, "AA^{-1}"]
\end{tikzcd}
$$

where AA^{-1} is the "do nothing function" from $\mathbb{R}^m$ to $\mathbb{R}^m$ and $A^{-1}A$ is the "do nothing function" from $\mathbb{R}^n$ to $\mathbb{R}^n$.

In matrix language we require

- A^{-1} is an $n \times m$ matrix
- $AA^{-1} = I_m$
- $A^{-1}A = I_n$,

where I_n is the identity matrix of size n,

$$
I_n := \left.\underbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & & & \ddots & \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}}_{n}\right\} n
$$

The important questions are:

① When does A^{-1} exist?
② How can we compute it?

(we skipped the FTLA)

The recently discussed FTLA tells us something important about ①.

☆ Claim: If A is not square then the inverse A^{-1} does not exist.

To see this let's recall how the matrix product is computed. If A & B are matrices such that the product exists, then we have three key formulas

$$(i,j)^{th} \text{ entry of } AB = (i^{th} \text{ row of } A)(j^{th} \text{ col of } B)$$

$$i^{th} \text{ row of } AB = (i^{th} \text{ row of } A)\, B$$

$$j^{th} \text{ column of } AB = A\,(j^{th} \text{ column of } B).$$

The 2nd & 3rd formulas tell us that

- if A has a row relation then so does AB.
- if B has a column relation then so does AB.

Example: Let A_{i*} be the i^{th} row of A and let $(AB)_{i*}$ be the ith row of AB so the formula says

$$(AB)_{i*} = A_{i*} B .$$

Now suppose that A has some row relation, say $A_{1*} + A_{2*} = A_{3*}$. Then AB has the same row relation because

$$\begin{aligned} A_{1*} + A_{2*} &= A_{3*} \\ (A_{1*} + A_{2*}) B &= A_{3*} B \\ A_{1*} B + A_{2*} B &= A_{3*} B \\ (AB)_{1*} + (AB)_{2*} &= (AB)_{3*} . \end{aligned}$$

///

Now suppose that A is $m \times n$ and B is $n \times m$ with

$$\begin{aligned} AB &= I_m \\ BA &= I_n \end{aligned}$$

We want to show that this is impossible when $m \neq n$. There are two cases.

Case 1: If $m > n$ then B is short and wide so its RREF will definitely have a non-pivot column. We conclude that

B has a column relation.

But then the product $AB = I_m$ must also have a column relation, which is impossible because the RREF of I_m is just I_m (which has no non-pivot columns).

///

Case 2: If $m < n$ then A is short and wide, so by the same reasoning

A has a column relation.

But then $BA = I_n$ has a column relation which is again impossible since $RREF(I_n) = I_n$ has no non-pivot columns.

///

This completes the proof of the claim.

Now let's discuss ②. If A is a square matrix then it may have an inverse. Let's try to compute it.

Example: Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

If the inverse $A^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ exists then it must satisfy

$$(*) \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the trick (j^{th} col AB) = A(j^{th} col B) we can break $(*)$ into two simultaneous linear systems:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\wr$

and then we can (try to) solve both of the systems separately.

First System:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right)$$

$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \longrightarrow \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Second System:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right)$$

$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

We conclude that A is invertible with inverse

$$A^{-1} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

[Well, there's an issue here. Certainly we know that

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

because that's the problem we were trying to solve. But it's not obvious why we should *also* have

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

You should perform the multiplication to check that this is true. In general, if A & B are square matrices such that $AB = I$, then it follows from the FTLA that we must *also* have $BA = I$, but this fact is more *subtle* than most people realize!]

Remark: Hey, we used the same elimination steps for both of those linear systems. Wouldn't it be more efficient to solve them at the same time?

Sure let's just put them "next to each other" and see what happens:

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right).$$

This cute trick can be summarized as

$$\left(A \mid I \right) \xrightarrow{RREF} \left(I \mid A^{-1} \right)$$

It might look strange, but it works (well, as long as A^{-1} exists)

To summarize our discussion of inverses:

Let A be an $m \times n$ matrix. We say that B is the inverse matrix of A if

$$AB = I_m \quad \& \quad BA = I_n .$$

Why do I say "the" inverse? Well, suppose we have another matrix C satisfying

$$AC = I_m \quad \& \quad CA = I_n .$$

Then it follows that

$$B = BI_m = B(AC) = (BA)C = I_nC = C .$$

We conclude that *if* the inverse of A exists, *then* it is unique. Since it's unique we can give it a special name:

we call it A^{-1}.

But *does* the inverse of A exist?

If A is *not square* we saw that A^{-1} does *not exist*.

So let A be square, say $m \times m$. If A^{-1} exists it will also be $m \times m$ and we can *try* to compute it with the following algorithm

$$\left(A \mid I_m \right) \xrightarrow{RREF} \left(I_m \mid A^{-1} \right).$$

The algorithm will *succeed* if and only if

$$RREF(A) = I_m.$$

In other words, the algorithm will *fail* if and only if

$\mathrm{RREF}(A) \neq I_m$.

Many textbooks summarize this with a theorem of the following sort.

☆ Invertible Matrix Theorem:

Let A be a square matrix. Then the following conditions are equivalent.

- A is invertible
- $\mathrm{RREF}(A) = I$
- A has no nontrivial column relation
- A has no nontrivial row relation.
- $\det(A) \neq 0$ [we'll discuss this later...]

The list can be expanded depending on how much abstract nonsense you know. [The version on Wolfram MathWorld has 23 equivalent conditions!]

The important points are these:

- We know exactly when a matrix is invertible.
- We know how to compute the inverse when it exists.

Now let me summarize the basic properties of matrix algebra for future reference.

Let A, B & C be matrices and let α & β be numbers. Then the following properties hold (as long as the matrices are <u>defined</u>):

- $(\alpha+\beta)A = \alpha A + \beta A$.
- $\alpha(\beta A) = (\alpha\beta)A$.
- $\alpha(A+B) = \alpha A + \alpha B$
- $\alpha(AB) = (\alpha A)B = A(\alpha B)$
- $(A+B)C = AC + BC$
- $A(B+C) = AB + AC$
- $A+(B+C) = (A+B)+C$
- $A(BC) = (AB)C$

[The last property ("associativity" of matrix multiplication) is surprisingly useful!]

These properties generalize the properties of vector algebra & the dot product, which in turn generalize the familiar properties of addition & multiplication of numbers.

Luckily all of the properties are very intuitive. The only difference from "classical arithmetic" is that in general we have

$$AB \neq BA \quad !$$

even when the matrices AB & BA are both defined and have the same shape.

Finally, let's look at the algebraic properties of inversion & transposition. Let A & B be matrices. When the following matrices exist we have

- $(A^{-1})^{-1} = A$
- $(A^T)^T = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(AB)^T = B^T A^T$
- $(A^T)^{-1} = (A^{-1})^T$
- $(A+B)^T = A^T + B^T$.

[WARNING: In general we have

$$(A+B)^{-1} \neq A^{-1} + B^{-1}$$

Indeed, if this were true then it would be true for 1×1 matrices. In other words, for all numbers a & b such that $a \neq 0$, $b \neq 0$ & $a+b \neq 0$ we would have

$$\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b},$$

and you know this is not true.]

Let's examine the 1st, 3rd & 5th properties.

1st: Suppose A^{-1} exists. Then by definition we have

$$AA^{-1} = I \quad \& \quad A^{-1}A = I.$$

But these two equations <u>also</u> tell us that A is the inverse of A^{-1}:

$$A = (A^{-1})^{-1}.$$

///

3rd: Suppose A^{-1}, B^{-1} & AB exist. Then by definition we have

$$AA^{-1} = I \quad \& \quad A^{-1}A = I$$
$$BB^{-1} = I \quad \& \quad B^{-1}B = I$$

Then using the "associativity" property of matrix multiplication gives

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$
$$= AIA^{-1} = AA^{-1} = I,$$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$$
$$= B^{-1}IB = B^{-1}B = I,$$

so we conclude that $B^{-1}A^{-1}$ is the inverse of AB, as desired.

5th: Suppose that A^{-1} exists, so by definition we have

$$AA^{-1} = I \quad \& \quad A^{-1}A = I.$$

Then applying the transpose to each equation gives

$$AA^{-1} = I \qquad \& \qquad A^{-1}A = I$$

$$(AA^{-1})^T = I^T \qquad (A^{-1}A)^T = I^T$$

$$(A^{-1})^T A^T = I \qquad A^T(A^{-1})^T = I ,$$

which tells us that $(A^{-1})^T$ is the inverse of A^T. In other words,

$$(A^T)^{-1} = (A^T)^{-1} .$$