Math 210 **Fall 2019**
Homework 6 **Drew Armstrong**

Let A be an $n \times n$ matrix. We say that a number λ is an *eigenvalue* of A if there exists a nonzero vector $\mathbf{x} \neq \mathbf{0}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. In this case we also say that $\mathbf{x}$ is an *eigenvector*. One can show that λ if an eigenvalue if and only if $\det(A - \lambda I) = 0$.[1]

Problem 1. Let A be a square matrix and suppose that we have $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$ for some vectors $\mathbf{x}, \mathbf{y}$ and scalars λ, μ.

(a) Show that $A(s\mathbf{x}) = \lambda(s\mathbf{x})$ for all scalars s.
(b) Show that $A^n(s\mathbf{x}) = \lambda^n(s\mathbf{x})$ for all scalars s and integers $n \geq 0$.
(c) Show that $A^n(s\mathbf{x} + t\mathbf{y}) = \lambda^n(s\mathbf{x}) + \mu^n(t\mathbf{y})$ for all scalars s, t and integers $n \geq 0$.

(a): By linearity of A we have $A(s\mathbf{x}) = s(A\mathbf{x}) = s(\lambda\mathbf{x}) = \lambda(s\mathbf{x})$.

(b): By linearity of A^n we have $A^n(s\mathbf{x}) = s(A^n\mathbf{x})$. If we can show that $A^n\mathbf{x} = \lambda^n\mathbf{x}$ then it will follow that

$$A^n(s\mathbf{x}) = s(A^n\mathbf{x}) = s(\lambda^n\mathbf{x}) = \lambda^n(s\mathbf{x}).$$

To prove that $A^n\mathbf{x} = \lambda^n\mathbf{x}$ for all $n \geq 0$ we will use the method of induction.[2] For the base case we observe that $A^0\mathbf{x} = I\mathbf{x} = \mathbf{x} = \lambda^0\mathbf{x}$ is correct.[3] Now fix some $n \geq 0$ and **assume for induction** that $A^n\mathbf{x} = \lambda^n\mathbf{x}$ is true. In this case we must also have

$$A^{n+1}\mathbf{x} = A(A^n\mathbf{x}) = A(\lambda^n\mathbf{x}) = \lambda^n(A\mathbf{x}) = \lambda^n(\lambda\mathbf{x}) = \lambda^{n+1}\mathbf{x}$$

as desired.

(c): From part (b) and the linearity of A^n we have

$$A^n(s\mathbf{x} + t\mathbf{y}) = A^n(s\mathbf{x}) + A^n(t\mathbf{y}) = \lambda^n(s\mathbf{x}) + \mu^n(t\mathbf{y}).$$

Problem 2. Geometry of Eigenvalues.

(a) If P is any matrix satisfying $P^2 = P$ (for example, a projection), show that the only possible eigenvalues are 0 and 1. Explain this geometrically.
(b) If F is any matrix satisfying $F^2 = I$ (for example, a reflection), show that the only possible eigenvalues are $+1$ and -1. Explain this geometrically.
(c) Use the quadratic formula to show that $\lambda = \cos\theta \pm \sin\theta\sqrt{-1}$ are the eigenvalues of the rotation matrix:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For which angles θ do you get real eigenvalues? Explain this geometrically.

(a): Assume that $P^2 = P$ and $P\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$. Then we must have

$$\begin{aligned} P^2\mathbf{x} &= P\mathbf{x} \\ \lambda^2\mathbf{x} &= \lambda\mathbf{x} \\ \lambda^2\mathbf{x} - \lambda\mathbf{x} &= \mathbf{0} \\ (\lambda^2 - \lambda)\mathbf{x} &= \mathbf{0} \end{aligned}$$

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[1] We allow the possibility that the eigenvalue λ is complex and the eigenvector $\mathbf{x}$ has complex entries.
[2] If you don't know what this is, don't worry.
[3] By convention we say that $A^0 = I$ is the identity matrix. If you don't like that, start with $n = 1$.

$$\begin{aligned} \lambda^2 - \lambda &= 0 \\ \lambda(\lambda - 1) &= 0 \\ \lambda &= 0 \text{ or } 1. \end{aligned}$$

Picture: Think of P as a projection onto a subspace. This subspace is the 1-eigenspace since every vector $\mathbf{x}$ in the subspace gets sent to itself. The 0-eigenspace is in the direction of the projection, i.e., the set of vectors that get projected to the origin $\mathbf{0}$. No other vector is an eigenvector because the projection changes its direction.

(b): Assume that $F^2 = I$ and $F\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$. Then we must have

$$\begin{aligned} F^2\mathbf{x} &= I\mathbf{x} \\ \lambda^2\mathbf{x} &= \mathbf{x} \\ \lambda^2\mathbf{x} - \mathbf{x} &= \mathbf{0} \\ (\lambda^2 - 1)\mathbf{x} &= \mathbf{0} \\ \lambda^2 - 1 &= 0 \\ (\lambda - 1)(\lambda + 1) &= 0 \\ \lambda &= 1 \text{ or } -1. \end{aligned}$$

Picture: Think of F as a reflection across a subspace. This subspace is the 1-eigenspace since every vector $\mathbf{x}$ in the subspace gets sent to itself. The -1-eigenspace is in the direction of the reflection, i.e., the set of vectors that get reflected to their negative. No other vector is an eigenvector because the reflection changes its direction.

(c): The characteristic equation is

$$\begin{aligned} \det(R_\theta - \lambda I) &= 0 \\ \det\begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} &= 0 \\ (\cos\theta - \lambda)(\cos\theta - \lambda) - \sin\theta(-\sin\theta) &= 0 \\ \lambda^2 - 2\cos\theta \cdot \lambda + \cos^2\theta + \sin^2\theta &= 0 \\ \lambda^2 - 2\cos\theta \cdot \lambda + 1 &= 0 \\ \lambda &= \frac{1}{2}[2\cos\theta \pm \sqrt{4\cos^2\theta - 4}] \\ \lambda &= \frac{1}{2}[2\cos\theta \pm \sqrt{4(\cos^2\theta - 1)}] \\ \lambda &= \frac{1}{2}[2\cos\theta \pm \sqrt{4\sin^2\theta(-1)}] \\ \lambda &= \frac{1}{2}[2\cos\theta \pm 2\sin\theta\sqrt{-1}] \\ \lambda &= \cos\theta \pm \sin\theta\sqrt{-1}. \end{aligned}$$

Picture: If $\sin\theta \neq 0$ (i.e., if $\theta \neq 0$ and $\theta \neq \pi$) then there are no real eigenvalues or eigenvectors. This reflects the fact that the direction of every vector changes under rotation. If $\theta = \pi$ then every vector gets rotated to its negative, hence every vector is a (-1)-eigenvector. If $\theta = 0$ then R_θ is the identity matrix, hence every vector is a 1-eigenvector.

Problem 3. Compute all eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} 8 & -2 \\ 15 & -3 \end{pmatrix}.$$

First we compute the eigenvalues:

$$\begin{aligned} \det(A - \lambda I) &= 0 \\ \det \begin{pmatrix} 8-\lambda & -2 \\ 15 & -3-\lambda \end{pmatrix} &= 0 \\ (8-\lambda)(-3-\lambda) - 15(-2) &= 0 \\ \lambda^2 - 5\lambda + 6 &= 0 \\ \lambda &= 2 \text{ or } 3. \end{aligned}$$

Now we compute the 2-eigenspace:

$$\left(\begin{array}{cc|c} 8-2 & -2 & 0 \\ 15 & -3-2 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 6 & -2 & 0 \\ 15 & -5 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 3 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

In other words, we have $A\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$ precisely when $3x - 1y = 0$. This is a line:

$$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Finally we compute the 3-eigenspace:

$$\left(\begin{array}{cc|c} 8-3 & -2 & 0 \\ 15 & -3-3 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 5 & -2 & 0 \\ 15 & -6 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 5 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

In other words, we have $A\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$ precisely when $5x - 2y = 0$. This is a line:

$$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

Problem 4. A Dynamical System. Suppose that a sequence of vectors $\mathbf{x}_n = (x_n, y_n)$ is defined by the following initial condition and recurrence:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 30 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n/2 + y_n \\ x_n/2 \end{pmatrix} \text{ for all integers } n \geq 0.$$

(a) Express the recurrence as a matrix equation $\mathbf{x}_{n+1} = A\mathbf{x}_n$. Then the n-th vector in the sequence is given explicitly by the matrix equation $\mathbf{x}_n = A^n\mathbf{x}_0$.

(b) Verify that $\mathbf{u} = (2, 1)$ and $\mathbf{v} = (-1, 1)$ are eigenvectors of the matrix A. [One could also find these from scratch, but I decided to give you a break.]

(c) Express the initial condition $\mathbf{x}_0$ as a linear combination of eigenvectors:

$$\mathbf{x}_0 = s\mathbf{u} + t\mathbf{v} \text{ for some scalars } s \text{ and } t.$$

(d) Use 1(c) and (a) to find an explicit formula for the vector $\mathbf{x}_n = A^n(s\mathbf{u} + t\mathbf{v})$.

(e) Use your formula from part (d) to compute the limit of the sequence:

$$\lim_{n\to\infty} \mathbf{x}_n.$$

Here is a word problem for you. There are some bears in a valley. Every winter half the bears go up the mountain and half stay in the valley.[4] At the same time, every bear that was on the mountain last year comes back down to the valley. If we begin with 30 bears in the valley, how many bears will be in the valley after n years?

(a): The transition/recurrence matrix is $A = \begin{pmatrix} 0.5 & 1 \\ 0.5 & 0 \end{pmatrix}$ because

$$\mathbf{x}_{n+1} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n/2 + y_n \\ x_n/2 \end{pmatrix} = \begin{pmatrix} 0.5 & 1 \\ 0.5 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = A\mathbf{x}_n.$$

(b): We verify that $\mathbf{u} = (2, 1)$ and $\mathbf{v} = (-1, 1)$ are eigenvectors with eigenvalues 1 and -0.5:

$$\begin{pmatrix} 0.5 & 1 \\ 0.5 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0.5 & 1 \\ 0.5 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix} = (-0.5) \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(c): To express our initial condition in terms of the eigenvectors we solve the following system:

$$\left(\begin{array}{cc|c} 2 & -1 & 30 \\ 1 & 1 & 0 \end{array} \right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -3 & 30 \end{array} \right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & -10 \end{array} \right).$$

We conclude that

$$\begin{pmatrix} 30 \\ 0 \end{pmatrix} = 10 \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 10 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(d): The distribution of bears after n years is given by

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= A^n \begin{pmatrix} 30 \\ 0 \end{pmatrix} \\ &= A^n \left[10 \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 10 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right] \\ &= 10 \cdot A^n \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 10 \cdot A^n \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= 10 \cdot 1^n \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 10 \cdot (-0.5)^n \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 20 + 10(-0.5)^n \\ 10 - 10(-0.5)^n \end{pmatrix}. \end{aligned}$$

(e): We conclude that

$$\lim_{n \to \infty} \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \end{pmatrix}.$$

After many years, we expect to have 20 bears in the valley and 10 bears on the mountain.

[4] Technically: Let's say that each bear has a 50% chance of staying and a 50% chance of moving.