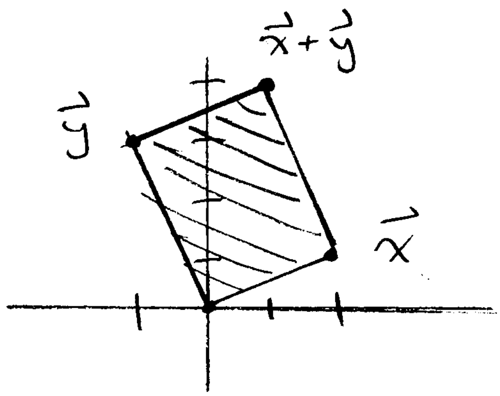
Math 210 F
Final Exam A

Drew Armstrong
Wed, Apr 27, 2016

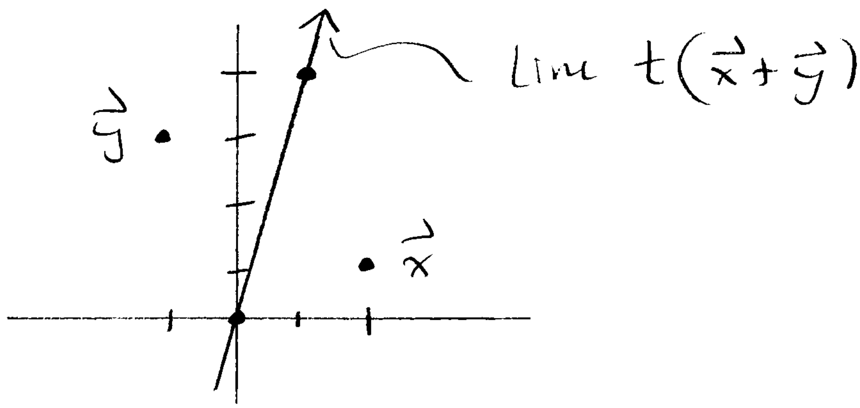
This is a closed book test. No electronic devices are allowed. If two students submit exams in which any solution has been copied, **both students will receive a score of zero.** There are 6 problems and 7 pages. Each page is worth 6 points, for a total of 42 points.

Problem 1. Consider the points $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ in the Cartesian plane.

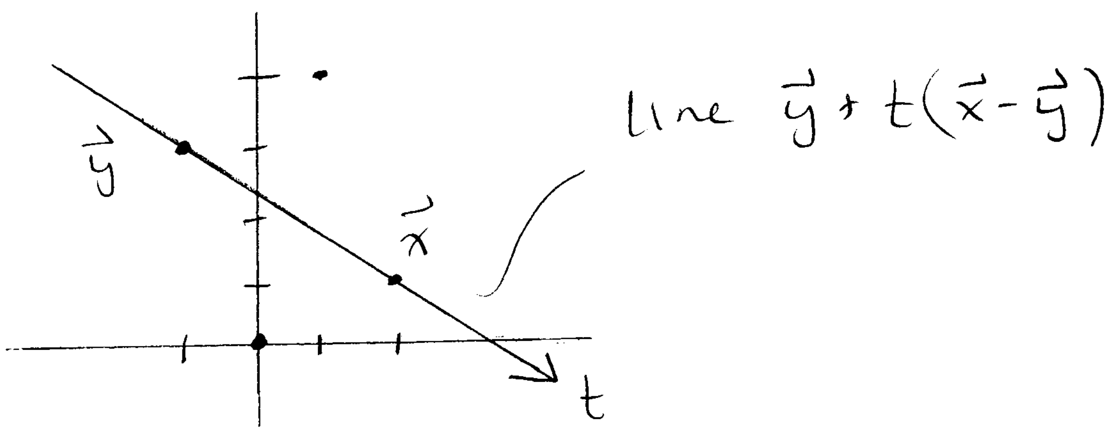
(a) Draw the collection of points $s\vec{x} + t\vec{y}$ where $0 \leq s \leq 1$ and $0 \leq t \leq 1$.



(b) Draw the collection of points $t\vec{x} + t\vec{y}$ for all t.



(c) Draw the collection of points $t\vec{x} + (1-t)\vec{y}$ for all t.



Problem 2. Consider the same vectors $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ from Problem 1.

(a) Compute the lengths of $\vec{x}$ and $\vec{y}$.

$$\|\vec{x}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$\|\vec{y}\| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

(b) Compute the cosine of the angle between $\vec{x}$ and $\vec{y}$.

$$\cos\theta = \frac{\vec{x} \circ \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|} = \frac{2(-1) + 1(3)}{\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{50}}$$

(c) Compute the otrhogonal projection of the point $\vec{x}$ onto the line $t\vec{y}$.

projection matrix $P = \frac{\vec{y}\vec{y}^T}{\vec{y}^T\vec{y}} = \frac{1}{10}\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$

projection of $\vec{x}$:

$$P\vec{x} = \frac{1}{10}\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 2-3 \\ -6+9 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Problem 3. Consider the vectors $\vec{u} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ in $\mathbb{R}^3$.

(a) Find a vector in $\mathbb{R}^3$ that is perpendicular to **both** of the vectors $\vec{u}$ and $\vec{v}$.

$$\begin{cases} \vec{x} \cdot \vec{u} = 0 \\ \vec{x} \cdot \vec{v} = 0 \end{cases} \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ -1 & 2 & 1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 5 & 3 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & 3/5 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1/5 & 0 \\ 0 & 1 & 3/5 & 0 \end{array}\right) \rightsquigarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = t \begin{pmatrix} -1/5 \\ -3/5 \\ 1 \end{pmatrix}$$

$$z = t$$

$$= -\frac{t}{5} \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}$$

(b) Use your answer from part (a) to find the equation of the plane $s\vec{u} + t\vec{v}$.

The plane $\perp$ to $(1, 3, -5)$ is

$$1x + 3y - 5z = 0.$$

(c) Compute the 3×3 matrix that projects orthogonally onto the plane from part (b).

$$P = \text{proj onto } \underline{\text{line}}\ t\begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix} = \frac{1}{35} \begin{pmatrix} 1 & 3 & -5 \\ 3 & 9 & -15 \\ -5 & -15 & 25 \end{pmatrix}$$

$$Q = \text{proj onto } \underline{\text{plane}}$$

$$= I - P$$

$$= \frac{1}{35} \begin{pmatrix} 35 & & \\ & 35 & \\ & & 35 \end{pmatrix} - \frac{1}{35} \begin{pmatrix} 1 & 3 & -5 \\ 3 & 9 & -15 \\ -5 & -15 & 25 \end{pmatrix}$$

$$= \frac{1}{35} \begin{pmatrix} 34 & -3 & 5 \\ -3 & 26 & 15 \\ 5 & 15 & 10 \end{pmatrix}$$

Problem 4. Consider the following system of 3 linear equations in 4 unknowns:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ 2x_1 + 2x_2 + 3x_3 + 4x_4 = 2 \\ 3x_1 + 3x_2 + 4x_3 + 5x_4 = 3 \end{cases}$$

(a) Put the system in Reduced Row Echelon Form (RREF).

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 & 2 \\ 3 & 3 & 4 & 5 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \begin{cases} x_1 + x_2 \qquad - x_4 = 1 \\ \qquad\quad x_3 + 2x_4 = 0 \end{cases}$$

(b) Tell me the pivot and non-pivot variables.

Pivot: x_1 & x_3 Free: $\underset{s}{x_2}$ & $\underset{t}{x_4}$

(c) Write down the complete solution of the system.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1-s+t \\ s \\ -2t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

Problem 5. Consider the matrix $T = \begin{pmatrix} 0.4 & 0.8 \\ 0.6 & 0.2 \end{pmatrix}$.

(a) Write down the characteristic equation of the matrix T. I'll just tell you that its two roots are 1 and -0.4 (you don't have to check this).

$$\det(T - \lambda I) = \det \begin{pmatrix} 0.4-\lambda & 0.8 \\ 0.6 & 0.2-\lambda \end{pmatrix} = 0$$

$$(0.4-\lambda)(0.2-\lambda) - (0.6)(0.8) = 0$$

(b) Find an eigenvector of T corresponding to eigenvalue $\lambda = 1$.

$$\left(\begin{array}{cc|c} 0.4-1 & 0.8 & 0 \\ 0.6 & 0.2-1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} -0.6 & 0.8 & 0 \\ 0.6 & -0.8 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 3 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow 3x - 4y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

(c) Find an eigenvector of T corresponding to eigenvalue $\lambda = -0.4$.

$$\left(\begin{array}{cc|c} 0.4+0.4 & 0.8 & 0 \\ 0.6 & 0.2+0.4 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 0.8 & 0.8 & 0 \\ 0.6 & 0.6 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \rightsquigarrow x + y = 0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

(d) Express $\begin{pmatrix}3\\4\end{pmatrix}$ as a linear combination of the eigenvectors from parts (b) and (c).

$$\begin{pmatrix}3\\4\end{pmatrix} = s\begin{pmatrix}4\\3\end{pmatrix} + t\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}4 & 1\\3 & -1\end{pmatrix}\begin{pmatrix}s\\t\end{pmatrix}$$

$$\begin{pmatrix}s\\t\end{pmatrix} = \begin{pmatrix}4 & 1\\3 & -1\end{pmatrix}^{-1}\begin{pmatrix}3\\4\end{pmatrix} = \frac{1}{-7}\begin{pmatrix}-1 & -1\\-3 & 4\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix}$$

$$= \frac{1}{-7}\begin{pmatrix}-7\\7\end{pmatrix} = \begin{pmatrix}1\\-1\end{pmatrix}$$

(e) Finally, consider the linear recurrence relation $\vec{v}_n = T\vec{v}_{n-1}$ with initial condition $\vec{v}_0 = (3, 4)$. Use all of your previous work to find a "closed form" solution for the n-th vector $\vec{v}_n$.

$$\vec{V}_n = T^n\begin{pmatrix}3\\4\end{pmatrix} = T^n\left(\begin{pmatrix}4\\3\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix}\right)$$

$$= T^n\begin{pmatrix}4\\3\end{pmatrix} - T^n\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= 1^n\begin{pmatrix}4\\3\end{pmatrix} - (-0.4)^n\begin{pmatrix}1\\-1\end{pmatrix}$$

Problem 6. Let A be a square matrix.

(a) If A is invertible, explain why $\lambda = 0$ **cannot** be an eigenvalue of A. [Hint: Suppose that we have $A\vec{x} = 0\vec{x} = \vec{0}$ for some vector $\vec{x} \neq \vec{0}$. Then ...]

Suppose A^{-1} exists. Then

$A\vec{x} = \vec{0}$ implies that

$$\vec{x} = I\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{0} = \vec{0}.$$

(b) If A is invertible and λ is an eigenvalue of A, explain why λ^{-1} is an eigenvalue of the inverse matrix A^{-1}. [Hint: Suppose that we have $A\vec{x} = \lambda\vec{x}$ for some vector $\vec{x} \neq \vec{0}$. Then ...]

If $A\vec{x} = \lambda\vec{x}$ and $\vec{x} \neq 0$ then

$$\cancel{A^{-1}A}\vec{x} = A^{-1}(\lambda\vec{x})$$

$$\vec{x} = \lambda(A^{-1}\vec{x})$$

$$\lambda^{-1}\vec{x} = A^{-1}\vec{x} \implies \lambda^{-1} \text{ is e.value of } A^{-1}.$$

(c) Suppose that A is a 2×2 matrix with eigenvalues $\lambda = 3$ and $\lambda = 4$. In this case, tell me the eigenvalues of the matrix A^n.

$$A\vec{x} = 3\vec{x}, \quad A\vec{y} = 4\vec{y}$$

$$A^n\vec{x} = 3^n\vec{x}, \quad A^n\vec{y} = 4^n\vec{y}$$

$\implies 3^n$ & 4^n are e.values of A^n

Math 210 E **Drew Armstrong**
Final Exam **Monday, May 6, 2013**

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Problem 1. Consider the plane $x + 2y - z = 0$ in $\mathbb{R}^3$.

(a) Tell me a normal vector to the plane.

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

(b) Tell me a normal vector of length 1.

$$\frac{1}{\|\vec{a}\|}\vec{a} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

(c) Compute the matrix Q that projects orthogonally onto the normal line.

$$Q = \vec{a}(\vec{a}^T\vec{a})^{-1}\vec{a}^T = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\left((1\ 2\ {-1})\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right)^{-1}(1\ 2\ {-1})$$

$$= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}(6)^{-1}(1\ 2\ {-1}) = \frac{1}{6}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}(1\ 2\ {-1})$$

$$= \frac{1}{6}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

(d) Compute the matrix P that projects orthogonally onto the plane. [Hint: Use your answer from part (c) to save time.]

$$P = I - Q$$

$$= \frac{1}{6}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{pmatrix}$$

Problem 2. The following matrix **rotates** vectors in $\mathbb{R}^2$ **counterclockwise** by 53.13°:

$$R = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.$$

(You can just believe this. You don't have to show it.)

(a) Rotate the column vector $(1, 1)$ **counterclockwise** by 53.13°.

$$R\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -1 \\ 7 \end{pmatrix}$$

(b) Compute the matrix that rotates vectors **clockwise** by 53.13°.

$\det(R) = 9/25 + 16/25 = 25/25 = 1$, so

$$R^{-1} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$

(c) Rotate the column vector $(1, 1)$ **clockwise** by 53.13°.

$$R^{-1}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 7 \\ -1 \end{pmatrix}$$

(d) Compute the matrix that rotates **counterclockwise** by 106.26°$(= 2 \times 53.13°)$.

$$R^2 = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$

$$= \frac{1}{25}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}$$

$$= \frac{1}{25}\begin{pmatrix} 9-16 & -12-12 \\ 12+12 & -16+9 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} -7 & -24 \\ 24 & -7 \end{pmatrix}$$

Problem 3. Consider the following three planes in $\mathbb{R}^3$:

$$x + y + z = 0, \tag{1}$$
$$x + 2y - z = 0, \tag{2}$$
$$x + 0y + z = 1. \tag{3}$$

(a) Compute the intersection of the **first and second planes**. [Hint: It's a line.]

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$$

$$\longrightarrow \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \\ \begin{array}{ccc} x & y & z \end{array} \end{array} \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3s \\ 2s \\ s \end{pmatrix} = s\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}.$$

Let $z = s$

(b) Compute the intersection of **second and third planes**. [Hint: It's a line.]

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & -2 & -1 \end{array}\right)$$

$$\longrightarrow \begin{array}{c} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1/2 \end{array}\right) \\ \begin{array}{ccc} x & y & z \end{array} \end{array} \longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-t \\ -1/2+t \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Let $z = t$

(c) Compute the intersection of the lines from parts (a) and (b). [Hint: It's a point.]

Multiple ways to do this. I'll intersect the line from (a) with the plane (3).

$$x + z = 1$$
$$(-3s) + (s) = 1$$
$$-2s = 1$$
$$s = -1/2$$

$$\longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1 \\ -1/2 \end{pmatrix}.$$

Problem 4. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

(a) Use the Gauss-Jordan method to compute the inverse of A.

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \begin{array}{l} R_1 \\ R_2 \\ R_3 - R_2 \end{array} \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \begin{array}{l} R_1 - R_3 \\ R_2 \\ R_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \begin{array}{l} R_1 - R_2 \\ R_2 \\ R_3 \end{array} = (I \mid A^{-1})$$

$$\Longrightarrow A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

(b) Solve the system $A\vec{x} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. [Hint: Use your answer from (a) to save time.]

$$\vec{x} = A^{-1} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Problem 5. Consider the following system:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & b \\ 0 & 1 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 0 \\ c \end{pmatrix}.$$

Tell me some values for b and c such that

(a) the system has a **unique solution** $\vec{x}$.

$b = 0$ and $c =$ anything.

Then the solution is

$$\vec{x} = A^{-1}\begin{pmatrix} 1 \\ 0 \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ c \end{pmatrix}$$ from problem 4.

(b) the system has **no solution** $\vec{x}$.

$b = 1$ and $c \neq 0$. Then

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & c \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & c \end{array}\right)$$

impossible.

(c) the system has **infinitely many solutions** $\vec{x}$.

$b = 1$ and $c = 0$. Then

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Now there are infinitely many solutions. (I won't bother finding them.)

A has shape $n \times m$

Problem 6. Consider the matrix $A = \begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \end{pmatrix}$, where $\vec{a}_i$ is the ith **column**. Suppose that A has n rows, so the columns are vectors in $\mathbb{R}^n$.

(a) Let $\vec{x}$ be a vector in $\mathbb{R}^n$. Write a single matrix equation to say that $\vec{x}$ is perpendicular to all the columns of A.

$$A^T \vec{x} = \vec{0}$$

(b) You can think of your matrix equation from part (a) as a system of **how many** linear equations, in **how many** unknowns?

A^T has shape $m \times n$, so $A^T\vec{x} = \vec{0}$

has m equations in n unknowns

(c) The solution to your equation in part (a) most likely has **how many dimensions**?

variables − # equations

$$n - m$$

(d) Now let $\vec{b}$ be any point in $\mathbb{R}^n$ and let $\vec{p}$ be the point in the column space of A that is **closest** to $\vec{b}$. Write a formula for $\vec{p}$ in terms of A and $\vec{b}$.

$$\vec{p} = A(A^T A)^{-1} A^T \vec{b}$$

Problem 7. Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$ with $\vec{u}^T\vec{v} \neq 0$, and consider the $n \times n$ matrix

$$A = \frac{\vec{u}\,\vec{v}^T}{\vec{u}^T\vec{v}}. = \underbrace{\left(\frac{1}{\vec{u}^T\vec{v}}\right)}_{1\times 1} \underbrace{\vec{u}\vec{v}^T}_{n\times n}$$

(a) For all vectors $\vec{x}$, show that $A\vec{x}$ is on the line generated by $\vec{u}$.

$$A\vec{x} = \frac{1}{\vec{u}^T\vec{v}}(\vec{u}\vec{v}^T)\vec{x} = \underbrace{\left(\frac{1}{\vec{u}^T\vec{v}}\right)}_{\text{number}}\vec{u}\underbrace{(\vec{v}^T\vec{x})}_{\text{number}} = \underbrace{\left(\frac{\vec{v}^T\vec{x}}{\vec{u}^T\vec{v}}\right)}_{\text{number}}\vec{u}$$

(b) The line generated by $\vec{u}$ is an eigenspace for A. What is the eigenvalue?

From part (a) we have

$$A\vec{u} = \underbrace{\left(\frac{\cancel{\vec{v}^T\vec{u}}}{\cancel{\vec{u}^T\vec{v}}}\right)}_{1}\vec{u} = 1\vec{u}$$

The eigenvalue is 1.

(c) Now let $\vec{x}$ be any vector **perpendicular** to $\vec{v}$. Show that $A\vec{x} = \vec{0}$.

Suppose $\vec{v}^T\vec{x} = 0$. Then from (a) we have

$$A\vec{x} = \left(\frac{\overset{0}{\cancel{\vec{v}^T\vec{x}}}}{\vec{u}^T\vec{v}}\right)\vec{u} = 0\vec{u} = \vec{0}.$$

(d) Is the matrix A invertible? If so, tell me its inverse.

NO, because there exists $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \vec{0}$. [If A were invertible we would get $\vec{x} = A^{-1}\vec{0} = \vec{0}$, contradiction.]