Math 210
Exam 2

Fall 2019
Drew Armstrong

There are 6 pages, each worth 6 points, for a total of 36 points. This is a closed book test. No electronic devices are allowed.

Problem 1.

(a) Find some 2×2 matrices A and B such that $AB \neq BA$.

Pretty much any two matrices will work.

(b) Find a 2×2 matrix A such that $A \neq 0$ and A^{-1} does not exist.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Any projection will work

(c) Find a 2×2 matrix A such that $A \neq I$ and $A^2 = I$.

$$\begin{pmatrix} -1 & 0 \\ 0 & \pm 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ or } \ldots$$

Any reflection will work.

Problem 2.

(a) Find a matrix A such that $A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$A = \begin{pmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{pmatrix}$$

(b) Find a matrix B such that $B\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ and $B\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

$$B = \begin{pmatrix}1 & s & 0\\ 0 & t & 1\end{pmatrix} \quad \text{for any } s, t.$$

(c) Using your matrices from parts (a) and (b), compute the matrix product BA.

$$BA = \begin{pmatrix}1 & s & 0\\ 0 & t & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Problem 3. Consider the matrix $A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$.

(a) Compute the characteristic polynomial $\det(A - \lambda I)$.

$$\det\begin{pmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{pmatrix} = (4-\lambda)(-3-\lambda) - 2(-5)$$
$$= \lambda^2 + \lambda 3 - \lambda 4 - 12 + 10$$
$$= \lambda^2 - \lambda - 2$$

(b) Compute the eigenvalues of A.

$$\lambda^2 - \lambda - 2 = 0$$
$$\lambda = \frac{1 \pm \sqrt{1+8}}{2} = -1 \text{ or } 2$$

(c) Find an eigenvector for each eigenvalue.

$$\lambda = -1: \quad \left(\begin{array}{cc|c} 4+1 & -5 & 0 \\ 2 & -3+1 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\lambda = 2: \quad \left(\begin{array}{cc|c} 4-2 & -5 & 0 \\ 2 & -3-2 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 2 & -5 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = t\begin{pmatrix} 5 \\ 2 \end{pmatrix}.$$

Problem 4. Consider the data points $\begin{pmatrix} t \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

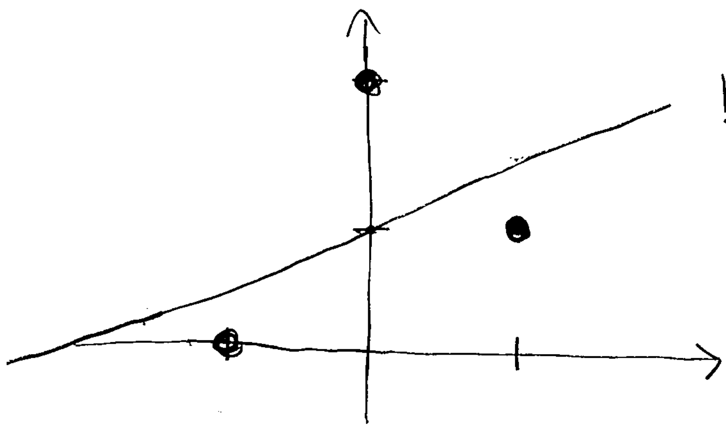
(a) Find the equation $b = C + tD$ of the (ordinary least squares) best fit line.

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Longrightarrow \begin{matrix} C = 1 \\ D = 1/2 \end{matrix}$$

(b) Draw the data points together with the best fit line.



$b = 1 + \frac{1}{2} t$

(c) Let $\hat{\mathbf{b}} = \begin{pmatrix} C - 1D \\ C + 0D \\ C + 1D \end{pmatrix}$. Fill in the blanks:

The point $\hat{\mathbf{b}}$ is the (orthogonal) projection of the point $(0, 2, 1)$

onto the plane $s(1,1,1) + t(-1,0,1)$.

Problem 5.

(a) Find the matrix P that projects (orthogonally) onto the line $t\begin{pmatrix}1\\1\\2\end{pmatrix}$.

$$P = \begin{pmatrix}1\\1\\2\end{pmatrix}\left[\begin{pmatrix}1 & 1 & 2\end{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix}\right]^{-1}\begin{pmatrix}1 & 1 & 2\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix}1\\1\\2\end{pmatrix}\begin{pmatrix}1 & 1 & 2\end{pmatrix} = \frac{1}{6}\begin{pmatrix}1 & 1 & 2\\1 & 1 & 2\\2 & 2 & 4\end{pmatrix}$$

(b) Find the matrix Q that projects (orthogonally) onto the plane $x + y + 2z = 0$.

$$Q = I - P = \frac{1}{6}\begin{pmatrix}6 & 0 & 0\\0 & 6 & 0\\0 & 0 & 6\end{pmatrix} - \frac{1}{6}\begin{pmatrix}1 & 1 & 2\\1 & 1 & 2\\2 & 2 & 4\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix}5 & -1 & -2\\-1 & 5 & -2\\-2 & -2 & 2\end{pmatrix}$$

(c) Compute the matrix product PQ. [Hint: Think of what it does.]

$$PQ = \begin{pmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{pmatrix}$$

Problem 6. Let $M = \mathbf{a}\mathbf{b}^T$ where $\mathbf{a}$ and $\mathbf{b}$ are some 2×1 vectors satisfying $\mathbf{b}^T\mathbf{a} = 1$.

(a) Show that $M^2 = M$.

$$M^2 = (\vec{a}\vec{b}^T)(\vec{a}\vec{b}^T)$$
$$= \vec{a}\underbrace{(\cancel{\vec{b}^T\vec{a}})}_{1}\vec{b}^T = \vec{a}\vec{b}^T = M$$

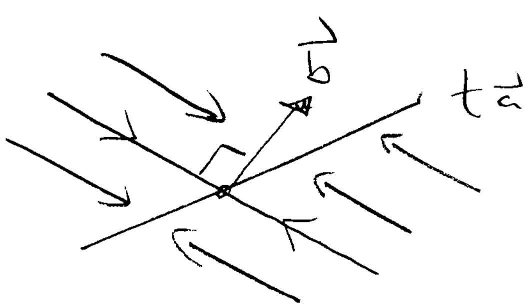
(b) Show that $\mathbf{a}$ is an eigenvector of M.

$$M\vec{a} = (\vec{a}\vec{b}^T)\vec{a}$$
$$= \vec{a}\underbrace{(\cancel{\vec{b}^T\vec{a}})}_{1} = \vec{a} = 1\vec{a}$$

(c) Let $\mathbf{x}$ be any nonzero vector perpendicular to $\mathbf{b}$. Show that $\mathbf{x}$ is an eigenvector of M.

$$M\vec{x} = (\vec{a}\vec{b}^T)\vec{x}$$
$$= \vec{a}\underbrace{(\cancel{\vec{b}^T\vec{x}})}_{0} = \vec{0} = 0\vec{x}$$

(d) **Bonus (1 point).** Give a geometric description of the function M.



Project onto line $t\vec{a}$ in direction $\perp$ to $\vec{b}$.