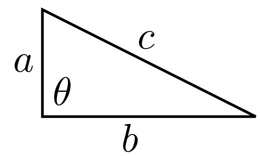
Sums of Squares (FNS 195-S) **Fall 2014**
Record of What We Did **Drew Armstrong**

The Pythagorean Theorem

• What is the Pythagorean Theorem? (and who was Pythagoras?)

Consider a triangle with side lengths a, b, c, with angle θ across from the side of length c.



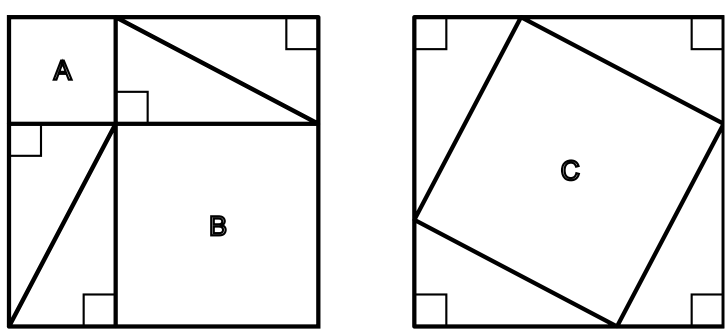
The Pythagorean Theorem says that **if** θ is a right angle (i.e., $90°$) **then** the side lengths satisfy the equation $a^2 + b^2 = c^2$.

Does it also work the other way? That is, **if** the equation $a^2 + b^2 = c^2$ is true, does it follow that θ is a right angle? Yes, this is also true, and it is important to recognize that it is a logically independent statement. In general, the statements "if P then Q" and "if Q then P" are not the same. Given an example of this.

• **Why** is the Pythagorean Theorem true? (and **is** it true?)

A **proof** is an attempt to persuade someone of the truth of a mathematical statement. We say that proof is valid if your intended audience is convinced by your argument. (Your proof is really valid if you succeed in convincing a professional mathematician.)

To say that the Pythagorean Theorem is **true** means that there exists a proof. The following pictures suggest a famous proof.



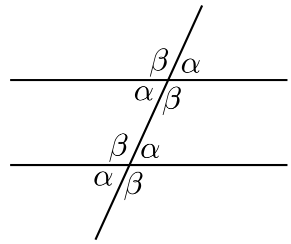
Please observe that this proof only works if the angles in our triangle sum to $180°$ (i.e., a straight line). Otherwise the figure on the right is not a square.

Is that a problem? Maybe.

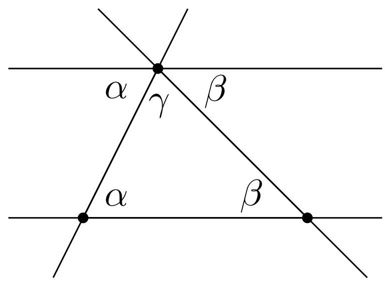
• Why do the angles in a triangle sum to $180°$? (and **do** they?)

Well, we need a **proof**. I can give a proof if you allow me to assume two things. **Fact 1:** Given a line ℓ and a point p not on ℓ, there exists a line through p that is **parallel** to ℓ. **Fact**

2: If a third line crosses two parallel lines, then the corresponding angles are equal, as in the following figure.



Now here's a proof. Start with a triangle. Say that it has angles α, β and γ.

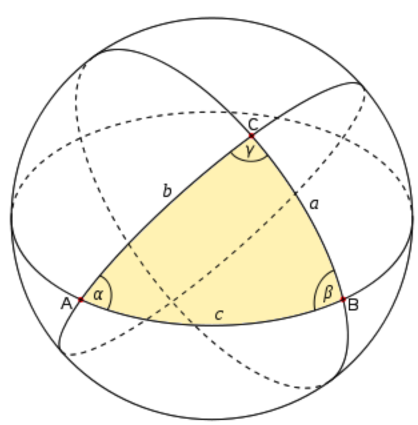


By **Fact 1**, there exists a line through the point at angle γ and parallel to the opposite side of the triangle. Then by **Fact 2** we know that the angles next to γ on the same side of this line are equal to α and β. We conclude that the angles sum to a straight line, i.e., $180°$.

So, **if** the two facts are true then the angles in any triangle sum to $180°$. But these two facts are not always true. For example, they're not true on the surface of a sphere. In fact, the sum of the angles in a spherical triangle is always **strictly greater** than $180°$.

• Find a triangle whose angles do **not** sum to $180°$.

As I mentioned, you can take any triangle on the surface of a sphere. What does this mean? A "straight line" should be the shortest path between two points. On the surface of a sphere the shortest path between two points is a segment of a great circle (equator). Thus a triangle on a sphere is bounded by three great circles.



We will see that the sum of the angles in a spherical triangle is always **strictly greater** than 180°. However, if the triangle is very small (or if the sphere is very large) then the sum of the angles will be **approximately equal to** 180°, as expected.

- What is the formula for the area of a triangle on the surface of a sphere?

Let $\Delta_{\alpha\beta\gamma}$ denote the area of the spherical triangle with angles α, β, γ, as shown above. Note that the great circles that bound this triangle divide the surface of the sphere into 8 triangles that come in pairs of equal area. Let Δ_α, Δ_β and Δ_γ denote the areas of the triangles that lie across the sides a, b and c (respectively) from our triangle. Let S denote the surface area of the whole sphere.

First note that we have

$$2(\Delta_{\alpha\beta\gamma} + \Delta_\alpha + \Delta_\beta + \Delta_\gamma) = S$$

because two copies of each triangle cover the whole sphere.

On the other hand, note the triangles $\Delta_{\alpha\beta\gamma}$ and Δ_α together make a "lune" of angle α. Since this lune covers $\alpha/2\pi$ of the whole sphere, it has area

$$\Delta_{\alpha\beta\gamma} + \Delta_\alpha = \frac{\alpha}{2\pi}S.$$

Similarly, we have

$$\Delta_{\alpha\beta\gamma} + \Delta_\beta = \frac{\beta}{2\pi}S \qquad \text{and} \qquad \Delta_{\alpha\beta\gamma} + \Delta_\gamma = \frac{\gamma}{2\pi}S.$$

Adding these three equations together gives

$$3\Delta_{\alpha\beta\gamma} + \Delta_\alpha + \Delta_\beta + \Delta_\gamma = \frac{(\alpha+\beta+\gamma)}{2\pi}S$$

and hence

$$6\Delta_{\alpha\beta\gamma} + 2\Delta_\alpha + 2\Delta_\beta + 2\Delta_\gamma = \frac{(\alpha+\beta+\gamma)}{\pi}S.$$

Finally, we compare this to the first equation to get

$$\begin{aligned} 4\Delta_{\alpha\beta\gamma} + S &= 4\Delta_{\alpha\beta\gamma} + 2(\Delta_{\alpha\beta\gamma} + \Delta_\alpha + \Delta_\beta + \Delta_\gamma) \\ &= 6\Delta_{\alpha\beta\gamma} + 2\Delta_\alpha + 2\Delta_\beta + 2\Delta_\gamma \\ &= \frac{(\alpha+\beta+\gamma)}{\pi}S. \end{aligned}$$

We conclude that

$$4\Delta_{\alpha\beta\gamma} = \frac{(\alpha+\beta+\gamma)}{\pi}S - S = \frac{(\alpha+\beta+\gamma-\pi)}{\pi}S,$$

and hence

$$\Delta_{\alpha\beta\gamma} = \frac{S}{4\pi}(\alpha+\beta+\gamma-\pi).$$

Discussion:

This result is due to Thomas Harriot (1603). It is remarkable because it shows that the area of a spherical triangle depends only on its angles. That is, if two triangles on a sphere are **similar**, then they are actually **congruent**. Furthermore, the area of a spherical triangle with angles α, β, γ is equal to a constant times its "spherical excess" $\alpha + \beta + \gamma - \pi$. Thus if the area $\Delta_{\alpha\beta\gamma}$ is very small then the "spherical excess" $\alpha + \beta + \gamma - \pi$ is also very small, and we have $\alpha + \beta + \gamma \approx \pi$. That is, a very small triangle on a sphere behaves approximately like a triangle on a flat plane. Equivalently, the surface of a sphere looks flat if you zoom in.