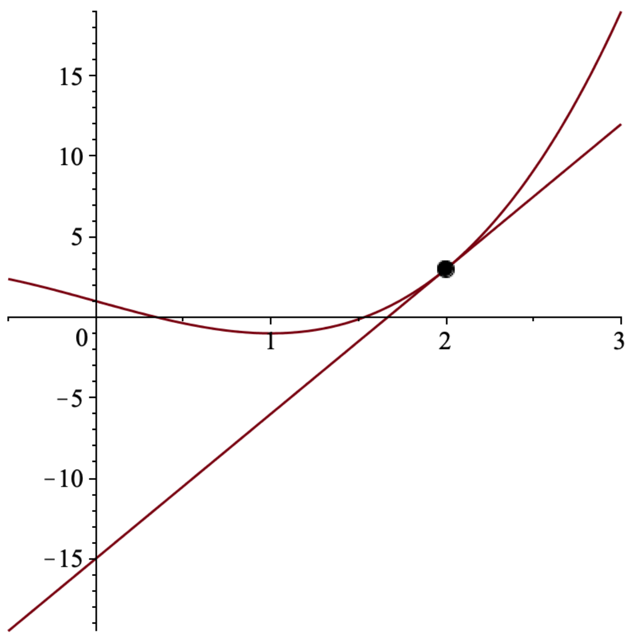
2.1.4. Find the equation of the tangent line to the curve $y = x^3 - 3x + 1$ at the point $(2, 3)$. The slope of the tangent is the derivative. We use the power rule:

$$\frac{dy}{dx} = \frac{d}{dx}(x^3) - 3\frac{d}{dx}(x) + \frac{d}{dx}1 = 3x^2 - 3 + 0 = 3(x^2 - 1).$$

The slope of the tangent at the point $(x, y) = (2, 3)$ is $3(x^2 - 1) = 3(2^2 - 1) = 3(3) = 9$. The line with slope 9 that passes through the point $(2, 3)$ has equation

$$\begin{aligned} \text{slope} &= \frac{\text{rise}}{\text{run}} \\ 9 &= \frac{y-3}{x-2} \\ 9(x-2) &= y - 3 \\ 9x - 18 &= y - 3 \\ y &= 9x - 15. \end{aligned}$$

Here is a picture (not to scale):



2.2.20. Compute the derivative of $f(x) = 1.5x^2 - x + 3.7$ directly from the definition.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(1.5(x+h)^2 - (x+h) + 3.7) - (1.5x^2 - x + 3.7)}{h} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{(1.5(\cancel{x^2} + 2xh + h^2) - (\cancel{x} + h) + \cancel{3.7}) - (1.5\cancel{x^2} - \cancel{x} + \cancel{3.7})}{h} \\
&= \lim_{h\to 0} \frac{3xh + 1.5h^2 - h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(3x + 1.5h - 1)}{\cancel{h}} \\
&= 3x + 0 - 1 \\
&= 3x - 1.
\end{aligned}
$$

2.2.26. Compute the derivative of $f(x) = x^{3/2}$ directly from the definition.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h} \cdot \frac{(x+h)^{3/2} + x^{3/2}}{(x+h)^{3/2} + x^{3/2}} \\
&= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h\left[(x+h)^{3/2} + x^{3/2}\right]} \\
&= \lim_{h\to 0} \frac{(\cancel{x^3} + 3x^2h + 3xh^2 + h^3) - \cancel{x^3}}{h\left[(x+h)^{3/2} + x^{3/2}\right]} \\
&= \lim_{h\to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}\left[(x+h)^{3/2} + x^{3/2}\right]} \\
&= \lim_{h\to 0} \frac{3x^2 + 3xh + h^2}{(x+h)^{3/2} + x^{3/2}} \\
&= \lim_{h\to 0} \frac{3x^2 + 0 + 0}{(x+0)^{3/2} + x^{3/2}} \\
&= \frac{3x^2}{2x^{3/2}} \\
&= \frac{3}{2}x^{1/2}
\end{aligned}
$$

Remark: Of course using the power rule is easier:

$$\frac{d}{dx}x^{3/2} = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{1/2}.$$

But this problem wanted you to practice using the limit definition of the derivative.

2.3.4. Differentiate $F(x) = (3/4)x^8$. We use the power rule:

$$F'(x) = \frac{3}{4}(x^8)' = \frac{3}{4} \cdot 8x^7 = 6x^7.$$

2.3.8. Differentiate $y = \sin t + \pi \cos t$. We use the formulas $(\sin t)' = \cos t$ and $(\cos t)' = -\sin t$:

$$\frac{dy}{dt} = \cos t + \pi(-\sin t) = \cos t - \pi \sin t.$$

Remark: I assume this exercise wants us to differentiate y as a function of t. Usually this is clear from the context.

2.3.10. Differentiate $h(x) = (x-2)(2x+3)$. We expand and use the power rule:

$$\begin{aligned} h(x) &= 2x^2 - 4x + 3x - 6 \\ h(x) &= 2x^2 - x - 6 \\ h'(x) &= 2 \cdot 2x - 1 + 0 \\ h'(x) &= 4x - 1. \end{aligned}$$

Alternatively, we can use the product rule:

$$\begin{aligned} h'(x) &= (x-2)(2x+3)' + (x-2)'(2x+3) \\ &= (x-2)(2+0) + (1+0)(2x+3) \\ &= 2x - 4 + 2x + 3 \\ &= 4x - 1. \end{aligned}$$

2.3.14. Differentiate $y = x^{5/3} - x^{2/3}$. We use the power rule:

$$\frac{dy}{dx} = \frac{5}{3}x^{\frac{5}{3}-1} - \frac{2}{3}x^{\frac{2}{3}-1} = \frac{5}{3}x^{2/3} - \frac{2}{3}x^{-1/3}.$$

This can't be simplified.

2.3.22. Differentiate $y = \frac{\sqrt{x}+x}{x^2}$. We can simplify and then use the power rule:

$$\begin{aligned} y &= \frac{\sqrt{x}}{x^2} + \frac{x}{x^2} \\ y &= \frac{x^{1/2}}{x^2} + \frac{x}{x^2} \\ y &= x^{1/2-2} + x^{1-2} \\ y &= x^{-3/2} + x^{-1} \\ \frac{dy}{dx} &= -\frac{3}{2}x^{-\frac{3}{2}-1} + (-1)x^{-1-1} \\ \frac{dy}{dx} &= -\frac{3}{2}x^{-5/2} - x^{-2} \end{aligned}$$

This can't be simplified. Alternatively, we can use the quotient rule:

$$\begin{aligned} y &= \frac{\sqrt{x}+x}{x^2} \\ \frac{dy}{dx} &= \frac{x^2(\sqrt{x}+x)' - (\sqrt{x}+x)(x^2)'}{(x^2)^2} \\ \frac{dy}{dx} &= \frac{x^2((1/2)x^{-1/2}+1) - (\sqrt{x}+x)(2x)}{(x^2)^2}. \end{aligned}$$

But here the simplification is much more unpleasant.

2.4.4. Differentiate $f(x) = \sqrt{x} \cdot \sin x$. We use the product rule:

$$f'(x) = \sqrt{x} \cdot (\sin x)' + (\sqrt{x})' \cdot \sin x$$

$$= \sqrt{x} \cdot \cos x + \frac{1}{2\sqrt{x}} \cdot \sin x.$$

This can't be simplified.

2.4.10. Differentiate $y = \sin\theta \cdot \cos\theta$. We use the product rule:

$$\begin{aligned}\frac{dy}{d\theta} &= (\sin\theta)' \cdot \cos\theta + \sin\theta \cdot (\cos\theta)' \\ &= \cos\theta \cdot \cos\theta + \sin\theta \cdot (-\sin\theta) \\ &= \cos^2\theta - \sin^2\theta.\end{aligned}$$

This simplifies using the trig identity $\cos^2\theta - \sin^2\theta = \cos(2\theta)$. Alternatively, we can use the trig identity $\sin(2\theta) = 2\sin\theta\cos\theta$ and then use the chain rule:

$$\begin{aligned}y &= \sin\theta \cdot \cos\theta \\ y &= \frac{1}{2}\sin(2\theta) \\ \frac{dy}{d\theta} &= \frac{1}{2}\cos(2\theta) \cdot (2\theta)' \\ &= \frac{1}{2}\cos(2\theta) \cdot 2 \\ &= \cos(2\theta).\end{aligned}$$

2.4.12. Differentiate $G(x) = \frac{x^2-2}{2x+1}$. We use the quotient rule:

$$\begin{aligned}G'(x) &= \frac{(2x+1)(x^2-2)' - (x^2-2)(2x+1)'}{(2x+1)^2} \\ &= \frac{(2x+1)(2x+0) - (x^2-2)(2+0)}{(2x+1)^2} \\ &= \frac{4x^2+2x-2x^2+4}{(2x+1)^2} \\ &= \frac{2x^2+2x+4}{(2x+1)^2}.\end{aligned}$$

This can't be simplified.

2.4.20. Differentiate $y = \frac{\cos x}{1-\sin x}$. We use the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1-\sin x)(\cos x)' - \cos x(1-\sin x)'}{(1-\sin x)^2} \\ &= \frac{(1-\sin x)(-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2} \\ &= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2} \\ &= \frac{-\sin x + 1}{(1-\sin x)^2} \\ &= \frac{\cancel{1-\sin x}}{\cancel{(1-\sin x)}(1-\sin x)}\end{aligned}$$

$$= \frac{1}{1-\sin x}.$$

2.5.8. Find the derivative of $F(x) = (4x - x^2)^{100}$. Use the power rule and the chain rule:

$$F'(x) = 100(4x - x^2)^{99}(4x - x^2)' = 100(4x - x^2)^{99}(4 - 2x).$$

This could be simplified a bit but we won't bother.

2.5.12. Find the derivative of $f(t) = \sqrt[3]{1 + \tan t}$. We use the power rule and the chain rule, and the fact that $(\tan t)' = 1/\cos^2 t$:

$$\begin{aligned} f'(t) &= (1 + \tan t)^{1/3} \\ &= \frac{1}{3}(1 + \tan t)^{-2/3} \cdot (1 + \tan t)' \\ &= \frac{1}{3}(1 + \tan t)^{-2/3} \cdot \left(0 + \frac{1}{\cos^2 t}\right). \end{aligned}$$

Remark: When I put this into my computer it said that

$$f'(t) = \frac{1 + \tan^2 t}{3(1 + \tan t)^{2/3}}.$$

Is this the same? Yes, because of the following identity:

$$1 + \tan^2 t = 1 + \frac{\sin^2 t}{\cos^2 t} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}.$$

Any expression involving trig functions can be written in a million ways.

2.5.18. Find the derivative of $f(x) = (x^2 + 1)^3(x^2 + 2)^6$. We could expand this and then use the power rule, but it's much quicker to apply the product rule:

$$\begin{aligned} f'(x) &= (x^2 + 1)^3 \left[(x^2 + 2)^6\right]' + \left[(x^2 + 1)^3\right]' (x^2 + 2)^6 \\ &= (x^2 + 1)^3 \cdot 6(x^2 + 2)^5(x^2 + 2)' + 3(x^2 + 1)^2(x^2 + 1)' \cdot (x^2 + 2)^6 \\ &= (x^2 + 1)^3 \cdot 6(x^2 + 2)^5(2x + 0) + 3(x^2 + 1)^2(2x + 0) \cdot (x^2 + 2)^6 \\ &= 12x(x^2 + 1)^3(x^2 + 2)^5 + 6x(x^2 + 1)^2(x^2 + 2)^6. \end{aligned}$$

This can be simplified a bit by taking out common factors:

$$\begin{aligned} f'(x) &= 12x(x^2 + 1)^3(x^2 + 2)^5 + 6x(x^2 + 1)^2(x^2 + 2)^6 \\ &= 6x(x^2 + 1)^2(x^2 + 2)^5 \left(2(x^2 + 1) + (x^2 + 2)\right) \\ &= 6x(x^2 + 1)^2(x^2 + 2)^5(3x^2 + 4). \end{aligned}$$

2.5.22. Find the derivative of $f(s) = \sqrt{\frac{s^2+1}{s^2+4}} = \left(\frac{s^2+1}{s^2+4}\right)^{1/2}$.

$$\begin{aligned} f'(s) &= \frac{1}{2}\left(\frac{s^2 + 1}{s^2 + 4}\right)^{-1/2} \cdot \left(\frac{s^2 + 1}{s^2 + 4}\right)' \\ &= \frac{1}{2}\left(\frac{s^2 + 1}{s^2 + 4}\right)^{-1/2} \cdot \frac{(s^2 + 4)(s^2 + 1)' - (s^2 + 1)(s^2 + 4)'}{(s^2 + 4)^2} \\ &= \frac{1}{2}\left(\frac{s^2 + 1}{s^2 + 4}\right)^{-1/2} \cdot \frac{(s^2 + 4)(2s + 0) - (s^2 + 1)(2s + 0)}{(s^2 + 4)^2}. \end{aligned}$$

There is no point simplifying this.

2.5.32. Differentiate the function $y = x\sin(1/x)$. We use the product rule and chain rule:

$$\begin{aligned} y' &= (x)'\sin(1/x) + x\,(\sin(1/x))' \\ &= 1\sin(1/x) + x\cos(1/x)(1/x)' \\ &= \sin(1/x) + x\cos(1/x)(-1/x^2). \end{aligned}$$

I guess we should simplify this a bit:

$$y' = \sin\left(\frac{1}{x}\right) - \frac{1}{x}\cos\left(\frac{1}{x}\right).$$

2.6.8. Find dy/dx by implicit differentiation, where $\cos(xy) = 1 + \sin y$. We apply d/dx to both sides and use the chain rule:

$$\begin{aligned} \frac{d}{dx}\cos(xy) &= \frac{d}{dx}1 + \frac{d}{dx}\sin y \\ -\sin(xy)\frac{d}{dx}(xy) &= 0 + \cos y \cdot \frac{dy}{dx}. \end{aligned}$$

Then we apply the product rule to xy:

$$\begin{aligned} -\sin(xy)\frac{d}{dx}(xy) &= 0 + \cos y \cdot \frac{dy}{dx} \\ -\sin(xy)\left(1y + x\frac{dy}{dx}\right) &= 0 + \cos y \cdot \frac{dy}{dx}. \end{aligned}$$

Finally we solve for dy/dx:

$$\begin{aligned} -x\sin(xy)\cdot\frac{dy}{dx} - y\sin(xy) &= \cos y \cdot \frac{dy}{dx} \\ (-x\sin(xy) - \cos y)\frac{dy}{dx} &= y\sin(xy) \\ \frac{dy}{dx} &= \frac{y\sin(xy)}{-x\sin(xy) - \cos y}. \end{aligned}$$

2.6.12. Find dy/dx by implicit differentiation, where $\sqrt{x+y} = 1 + x^2y^2$. First we apply d/dx to both sides:

$$\begin{aligned} \frac{1}{2}(x+y)^{-1/2}\cdot\frac{d}{dx}(x+y) &= 0 + \frac{d}{dx}(x^2y^2) \\ \frac{1}{2}(x+y)^{-1/2}\cdot\left(1 + \frac{dy}{dx}\right) &= \frac{d}{dx}(x^2)\cdot y^2 + x^2\cdot\frac{d}{dx}(y^2) \\ \frac{1}{2}(x+y)^{-1/2}\cdot\left(1 + \frac{dy}{dx}\right) &= (2x)\cdot y^2 + x^2\cdot\left(2y\cdot\frac{dy}{dx}\right). \end{aligned}$$

Then we solve for dy/dx:

$$\begin{aligned} \frac{1}{2}(x+y)^{-1/2}\cdot\left(1 + \frac{dy}{dx}\right) &= (2x)\cdot y^2 + x^2\cdot\left(2y\cdot\frac{dy}{dx}\right) \\ \frac{1}{2}(x+y)^{-1/2} + \frac{1}{2}(x+y)^{-1/2}\cdot\frac{dy}{dx} &= 2xy^2 + 2x^2y\cdot\frac{dy}{dx} \\ \left(\frac{1}{2}(x+y)^{-1/2} - 2x^2y\right)\cdot\frac{dy}{dx} &= 2xy^2 - \frac{1}{2}(x+y)^{-1/2} \end{aligned}$$

$$\frac{dy}{dx} = \frac{2xy^2 - \frac{1}{2}(x+y)^{-1/2}}{\frac{1}{2}(x+y)^{-1/2} - 2x^2y}.$$

I apologize for how horrible this problem is.

2.6.16. Find dy/dx by implicit differentiation, where $\tan(x-y) = \frac{y}{1+x^2}$. First we apply d/dx to both sides:

$$\begin{aligned}
\frac{d}{dx}\tan(x-y) &= \frac{d}{dx}\frac{y}{1+x^2} \\
\frac{1}{\cos^2(x-y)} \cdot \frac{d}{dx}(x-y) &= \frac{(1+x^2)\cdot\frac{dy}{dx} - y \cdot \frac{d}{dx}(1+x^2)}{(1+x^2)^2} \\
\frac{1}{\cos^2(x-y)} \cdot \left(1 - \frac{dy}{dx}\right) &= \frac{(1+x^2)\cdot\frac{dy}{dx} - y\cdot(0+2x)}{(1+x^2)^2}.
\end{aligned}$$

Then we solve for dy/dx:

$$\begin{aligned}
\frac{(1+x^2)^2}{\cos^2(x-y)} \cdot \left(1 - \frac{dy}{dx}\right) &= (1+x^2)\cdot\frac{dy}{dx} - y\cdot(0+2x) \\
\frac{(1+x^2)^2}{\cos^2(x-y)} - \frac{(1+x^2)^2}{\cos^2(x-y)} \cdot \frac{dy}{dx} &= (1+x^2)\cdot\frac{dy}{dx} - 2xy \\
\left(-\frac{(1+x^2)^2}{\cos^2(x-y)} - (1+x^2)\right)\cdot\frac{dy}{dx} &= -2xy - \frac{(1+x^2)^2}{\cos^2(x-y)} \\
\frac{dy}{dx} &= \frac{-2xy - \frac{(1+x^2)^2}{\cos^2(x-y)}}{-\frac{(1+x^2)^2}{\cos^2(x-y)} - (1+x^2)}.
\end{aligned}$$

Again, I apologize for how horrible this is. The first part of the problem (applying d/dx) is much more important than the second part (solving for dy/dx). Needless to say, the problems on the exam will be much simpler than this one.

2.6.20. Here's a problem that's much more appropriate for the exam. Use implicit differentiation to find the equation of the tangent line to the curve $x^2 + 2xy - y^2 + x = 2$ at the point $(1,2)$. First we apply d/dx to both sides of the equation and solve for dy/dx:[1]

$$\begin{aligned}
(x^2 + 2xy - y^2 + x)' &= (2)' \\
(x^2)' + 2(xy)' - (y^2)' + (x)' &= 0 \\
2x + 2(1y + xy') - (2yy') + 1 &= 0 \\
2x + 2y + 2xy' - 2yy' + 1 &= 0 \\
(2x - 2y)y' &= -1 - 2x - 2y \\
y' &= \frac{-1 - 2x - 2y}{2x - 2y}.
\end{aligned}$$

The slope of the tangent at the point $(x,y) = (1,2)$ is

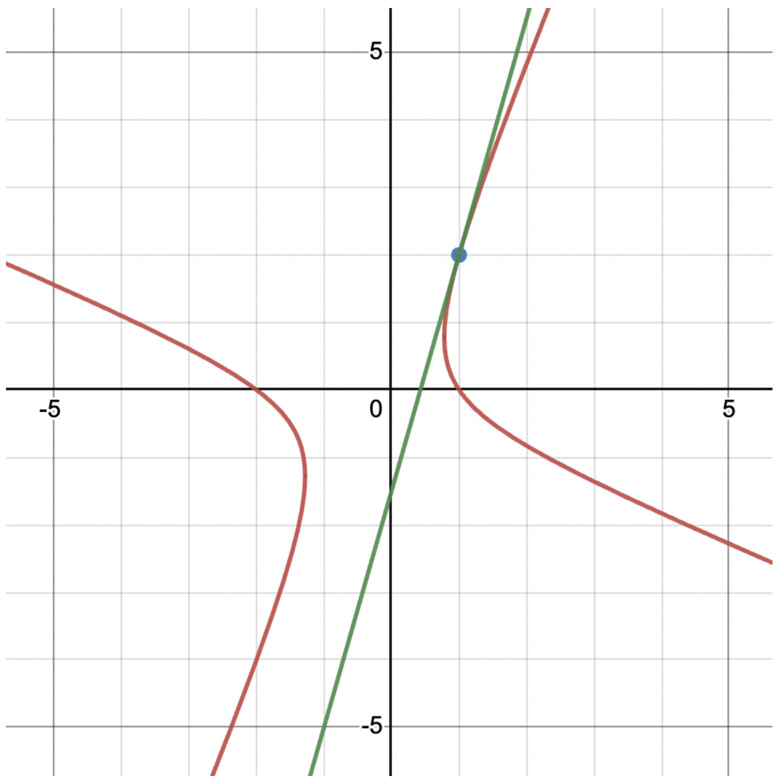
$$\frac{dy}{dx} = \frac{-1-2x-2y}{2x-2y} = \frac{-1-2(1)-2(2)}{2(1)-2(2)} = \frac{-7}{-2} = \frac{7}{2}.$$

[1] It saves a lot of time to write "prime" instead of d/dx.

Therefore the equation of the tangent line at $(1, 2)$ is

$$\begin{aligned}\frac{7}{2} &= \frac{y-2}{x-1}\\ \frac{7}{2}(x-1) &= y-2\\ y &= \frac{7}{2}x - \frac{7}{2} + 2\\ y &= \frac{7}{2}x - \frac{3}{2}.\end{aligned}$$

I know that this is correct because I plugged it into Desmos and got the following picture:



If the answer was incorrect then the green line would almost certainly not be tangent to the red curve. Remark: I only included the picture for illustration. You do not need to sketch curves on Exam 2.