Math 161 **Summer 2023**
Homework 1 Solutions **Drew Armstrong**

1.3.13. Consider the function $f(x) = \frac{\sin x}{x+\tan x}$. Here is a table of values:

x	$f(x)$
$+1$	0.329033
-1	0.329033
$+0.5$	0.458209
-0.5	0.458209
$+0.2$	0.493331
-0.2	0.493331
$+0.1$	0.498333
-0.1	0.498333
$+0.05$	0.499583
-0.05	0.499583
$+0.01$	0.499983
-0.01	0.499983

Based on this, we guess that $\lim_{x\to 0} f(x) = 1/2$.

Remark: Exercise **1.4.55** (which I didn't assign) asks for a proof of this. Here is a solution. Recall that $\sin x/x \to 1$ and $\tan x/x \to 1$ as $x \to 0$. Hence we have

$$\begin{aligned}\lim_{x\to 0} \frac{\sin x}{x+\tan x} &= \lim_{x\to 0} \frac{(\sin x)/x}{(x+\tan x)/x} \\ &= \lim_{x\to 0} \frac{(\sin x/x)}{1+(\tan x/x)} \\ &= \frac{1}{1+1} \\ &= 1/2.\end{aligned}$$

If you didn't memorize the formula $\lim_{x\to 0} \tan x/x = 1$, here is a derivation:

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x}{x} &= \lim_{x\to 0} \frac{\sin x/\cos x}{x} \\ &= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} \\ &= 1 \cdot 1 \\ &= 1.\end{aligned}$$

1.4.12. We factor the numerator and denominator, then cancel:

$$\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4} = \lim_{x\to 4} \frac{x\cancel{(x-4)}}{(x+1)\cancel{(x-4)}} = \lim_{x\to 4} \frac{x}{x+1} = \frac{4}{4+1} = \frac{4}{5}.$$

1.4.18. First we note that

$$\begin{aligned}(2+h)^3 &= (2+h)(2+h)^2 \\ &= (2+h)(4+4h+h^2) \\ &= 8+8h+2h^2+4h+4h^2+h^3\end{aligned}$$

$$= 8 + 12h + 6h^2 + h^3.$$

Then we have

$$\begin{aligned}
\lim_{h\to 0} \frac{(2+h)^3 - 8}{h} &= \lim_{h\to 0} \frac{(\cancel{8} + 12h + 6h^2 + h^3) - \cancel{8}}{h} \\
&= \lim_{h\to 0} \frac{12h + 6h^2 + h^3}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(12 + 6h + h^2)}{\cancel{h}} \\
&= \lim_{h\to 0} 12 + 6h + h^2 \\
&= 12 + 0 + 0 \\
&= 12.
\end{aligned}$$

1.4.22. This time our only hope is to multiply top and bottom by the "conjugate" of the numerator:

$$\begin{aligned}
\lim_{u\to 2} \frac{\sqrt{4u+1} - 3}{u-2} &= \lim_{u\to 2} \frac{\sqrt{4u+1} - 3}{u-2} \cdot \frac{\sqrt{4u+1} + 3}{\sqrt{4u+1} + 3} \\
&= \lim_{u\to 2} \frac{(\sqrt{4u+1})^2 - 3^2}{(u-2)\left(\sqrt{4u+1} + 3\right)} \\
&= \lim_{u\to 2} \frac{4u + 1 - 9}{(u-2)\left(\sqrt{4u+1} + 3\right)} \\
&= \lim_{u\to 2} \frac{4u - 8}{(u-2)\left(\sqrt{4u+1} + 3\right)} \\
&= \lim_{u\to 2} \frac{4\cancel{(u-2)}}{\cancel{(u-2)}\left(\sqrt{4u+1} + 3\right)} \\
&= \lim_{u\to 2} \frac{4}{\left(\sqrt{4u+1} + 3\right)} \\
&= \frac{4}{\left(\sqrt{4\cdot 2+1} + 3\right)} \\
&= \frac{4}{\left(\sqrt{9} + 3\right)} \\
&= 4/6 \\
&= 2/3.
\end{aligned}$$

1.4.28. First we subtract the fractions in the numerator:

$$\begin{aligned}
\frac{1}{(x+h)^2} - \frac{1}{x^2} &= \frac{1}{(x+h)^2} \cdot \frac{x^2}{x^2} - \frac{1}{x^2} \cdot \frac{(x+h)^2}{(x+h)^2} \\
&= \frac{x^2 - (x+h)^2}{x^2(x+h)^2}.
\end{aligned}$$

Now we try to compute the limit:

$$\lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h}$$

$$= \lim_{h\to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2}$$

If we now expand the numerator then we see that there is secretly a factor of h:

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} &= \lim_{h\to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} - (\cancel{x^2} + 2xh + h^2)}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-2xh - h^2}{hx^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{\cancel{h}(-2x - h)}{\cancel{h}x^2(x+h)^2} \\
&= \lim_{h\to 0} \frac{-2x - h}{x^2(x+h)^2} \\
&= \frac{-2x - 0}{x^2(x+0)^2} \\
&= -2x/x^4 \\
&= -2/x^3.
\end{aligned}$$

1.4.50. Here we use the fact that $\sin(4x)/x \to 4$ and $\sin(6x)/x \to 6$ as $x \to 0$ to get

$$\lim_{x\to 0} \frac{\sin(4x)}{\sin(6x)} = \lim_{x\to 0} \frac{(\sin(4x)/x)}{(\sin(6x)/x)} = \frac{4}{6} = \frac{2}{3}.$$

Remark: To see that $\sin(ax)/x \to a$ as $x \to 0$ we can use the substitution $y = ax$. In this case note that $y \to 0$ as $x \to 0$, hence

$$\lim_{x\to 0} \frac{\sin(ax)}{x} = \lim_{y\to 0} \frac{\sin(y)}{y/a} = \lim_{y\to 0} a \cdot \frac{\sin y}{y} = a \cdot 1 = a.$$

1.4.52. Here we use the fact that $(\sin\theta)/\theta \to 1$ and $(\cos\theta - 1)/\theta \to 0$ as $\theta \to 0$ to get

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta\to 0} \frac{(\cos\theta - 1)/\theta}{\sin\theta/\theta} = \frac{0}{1} = 0.$$

Remark: If we didn't memorize the formula $\lim_{\theta\to 0}(\cos\theta - 1)/\theta = 0$, we could derive it by multiplying top and bottom by the "conjugate" expression:

$$\begin{aligned}
\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} \cdot \frac{\cos\theta + 1}{\cos\theta + 1} &= \lim_{\theta\to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} \\
&= \lim_{\theta\to 0} \frac{-\sin^2\theta}{\theta(\cos\theta + 1)} \\
&= \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \frac{-\sin\theta}{\cos\theta} \\
&= 1 \cdot 0 \\
&= 0.
\end{aligned}$$

Here we used the identity $\cos^2\theta + \sin^2\theta = 1$.

1.4.54. Here we use the fact that $\sin(3x)/x \to 3$ and $\sin(5x)/x \to 5$ as $x \to 0$ to get

$$\lim_{x\to 0} \frac{\sin(3x)\sin(5x)}{x^2} = \lim_{x\to 0} \frac{\sin(3x)}{x} \cdot \frac{\sin(5x)}{x} = 3 \cdot 5 = 15.$$

1.4.56. First we note that $\sin(x^2)/x^2 \to 1$ as $x \to 0$. To see this we make the substitution $y = x^2$ and note that $y \to 0$ (from the right) as $x \to 0$. Hence

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^2} = \lim_{y\to 0^+} \frac{\sin(y)}{y} = 1.$$

Applying this to the current problem gives

$$\lim_{x\to 0} \frac{\sin(x^2)}{x} = \lim_{x\to 0} x \cdot \frac{\sin(x^2)}{x^2} = 0 \cdot 1 = 0.$$

1.6.14. When x is just to the left of -3 then $x+2$ is approximately -1 and $x+3$ is a tiny **negative** number, hence

$$\lim_{x\to -3^-} \frac{x+2}{x+3} = \frac{-1}{\text{tiny negative number}} = +\infty.$$

Remark: For example, the number $x = -3.001$ is slightly to the left of -3 and $-3.001 - 3 = -0.001$, which is a tiny negative number.

1.6.18. First we factor the numerator and denominator:

$$\frac{x^2 - 2x}{x^2 - 4x + 4} = \frac{x\cancel{(x-2)}}{(x-2)\cancel{(x-2)}} = \frac{x}{x-2}.$$

If x is slightly to the left of 2 then x is close to 2 and $x - 2$ is a tiny **negative** number, hence

$$\lim_{x\to 2^-} \frac{x^2-2x}{x^2-4x+4} = \lim_{x\to 2^-} \frac{x}{x-2} = \frac{2}{\text{tiny negative number}} = -\infty.$$

1.6.24. We use the fact that $1/x^4 \to 0$ as $x \to \infty$ to get

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2}{\sqrt{x^4+1}} &= \lim_{x\to\infty} \frac{x^2/x^2}{(\sqrt{x^4+1})/x^2} \\
&= \lim_{x\to\infty} \frac{1}{(\sqrt{x^4+1})/x^2} \\
&= \lim_{x\to\infty} \frac{1}{(\sqrt{x^4+1})/\sqrt{x^4}} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{\frac{x^4+1}{x^4}}} \\
&= \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^4}}} \\
&= 1/\sqrt{1+0} \\
&= 1.
\end{aligned}$$

1.6.28. The quantity $\sin^2 x = (\sin x)^2$ oscillates but stays bounded between 0 and 1:

$$0 \le \sin^2 x \le 1.$$

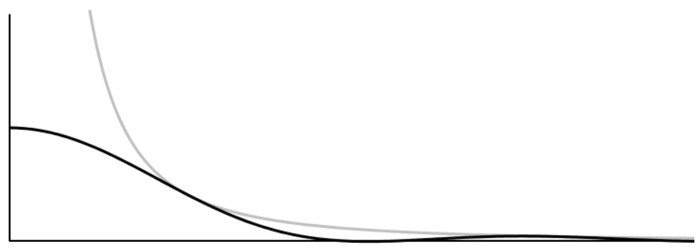
If $x \ne 0$ then dividing these inequalities by the positive number x^2 gives

$$\frac{0}{x^2} \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2}.$$

Since $1/x^2 \to 0$ as $x \to \infty$, the quantity in the middle gets squeezed:

$$\lim_{x\to\infty} \frac{\sin^2 x}{x^2} = 0.$$

Here is a picture of the graph:



1.6.30. To investigate the behavior of $(1+x^6)/(x^4+1)$ we will divide the numerator and denominator by x^4:

$$\frac{1+x^6}{x^4+1} = \frac{(1+x^6)/x^4}{(x^4+1)/x^4} = \frac{x^2+1/x^4}{1+1/x^4}.$$

Since $1/x^4 \to 0$ as $x \to -\infty$, the denominator $1+1/x^4$ goes to 1. But the numerator x^2+1/x^4 goes to $+\infty$ as $x \to -\infty$, hence

$$\lim_{x\to-\infty} \frac{1+x^6}{x^4+1} = \lim_{x\to-\infty} \frac{x^2+1/x^4}{1+1/x^4} = \frac{+\infty}{1} = +\infty.$$

Remark: We could also divide the numerator and denominator by x^6:

$$\frac{1+x^6}{x^4+1} = \frac{(1+x^6)/x^6}{(x^4+1)/x^6} = \frac{1+1/x^6}{1/x^2+1/x^6}.$$

Now the numerator goes to 1 and the denominator is a tiny **positive** number, hence

$$\lim_{x\to-\infty} \frac{1+x^6}{x^4+1} = \lim_{x\to-\infty} \frac{1+1/x^6}{1/x^2+1/x^6} = \frac{1}{\text{tiny positive number}} = +\infty.$$

Remark: Here we used the fact that

$$(\text{negative number})^{\text{even power}} = (\text{positive number}).$$

A.1. (a): According to the given identity we have

$$1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}.$$

If $|r| < 1$ then $r^{n+1} \to 0$ as $n \to \infty$, hence

$$\lim_{n\to\infty}(1 + r + r^2 + \cdots + r^n) = \lim_{n\to\infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1 - 0}{1 - r} = \frac{1}{1 - r}.$$

The limit on the left hand side is usually written as an infinite sum:

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1 - r}.$$

(b): When $r = 1/10$ we have

$$\begin{aligned} 1 + r + r^2 + r^3 + \cdots &= \frac{1}{1 - r} \\ 1 + (1/10)^2 + (1/10)^3 + \cdots &= \frac{1}{1 - 1/10} \\ 1 + \frac{1}{10} + \frac{1}{100} + \cdots &= \frac{1}{9/10} \\ 1 + \frac{1}{10} + \frac{1}{100} + \cdots &= \frac{10}{9}. \end{aligned}$$

Remark: In terms of decimals, the series on the left is $1 + 0.1 + 0.01 + \cdots = 1.1111\cdots$. This is exactly what my calculator says when I type in $10/9$.

A.2. The number e is defined as the following limit:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n.$$

Now consider any positive real number $r > 0$. We want to compute the limit

$$\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n.$$

To do this we make the substitution $n = mr$ to get

$$\left(1 + \frac{r}{n}\right)^n = \left(1 + \frac{\cancel{r}}{m\cancel{r}}\right)^{mr} = \left[\left(1 + \frac{1}{m}\right)^m\right]^r.$$

Since $r > 0$ we note that $m \to \infty$ as $n \to \infty$, hence

$$\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n = \lim_{m\to\infty}\left[\left(1 + \frac{1}{m}\right)^m\right]^r = \left[\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m\right]^r = e^r.$$

Example: Suppose you invest \$1 in a bank account with 5% yearly rate of return. This corresponds to the value $r = 0.05$. After one year with simple interest you will have \$1.05, so you gain 5 cents. After one year with continuously compounded interest you will have $e^r = e^{0.05} = 1.051271$ dollars, so you gain 5.1271 cents. (Rounded to the nearest cent this is still just 5 cents.)