Math 161 — Summer 2015
Homework 5 Solutions — Drew Armstrong

Book Problems:

- Chap 4.5 Exercises 2, 8, 14
- Chap 5.2 Exercises 16, 20, 56
- Chap 5.4 Exercises 42, 44, 46
- Chap 5.6 Exercises 8, 16
- Chap 6.1 Exercises 2, 14, 30

Additional Problems:

A1. Let $r > 0$ be constant. In this problem you will evaluate the following integral in two different ways:

$$\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

(a) Interpret this integral as the area of a shape you know.
(b) Use the substitution $x = r \sin\theta$ and the trigonometric identities

$$1 - \sin^2\theta = \cos^2\theta \qquad \text{and} \qquad \cos^2\theta = \frac{1}{2}\cos(2\theta) + \frac{1}{2}.$$

Then use the substitution $u = 2\theta$.

Solutions:

4.5.2. Evaluate the integral $\int x^3(2+x^4)^5\, dx$ using the substitution $u = 2 + x^4$.

Since $u = 2 + x^4$ we have $du = 4x^3\, dx$ and hence

$$\begin{aligned}
\int x^3(2+x^4)^5\, dx &= \int x^3(u)^5\, dx \\
&= \int u^5\, x^3 dx \\
&= \int u^5 \left(\frac{du}{4}\right) \\
&= \frac{1}{4}\int u^5\, du \\
&= \frac{1}{4}\cdot\frac{1}{6}u^6 + C \\
&= \frac{1}{24}(2+x^4)^6 + C,
\end{aligned}$$

where C is an arbitrary constant.

4.5.8. Evaluate the integral $\int x^2\cos(x^3)\, dx$.

We will use the substitution $u = x^3$, so that $du = 3x^2\, dx$. Then we have

$$\int x^2\cos(x^3)\, dx = \int x^2\cos(u)\, dx$$

$$
\begin{aligned}
&= \int \cos(u)\, x^2 dx \\
&= \int \cos(u) \left(\frac{du}{3}\right) \\
&= \frac{1}{3} \int \cos(u)\, du \\
&= \frac{1}{3} \sin(u) + C \\
&= \frac{1}{3} \sin(x^3) + C,
\end{aligned}
$$

where C is an arbitrary constant.

4.5.14. Evaluate the integral $\int \frac{x}{(x^2+1)^2}\, dx$.

We will use the substitution $u = x^2 + 1$, so that $du = 2x\, dx$. Then we have

$$
\begin{aligned}
\int \frac{x}{(x^2+1)^2}\, dx &= \int \frac{x}{u^2}\, dx \\
&= \int \frac{1}{u^2}\, x dx \\
&= \int \frac{1}{u^2} \left(\frac{du}{2}\right) \\
&= \frac{1}{2} \int u^{-2}\, du \\
&= \frac{1}{2} \cdot \frac{u^{-1}}{-1} + C \\
&= -\frac{1}{2u} + C \\
&= -\frac{1}{2(x^2+1)} + C,
\end{aligned}
$$

where C is an arbitrary constant.

5.2.16. Differentiate $f(x) = x\ln(x) - x$.

We use the product rule to compute

$$
\begin{aligned}
f'(x) &= (x\ln(x) - x)' \\
&= (x\ln(x))' - 1 \\
&= (x)' \ln(x) + x(\ln(x))' - 1 \\
&= \ln(x) + x \cdot \frac{1}{x} - 1 \\
&= \ln(x) + 1 - 1 \\
&= \ln(x).
\end{aligned}
$$

[Remark: Hey, we just discovered by accident that

$$
\boxed{\int \ln(x)\, dx = x\ln(x) - x + C.}
$$

That was lucky!]

5.2.20. Differentiate $y = \frac{1}{\ln(x)}$.

First we write $y = (\ln(x))^{-1}$. Then we use the chain rule to get

$$\frac{dy}{dx} = (-1)(\ln(x))^{-2} \cdot (\ln(x))' = -\frac{1}{(\ln(x))^2} \cdot \frac{1}{x} = -\frac{1}{x(\ln(x))^2}.$$

5.2.56. Evaluate the integral $\int_0^3 \frac{dx}{5x+1}$.

We use the substitution $u = 5x + 1$, so that $du = 5\,dx$. Then we have

$$\begin{aligned}
\int_{x=0}^{x=3} \frac{dx}{5x+1}\,dx &= \int_{x=0}^{x=3} \frac{dx}{u} \\
&= \int_{x=0}^{x=3} \frac{du/5}{u} \\
&= \frac{1}{5}\int_{u=1}^{u=16} \frac{1}{u}\,du \\
&= \frac{1}{5}\ln|u|\Big|_{u=1}^{u=16} \\
&= \frac{1}{5}(\ln(16) - \ln(1)) \\
&= 0.5545
\end{aligned}$$

5.4.42. Evaluate the integral $\int (x^5 + 5^x)\,dx$.

Here we just have to remember or look up the rules:

$$\begin{aligned}
\int (x^5 + 5^x)\,dx &= \int x^5\,dx + \int 5^x\,dx \\
&= \frac{1}{6}\cdot x^6 + \frac{1}{\ln(5)}\cdot 5^x + C,
\end{aligned}$$

where C is an arbitrary constant.

5.4.44. Evaluate the integral $\int x2^{x^2}\,dx$.

Here we use the substitution $u = x^2$, so that $du = 2x\,dx$. Then we have

$$\begin{aligned}
\int x2^{x^2}\,dx &= \int x2^u\,dx \\
&= \int 2^u\,xdx \\
&= \int 2^u\left(\frac{du}{2}\right) \\
&= \frac{1}{2}\int 2^u\,dx \\
&= \frac{1}{2}\cdot\frac{1}{\ln(2)}\cdot 2^u + C
\end{aligned}$$

$$= \frac{2^{x^2}}{2\ln(2)} + C,$$

where C is an arbitrary constant.

5.4.46. Evaluate the integral $\int \frac{2^x}{2^x+1}\,dx$.

Here we use the substitution $u = 2^x + 1$, so that $du = \ln(2)\cdot 2^x\,dx$. Then we have

$$\begin{aligned}
\int \frac{2^x}{2^x+1}\,dx &= \int \frac{2^x}{u}\,dx \\
&= \int \frac{1}{u}\,2^x dx \\
&= \int \frac{1}{u}\left(\frac{du}{\ln(2)}\right) \\
&= \frac{1}{\ln(2)}\int \frac{1}{u}\,du \\
&= \frac{1}{\ln(2)}\cdot \ln|u| + C \\
&= \frac{\ln|2^x+1|}{\ln(2)} + C \\
&= \frac{\ln(2^x+1)}{\ln(2)} + C \\
&= \log_2(2^x+1) + C,
\end{aligned}$$

where C is an arbitrary constant. [Remark: The last two steps of simplification were not necessary.]

5.6.8. Simplify the expression $\tan(\sin^{-1} x)$.

There are two ways to do this problem.

(1) Well, one thing we do know is that $\sin(\sin^{-1} x) = x$. [This is the **definition** of $\sin^{-1}$.] So we have

$$\tan(\sin^{-1} x) = \frac{\sin(\sin^{-1} x)}{\cos(\sin^{-1}(x))} = \frac{x}{\cos(\sin^{-1} x)}.$$

Now we have to compute $\cos(\sin^{-1} x)$. First we recall that

$$\cos^2\theta + \sin^2\theta = 1$$

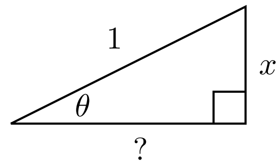
for any θ. Then we substitute $\theta = \sin^{-1} x$ to get

$$\begin{aligned}
\cos^2(\sin^{-1}) + \sin^2(\sin^{-1} x) &= 1 \\
\cos^2(\sin^{-1}) + x^2 &= 1 \\
\cos^2(\sin^{-1}) &= 1 - x^2 \\
\cos(\sin^{-1} x) &= \sqrt{1-x^2}.
\end{aligned}$$

Finally we have

$$\tan(\sin^{-1} x) = \frac{x}{\cos(\sin^{-1} x)} = \frac{x}{\sqrt{1-x^2}}.$$

(2) Let $\theta = \sin^{-1} x$, so that $x = \sin\theta$. Now let's draw a right angled tringle with angle θ and "hypotenuse" of length 1. Since $x = \sin\theta$, the length of the "opposite" side must be x. Let ? be the length of the "adjacent" side.



The Pythagorean Theorem tells us that

$$\begin{aligned} ?^2 + x^2 &= 1^2 \\ ?^2 &= 1 - x^2 \\ ? &= \sqrt{1 - x^2}. \end{aligned}$$

Finally, we have

$$\tan(\sin^{-1}) = \tan\theta = \frac{\text{“opposite”}}{\text{“adjacent”}} = \frac{x}{\sqrt{1 - x^2}}.$$

5.6.16. Find the derivative of the function $\tan^{-1}(x^2)$.

First we have to remember the formula

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1 + x^2}.$$

[If we didn't remember the formula then we would have to rediscover it.] Then we use the chain rule to compute

$$\frac{d}{dx}\tan^{-1}(x^2) = \frac{1}{1 + (x^2)^2} \cdot \frac{d}{dx}x^2 = \frac{2x}{1 + x^4}.$$

6.1.2. Evaluate the integral $\int \theta \cos\theta \, d\theta$ using integration by parts, with $u = \theta$ and $dv = \cos\theta \, d\theta$.

Since $u = \theta$ we have $du = d\theta$, and since $dv = \cos\theta \, d\theta$ we have $v = \sin\theta$. Then integration by parts gives

$$\begin{aligned} \int udv &= uv - \int vdu \\ \int \theta \cos\theta \, d\theta &= \theta \sin\theta - \int \sin\theta \, d\theta \\ &= \theta \sin\theta - (-\cos\theta) + C \\ &= \theta \sin\theta + \cos\theta + C, \end{aligned}$$

where C is an arbitrary constant.

6.1.14. Evaluate the integral $\int e^{-\theta} \cos(2\theta) \, d\theta$.

We will use integration by parts with $f(\theta) = \cos(2\theta)$ and $g'(\theta) = e^{-\theta}$, so that $f'(\theta) = -2\sin(2\theta)$ and $g(\theta) = -e^{-\theta}$. Then we have

$$\int f(\theta) g'(\theta) \, d\theta = f(\theta) g(\theta) - \int f'(\theta) g(\theta) \, d\theta$$

$$\begin{aligned}\int e^{-\theta}\cos(2\theta)\,d\theta &= -e^{-\theta}\cos(2\theta) - \int 2e^{-\theta}\sin(2\theta)\,d\theta\\ &= -e^{-\theta}\cos(2\theta) - 2\int e^{-\theta}\sin(2\theta)\,d\theta.\end{aligned}$$

Did that help? Now we have to evaluate the integral $\int e^{-\theta}\sin(2\theta)\,d\theta$. Okay, let's do it! Let $F(\theta) = \sin(2\theta)$ and $G'(\theta) = e^{-\theta}$, so that $F'(\theta) = 2\cos(2\theta)$ and $G(\theta) = -e^{-\theta}$ Then we have

$$\begin{aligned}\int F(\theta)G'(\theta)\,d\theta &= F(\theta)G(\theta) - \int F'(\theta)G(\theta)\,d\theta\\ \int e^{-\theta}\sin(2\theta)\,d\theta &= -e^{-\theta}\sin(2\theta) - \int 2(-e^{-\theta})\cos(2\theta)\,d\theta\\ &= -e^{-\theta}\sin(2\theta) + 2\int e^{-\theta}\cos(2\theta)\,d\theta\end{aligned}$$

Now we're back to where we started. But that's a good thing! Define

$$A := \int e^{-\theta}\cos(2\theta)\,d\theta.$$

Putting our two equations together gives

$$\begin{aligned}A &= -e^{-\theta}\cos(2\theta) - 2\int e^{-\theta}\sin(2\theta)\,d\theta\\ A &= -e^{-\theta}\cos(2\theta) - 2(-e^{-\theta}\sin(2\theta) + 2A)\\ A &= -e^{-\theta}\cos(2\theta) + 2e^{-\theta}\sin(2\theta) - 4A\\ 5A &= e^{-\theta}(2\sin(2\theta) - \cos(2\theta))\\ A &= \frac{1}{5}e^{-\theta}(2\sin(2\theta) - \cos(2\theta)).\end{aligned}$$

We conclude that

$$\int e^{-\theta}\cos(2\theta)\,d\theta = \frac{1}{5}e^{-\theta}(2\sin(2\theta) - \cos(2\theta)) + C.$$

where C is an arbitrary constant.

[Remark: Good thing we didn't lose our confidence when the first integration by parts didn't work.]

6.1.30. First make a substitution and then use integration by parts to evaluate $\int_1^4 e^{\sqrt{x}}\,dx$.

First we let $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}}dx$. Then we have

$$\begin{aligned}\int_{x=1}^{x=4} e^{\sqrt{x}}\,dx &= \int_{x=1}^{x=4} e^u\,2\sqrt{x}\,du\\ &= \int_{u=1}^{u=2} 2ue^u\,du.\end{aligned}$$

Okay. Now we let $f(u) = 2u$ and $g'(u) = e^u$, so that $f'(u) = 2$ and $g(u) = e^u$. Then we have

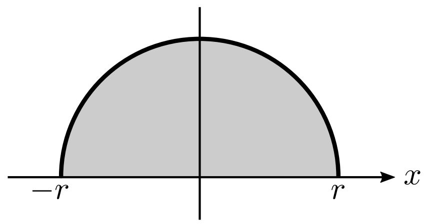
$$\begin{aligned}\int_{u=1}^{u=2} f(u)g'(u) &= f(u)g(u)|_{u=1}^{u=2} - \int_{u=1}^{u=2} f'(u)g(u)\,du\\ \int_1^2 2ue^u\,du &= 2ue^u|_1^2 - \int_1^2 2e^u\,du\\ &= (2(2)e^2 - 2(1)e^1) - 2(e^2 - e^1)\end{aligned}$$

$$
\begin{aligned}
&= 4e^2 - 2e - 2e^2 + 2e \\
&= 2e^2.
\end{aligned}
$$

And that's the answer.

A1. Compute the integral $\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$ in two ways.

(a) First we notice that this is just the area of a semicircle of radius r:



Hence $\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx = \frac{\pi r^2}{2}$.

(b) Second, we will follow the hints to evaluate the integral by hand. Let $x = r\sin\theta$, so that $dx = r\cos\theta\, d\theta$. Then we have

$$
\begin{aligned}
\int \sqrt{r^2 - x^2}\, dx &= \int \sqrt{r^2 - r^2\sin^2\theta}\, dx \\
&= \int \sqrt{r^2(1 - \sin^2\theta)}\, dx \\
&= \int \sqrt{r^2\cos^2\theta}\, dx \\
&= \int r\cos\theta\, dx \\
&= \int r\cos\theta(r\cos\theta\, d\theta) \\
&= r^2 \int \cos^2\theta\, d\theta \\
&= r^2 \int \left(\frac{1}{2}\cos(2\theta) + \frac{1}{2}\right) d\theta \\
&= \frac{r^2}{2} \int (\cos(2\theta) + 1)\, d\theta.
\end{aligned}
$$

Then we make the substitution $u = 2\theta$, so that $du = 2d\theta$, to get

$$
\begin{aligned}
\frac{r^2}{2} \int (\cos(2\theta) + 1)\, d\theta &= \frac{r^2}{2} \int (\cos(u) + 1)\, d\theta \\
&= \frac{r^2}{2} \cdot \frac{1}{2} \int (\cos(u) + 1)\, du \\
&= \frac{r^2}{4}(\sin(u) + u) + C,
\end{aligned}
$$

where C is an arbitrary constant. Finally, since $x = r\sin\theta$ we note that x goes from $-r$ to r as θ goes from $-\pi/2$ to $\pi/2$; and since $u = 2\theta$ we note that θ goes from $-\pi/2$

to $\pi/2$ as u goes from $-\pi$ to π. We conclude that

$$\begin{aligned}
\int_{x=-r}^{x=r} \sqrt{r^2 - x^2}\, dx &= \frac{r^2}{2} \int_{\theta=-\pi/2}^{\theta=\pi/2} (\cos(2\theta) + 1)\, d\theta \\
&= \frac{r^2}{4} \int_{u=-\pi}^{u=\pi} (\cos(u) + 1)\, du \\
&= \frac{r^2}{4} (\sin(u) + u)\Big|_{u=-\pi}^{u=\pi} \\
&= \frac{r^2}{4} \left[(\sin(\pi) + \pi) - (\sin(-\pi) + (-\pi)) \right] \\
&= \frac{r^2}{4} \left[(0 + \pi) - (0 - \pi) \right] \\
&= \frac{r^2}{4} \left[2\pi \right] \\
&= \frac{\pi r^2}{2}.
\end{aligned}$$

Which method do you prefer?

[Remark: That was the final homework problem of the course. Now we have come full circle. On HW1 Problem 1 we discussed Archimedes' proof that the area of a circle is πr^2. Here we used the methods of Calculus to come up with a completely different proof. Calculus can be used to solve a wide array of problems. And once you have some practice, it doesn't really require that much effort. We can all be Archimedes now.]