Math 161 **Summer 2015**
Homework 3 Solutions **Drew Armstrong**

Book Problems:

- Chap 2.5 Exercises 8, 22, 24, 38
- Chap 2.8 Exercises 11, 12, 22, 24
- Chap 3.3 Exercises 22, 34
- Chap 3.5 Exercises 2, 4

Solutions:

2.5.8. Let $y = (4x - x^2)^{100}$. Compute $\frac{dy}{dx}$.

We'll use the chain rule. Let $u = 4x - x^2$ so that $y = u^{100}$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 100\,u^{99} \cdot (4 - 2x) \\ &= 100\,(4x - x^2)^{99}(4 - 2x).\end{aligned}$$

2.5.22. Let $f(s) = \sqrt{\dfrac{s^2+1}{s^2+4}}$. Compute $f'(s)$.

We'll use the chain rule, and this time we won't be so fussy about it.

$$\begin{aligned}f'(s) &= \frac{1}{2}\left(\frac{s^2+1}{s^2+4}\right)^{-\frac{1}{2}} \cdot \left(\frac{s^2+1}{s^2+4}\right)' \\ &= \frac{1}{2}\left(\frac{s^2+1}{s^2+4}\right)^{-\frac{1}{2}} \cdot \frac{(s^2+4)(s^2+1)' - (s^2+1)(s^2+4)'}{(s^2+4)^2} \\ &= \frac{1}{2}\left(\frac{s^2+1}{s^2+4}\right)^{-\frac{1}{2}} \cdot \frac{(s^2+4)(2s) - (s^2+1)(2s)}{(s^2+4)^2} \\ &= \frac{1}{2}\left(\frac{s^2+1}{s^2+4}\right)^{-\frac{1}{2}} \cdot \frac{2s((s^2+4) - (s^2+1))}{(s^2+4)^2} \\ &= \frac{1}{2}\left(\frac{s^2+1}{s^2+4}\right)^{-\frac{1}{2}} \cdot \frac{6s}{(s^2+4)^2}.\end{aligned}$$

We won't bother to simplify it further. (I don't even know why I even simplified it this far.)

2.5.24. Let $f(x) = \dfrac{x}{\sqrt{7-3x}}$. Compute $f'(x)$.

I'll use the quotient rule:

$$\begin{aligned} f'(x) &= \frac{\sqrt{7-3x}\cdot(x)' - x\cdot(\sqrt{7-3x})'}{(\sqrt{7-3x})^2} \\ &= \frac{\sqrt{7-3x}\cdot(1) - x\cdot(\frac{1}{2\sqrt{7-3x}}\cdot(-3))}{7-3x} \\ &= \frac{\sqrt{7-3x} + \frac{3x}{2\sqrt{7-3x}}}{7-3x} \end{aligned}$$

We don't need to simplify, but if we want to we can multiply the numerator and denominator both by $\sqrt{7-3x}$ to get

$$\begin{aligned} f'(x) &= \frac{\sqrt{7-3x} + \frac{3x}{2\sqrt{7-3x}}}{7-3x}\cdot\frac{\sqrt{7-3x}}{\sqrt{7-3x}} \\ &= \frac{(7-3x)+\frac{3x}{2}}{(7-3x)\sqrt{7-3x}} \\ &= \frac{7-\frac{3x}{2}}{(7-3x)^{3/2}} \\ &= \frac{14-3x}{2(7-3x)^{3/2}}. \end{aligned}$$

2.5.38. Let $y = \sqrt{x+\sqrt{x+\sqrt{x}}}$. Compute $\frac{dy}{dx}$.

Just do the chain rule a lot:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-\frac{1}{2}}\cdot\frac{d}{dx}\left(x+\sqrt{x+\sqrt{x}}\right) \\ &= \frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-\frac{1}{2}}\cdot\left(1+\frac{1}{2}\left(x+\sqrt{x}\right)^{-\frac{1}{2}}\cdot\frac{d}{dx}\left(x+\sqrt{x}\right)\right) \\ &= \frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-\frac{1}{2}}\cdot\left(1+\frac{1}{2}\left(x+\sqrt{x}\right)^{-\frac{1}{2}}\cdot\left(1+\frac{1}{2}x^{-\frac{1}{2}}\right)\right) \end{aligned}$$

Don't bother simplifying. There's no way to make this look good.

2.8.11. Use linear approximation to estimate $(1.999)^4$.

We want to estimate the value of $f(x) = x^4$ for x near 2. The formula is

$$\boxed{f(x) \approx f(2) + f'(2)(x-2) \text{ when } x \approx 2.}$$

Since $f'(x) = 4x^3$, the boxed formula says

$$\begin{aligned} x^4 &\approx (2)^4 + 4(2)^3(x-2) \\ x^4 &\approx 16 + 32(x-2) \\ x^4 &\approx -48 + 32x. \end{aligned}$$

Of course, this is only accuate when $x \approx 2$. Luckily, $1.999 \approx 2$ so we get

$$(1.999)^4 \approx -48 + 32(1.999) = 15.968$$

How good is this estimate? My computer says $(1.999)^4 = 15.96802$, so it's pretty good.

2.8.12. Use linear approximation to estimate $\sqrt[3]{1001} = (1001)^{1/3}$.

We want to estimate the value of $f(x) = x^{1/3}$ for x near 1000. The formula is

$$\boxed{f(x) \approx f(1000) + f'(1000)(x - 1000) \text{ when } x \approx 1000.}$$

Since $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$, the boxed formula says

$$\begin{aligned}
x^{1/3} &\approx (1000)^{1/3} + \frac{1}{3}(1000)^{-\frac{2}{3}}(x - 1000) \\
x^{1/3} &\approx 10 + \frac{1}{3} \cdot \frac{1}{100}(x - 1000) \\
x^{1/3} &\approx \frac{20}{3} + \frac{x}{300}
\end{aligned}$$

Of course, this is only accurate when $x \approx 1000$. Luckily, $1001 \approx 1000$ so we get

$$(1001)^{1/3} \approx \frac{20}{3} - \frac{1001}{300} = \frac{3001}{300} = 10.00333\cdots.$$

How good is this estimate? My computer says $(1001)^{1/3} = 10.0033322$, so it's pretty good.

2.8.22. The radius of a circular disk is given as 24 cm with a maximum error in the measurement of 0.2 cm. Use differentials to estimate the maximum error in the area of the disk. What is the relative error? What is the percentage error?

Let r stand for the radius of the disk, so that $r = 24$ and $dr = 0.2$. Let A stand for the area of the disk. Then we have $A = \pi r^2$. We want to compute the differential dA. Well,

$$\begin{aligned}
\frac{dA}{dr} &= 2\pi r \\
dA &= 2\pi r\, dr
\end{aligned}$$

so we have $dA = 2\pi(24)(0.2) = 30.16 \text{ cm}^2$. This is our estimate for the maximum error in the area of the disk. To compute the relative error we first need to know that $A = \pi r^2 = \pi(24)^2 = 1809.56 \text{ cm}^2$. Thus the relative (and percentage) error is

$$\frac{dA}{A} = \frac{30.16}{1809.56} = 0.017 = 1.7\%.$$

2.8.24. Use differentials to estimate the amount of paint needed to apply a coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 m.

This is a fun one. Let D represent the diameter of the dome, so $D = 50$ m. When we add the coat of paint (I'm assuming on the outside of the dome, but it could be on the inside I guess), the diameter of the dome will change by $dD = 2(0.05) = 0.1$ cm.

Now we want to know how much paint is needed. Paint is measured by **volume**, so let V represent volume. Of what? This is tricky. We let V represent the volume of the full hemisphere. Then dV is the change in the volume of the hemisphere after we add the paint. In other words, dV is the volume of paint!

The volume of a hemisphere is 1/2 the volume of a sphere:

$$V = \frac{1}{2} \cdot \frac{4}{3}\pi \left(\frac{D}{2}\right)^3 = \frac{\pi}{12} D^3.$$

The differential is

$$\begin{aligned} \frac{dV}{dD} &= \frac{\pi}{12} \cdot 3D^2 = \frac{\pi}{4} D^2 \\ dV &= \frac{\pi}{4} D^2 \, dD \end{aligned}$$

Thus the amount of paint required is

$$dV = \frac{\pi}{4}(50)^2(0.001) = 1.963 \text{ m}^3.$$

(Note that I rewrote 0.1 cm as 0.001 m.) In summary, we will need 1.963 cubic meters of paint to cover the dome. Google tells me that this is 518.57 US gallons. Is that more or less than you expected?

3.3.22. Let $f(x) = 36x + 3x^2 - 2x^3$. Determine when $f'(x)$ and $f''(x)$ are positive, negative, and zero. Use this information to sketch the graph.

First we compute

$$\begin{aligned} f'(x) &= 36 + 3 \cdot 2x - 2 \cdot 3x^2 \\ &= 36 + 6x - 6x^2 \\ &= 6(6 + x - x^2) \\ &= 6(2 + x)(3 - x). \end{aligned}$$

We conclude that $f'(x) = 0$ when $x = -2$ or $x = 3$. These are called the "critical numbers". In between we have $f'(x) < 0$ (f is decreasing) when $x < -2$, $f'(x) > 0$ (f is increasing) when $-2 < x < 3$, and $f'(x) < 0$ (f is decreasing) when $3 < x$.

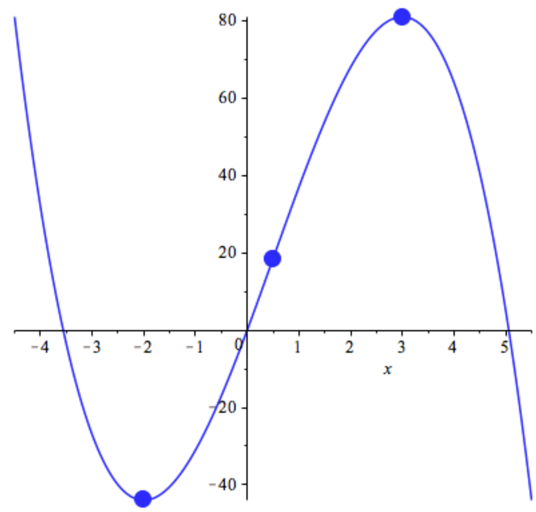
Next we compute

$$\begin{aligned} f''(x) &= (36 + 6x - 6x^2)' \\ &= 0 + 6 - 6 \cdot 2x \\ &= 6 - 12x \\ &= 6(1 - 2x). \end{aligned}$$

We conclude that $f''(x) = 0$ when $x = 1/2$ (this is an inflection point). When $x < 1/2$ we have $f''(x) > 0$ (f is concave up), and when $1/2 < x$ we have $f''(x) < 0$ (f is concave down).

Evaluating $f''(x)$ at the critical numbers gives $f''(-2) > 0$ and $f''(3) < 0$, so by the second derivative test we conclude that $f(x)$ has a local minimum at $x = -2$ and a local maximum at $x = 3$.

In summary, the graph of $f(x)$ has a minimum at $(-2, f(-2)) = (-2, -44)$, an inflection at $(1/2, f(1/2)) = (1/2, 37/2)$ and a maximum at $(3, f(3)) = (3, 81)$. Here is a sketch showing the graph of $f(x)$, the extrema, and the inflection points, made by my computer:



3.3.34. Let $f(x) = (x^2 - 4)/(x^2 + 4)$. Determine when $f'(x)$ and $f''(x)$ are positive, negative, and zero. Use this information to sketch the graph.

First we use the quotient rule to compute

$$\begin{aligned} f'(x) &= \frac{(x^2+4)(x^2-4)' - (x^2-4)(x^2+4)'}{(x^2+4)^2} \\ &= \frac{(x^2+4)(2x) - (x^2-4)(2x)}{(x^2+4)^2} \\ &= \frac{2x((x^2+4) - (x^2-4))}{(x^2+4)^2} \\ &= \frac{16x}{(x^2+4)^2} \end{aligned}$$

Since $(x^2 + 4)^2$ is always strictly positive, we conclude that: $f'(x) < 0$ when $x < 0$, $f'(x) = 0$ when $x = 0$, and $f'(x) > 0$ when $x > 0$. The only critical number is $x = 0$.

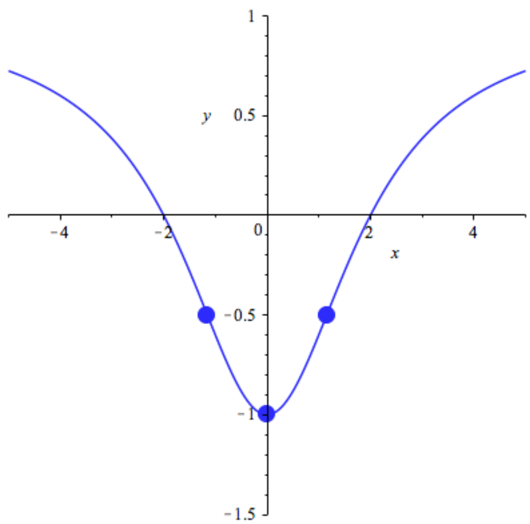
Next we use the quotient rule again to compute

$$\begin{aligned} f''(x) &= \frac{(x^2+4)^2(16x)' - 16x((x^2+4)^2)'}{[(x^2+4)^2]^2} \\ &= \frac{16(x^2+4)^2 - 16x \cdot 2(x^2+4)(2x)}{(x^2+4)^4} \\ &= \frac{16(x^2+4)[(x^2+4) - 4x^2]}{(x^2+4)^4} \\ &= \frac{16(x^2+4)(4-3x^2)}{(x^2+4)^4} \\ &= \frac{16(4-3x^2)}{(x^2+4)^3}. \end{aligned}$$

Since $(x^2 + 4)^3$ is always strictly positive, we conclude that $f''(x) = 0$ when $4 - 3x^2 = 0$, i.e., when $x = +\sqrt{4/3}$ or $x = -\sqrt{4/3}$. When $x < -\sqrt{4/3}$ we have $f''(x) < 0$ (f is concave down), when $-\sqrt{4/3} < x < +\sqrt{4/3}$ we have $f''(x) > 0$ (f is concave up), and when $+\sqrt{4/3} < x$ we have $f''(x) < 0$ (f is concave down).

Evaluating $f''(x)$ at the critical value $x = 0$ gives $f''(0) > 0$, so by the second derivative test we conclude that $f(x)$ has a local minimum at $x = 0$. This is the only local extremum.

In summary, the graph of $f(x)$ has a local minimum at $(0, f(0)) = (0, -1)$. It has inflection points at $(-\sqrt{4/3}, f(-\sqrt{4/3})) = (-\sqrt{4/3}, -1/2)$ and $(+\sqrt{4/3}, f(+\sqrt{4/3})) = (+\sqrt{4/3}, -1/2)$. Here is a sketch showing the graph of $f(x)$, the extrema, and the inflection points, made by my computer:



It might help your sketch to observe that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 1$. We say that the line $y = -1$ is a horizontal asymptote.

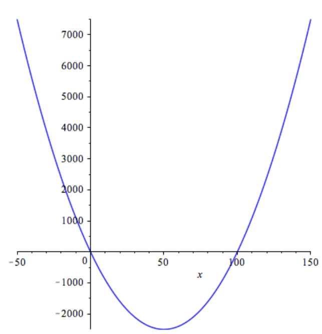
3.5.2. Find two numbers whose difference is 100 and whose product is a minimum.

Call the two numbers x and y. We assume that $x - y = 100$, and we want to minimize the "product function" $P(x, y) = xy$. First observe that we can use the condition $x - y = 100$ to express $P(x, y)$ as a function of x alone:

$$P(x) = xy = x(x - 100) = x^2 - 100x.$$

To minize $P(x)$ we first compute the derivative $P'(x) = 2x - 100$. Setting this equal to zero gives $2x - 100 = 0$, hence $x = 50$. Is this a minimum or a maximum? The second derivative is $P''(x) = 2$. Since $P''(50) = 2 > 0$, the second derivative test says that $P(x)$ has a minimum when $x = 50$. The other number is $y = x - 100 = 50 - 100 = -50$.

The minimum possible value of the product function is $P(x)$ is $P(50) = -2500$. Here is the graph of $P(x)$:



3.5.4. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

Call the two numbers x and y. We assume that $x + y = 16$, and we want to minimize the "sum of squares function" $S(x, y) = x^2 + y^2$. We can use the condition $x + y = 16$ to express $S(x, y)$ as a function of x alone:

$$S(x) = x^2 + y^2 = x^2 + (16 - x)^2 = 2x^2 - 32x + 256.$$

To minimize $S(x)$ we first compute the derivative $S'(x) = 4x - 32$. Setting this equal to zero gives $4x - 32 = 0$, hence $x = 8$. The second derivative is $S''(x) = 4$. Since $S''(8) = 4 > 0$, the second derivative test says that $S(x)$ has a minimum when $x = 8$. The other number is $y = 16 - x = 16 - 8 = 8$.

The minimum possible value of the sum of squares function is $S(8) = 128$. Here is the graph of $S(x)$: