Book Problems:

- Chap 2.1 Exercises 4, 14, 18
- Chap 2.2 Exercises 4, 6, 7
- Chap 2.3 Exercises 2, 4, 8, 16, 20

Additional Problems:

A1. Recall that the number e is defined by the limit

$$e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

In class we interpreted this as the amount of money you will have after one year if you invest \$1 in a bank account with 100% yearly rate of return. Using the same reasoning we can interpret the limit

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = \ ?$$

as the amount you will have after one year if you invest \$1 in a bank account with yearly rate of return $r > 0$. (The rate $r = 1$ corresponds to 100%.) Use the **substitution method** to evaluate this limit. [Hint: Let $n = mr$ and note that $n \to \infty$ as $m \to \infty$.]

Solutions:

2.1.4. To find the equation of the tangent line to the curve $y = x^3 - 3x + 1$ at the point $(2, 3)$ we first compute its slope. I'll do it the long way because we didn't learn the tricks yet in Chapter 2.1. By definition we have

$$\begin{aligned}
\frac{dy}{dx}(2) &= \lim_{h\to 0}\frac{((2+h)^3 - 3(2+h)+1) - (2^3 - 3(2)+1)}{h} \\
&= \lim_{h\to 0}\frac{\cancel{8} + 12h + 6h^2 + h^3 - \cancel{6} - 3h + \cancel{1} - \cancel{3}}{h} \\
&= \lim_{h\to 0}\frac{9h + 6h^2 + h^3}{h} \\
&= \lim_{h\to 0}\frac{\cancel{h}(9 + 6h + h^2)}{\cancel{h}} \\
&= 9.
\end{aligned}$$

The tangent line has slope 9 and passes through the point $(2, 3)$ so it has equation

$$\begin{aligned}
(y-3)/(x-2) &= 9 \\
y - 3 &= 9(x-2) \\
y &= 9(x-2) + 3 \\
y &= 9x - 15.
\end{aligned}$$

2.1.14. Let $s(t) = t^2 - 8t + 18$ be the position of a particle at time t.

(a) The average velocity of the particle over time interval $[t_1, t_2]$ is $\frac{s(t_2)-s(t_1)}{t_2-t_1}$.

(i) The average velocity of the particle over time interval $[3, 4]$ is $\frac{s(4)-s(3)}{1} = -1$.

(ii) The average velocity of the particle over time interval $[3.5, 4]$ is $\frac{s(4)-s(3.5)}{0.5} = -0.5$.

(iii) The average velocity of the particle over time interval $[4, 5]$ is $\frac{s(5)-s(4)}{1} = 1$.

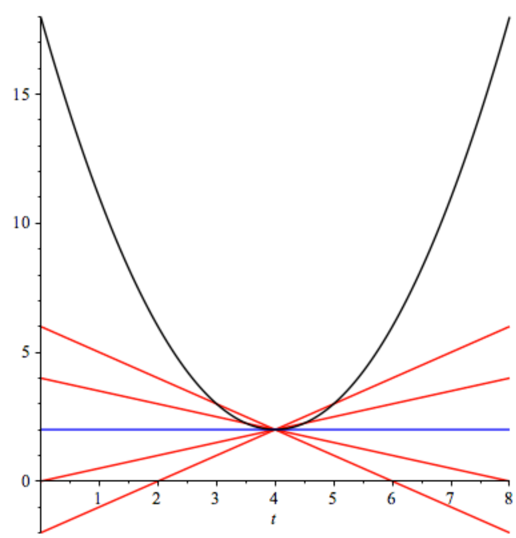
(iv) The average velocity of the particle over time interval $[4, 4.5]$ is $\frac{s(4.5)-s(4)}{1} = 0.5$.

(b) The instantaneous velocity of the particle at time t is

$$\begin{aligned}
s'(t) &= \lim_{h\to 0} \frac{s(t+h) - s(t)}{h} \\
&= \lim_{h\to 0} \frac{((t+h)^2 - 8(t+h) + 18) - (t^2 - 8t + 18)}{h} \\
&= \lim_{h\to 0} \frac{\cancel{t^2} + 2th + h^2 - \cancel{8t} - 8h + \cancel{18} - \cancel{t^2} + \cancel{8t} - \cancel{18}}{h} \\
&= \lim_{h\to 0} \frac{2th + h^2 - 8h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2t + h - 8)}{\cancel{h}} \\
&= \lim_{h\to 0} 2t + h - 8 \\
&= 2t - 8.
\end{aligned}$$

Therefore the instantaneous velocity of the particle at $t = 4$ is $s'(4) = 2 \cdot 4 - 8 = 0$.

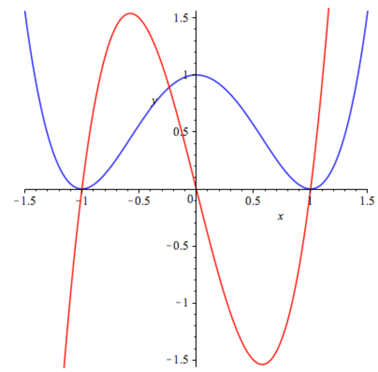
(c) Here is a picture of the graph of $s(t)$ (in black) showing the secant lines (in red) and the tangent line (in blue) whose slopes we computed in (a) and (b).



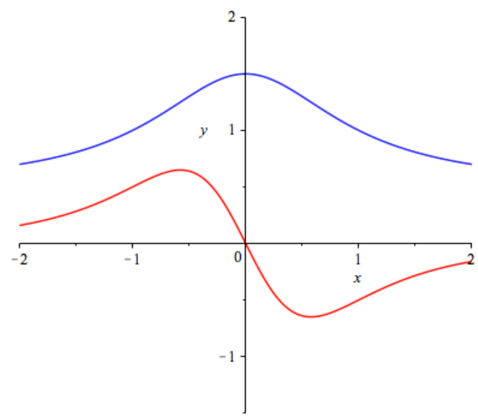
2.1.18. If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.

Well, since the point $(4, 3)$ is assumed to be on the graph, we must have $(4, 3) = (4, f(4))$. Hence $f(4) = 3$. Next, we know that the tangent line at $x = 4$ contains the points $(4, 3)$ and $(0, 2)$ so it must have slope $(3 - 2)/(4 - 0) = 1/4$. But by definition the slope of the tangent line at $x = 4$ is $f'(4)$. Hence $f'(4) = 1/4$.

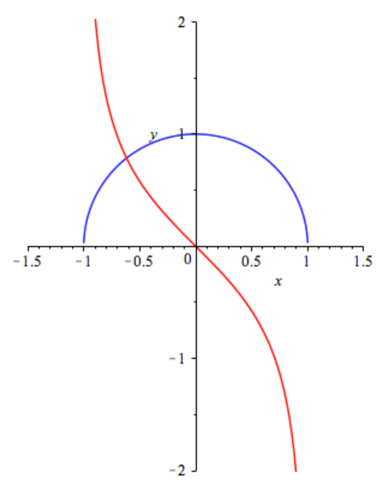
2.2.4. Here is the graph of f (in blue) and the graph of f' (in red).



2.2.6. Here is the graph of f (in blue) and the graph of f' (in red).



2.2.7. Here is the graph of f (in blue) and the graph of f' (in red).



2.3.2. Let $f(x) = \pi^2$. Then since π^2 is just a constant we have

$$f'(x) = 0.$$

2.3.4. Let $F(x) = \frac{3}{4}x^8$. Using the "power rule" and "constant multiple rule" gives

$$F'(x) = \left(\frac{3}{4}x^8\right)' = \frac{3}{4}\left(x^8\right)' = \frac{3}{4} \cdot 8x^7 = 6x^7.$$

2.4.8. Let $y = \sin t + \pi \cos t$. Then using some rules gives

$$\frac{dy}{dt} = \frac{d}{dt}(\sin t + \pi \cos t) = \frac{d}{dt}(\sin t) + \pi \frac{d}{dt}(\cos t) = \cos t - \pi \sin t.$$

2.4.16. Let $y = \sqrt{x}(x-1)$. Before differentiating, let's expand the expression to get

$$y = x^{1/2}(x-1) = x^{1/2} \cdot x - x^{1/2} = x^{3/2} - x^{1/2}.$$

Now we can differentiate using the power rule to get

$$\frac{dy}{dx} = \frac{3}{2} \cdot x^{3/2-1} - \frac{1}{2} \cdot x^{1/2-1} = \frac{3}{2} \cdot x^{1/2} - \frac{1}{2} \cdot x^{-1/2}.$$

2.4.20. Let $g(u) = \sqrt{2}\,u + \sqrt{3u}$. First we expand the expression:

$$g(u) = \sqrt{2}\,u^1 + \sqrt{3}\sqrt{u} = \sqrt{2}\,u^1 + \sqrt{3}\,u^{1/2}$$

Then we use the power rule:

$$g'(u) = \sqrt{2} \cdot 1u^0 + \sqrt{3} \cdot \frac{1}{2}u^{-1/2} = \sqrt{2} + \frac{\sqrt{3}}{2\sqrt{u}}.$$

A1. We know that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = e \approx 2.718$. Let $r > 0$ be a positive constant. We want to compute the limit

$$\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n.$$

To do this we will make the **substitution** $n = rm$. Since r is positive this means that $m \to \infty$ as $n \to \infty$. Then we have

$$\begin{aligned}
\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n &= \lim_{m\to\infty}\left(1 + \frac{r}{rm}\right)^{rm} \\
&= \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^{mr} \\
&= \lim_{m\to\infty}\left[\left(1 + \frac{1}{m}\right)^m\right]^r \\
&= \left[\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m\right]^r \\
&= e^r.
\end{aligned}$$

In summary, if you put $\$1$ in a bank account with yearly rate of return r then at the end of the year you will have $\$\,e^r$. For example, suppose the yearly rate of return is 3.6%, which corresponds to $r = 0.036$. Then at the end of the year you will have

$$\$\,e^{0.036} = \$\,1.036655846\,.$$

That's (very) slightly more than the $\$1.036$ you would get without using compound interest. You'll notice a bigger difference if you (1) deposit more money, (2) have a larger rate of return, or (3) wait longer. In general, if you deposit $\$\,P$ at yearly rate of return r and you wait t years, then you will get

$$\$\,Pe^{rt}.$$