Traveling Wave Solutions in Partially Degenerate Cooperative Reaction-Diffusion Systems

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Reaction-diffusion system

$$\frac{\partial \mathbf{u}}{\partial t} = D\frac{\partial^2 \mathbf{u}}{\partial x^2} - E\frac{\partial \mathbf{u}}{\partial x} + \mathbf{f}(\mathbf{u}), \qquad (1)$$

where

$$\mathbf{u}(t,x) = (u_1(t,x), u_2(t,x), ..., u_k(t,x))$$
$$D = \mathrm{diag}(d_1, ..., d_k),\ d_i \geq 0$$
$$E = \mathrm{diag}(e_1, ..., e_k)$$
$$\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), f_2(\mathbf{u}), ..., f_k(\mathbf{u}))$$

Hypotheses I

i. There is a proper subset Σ_0 of $\{1, ..., k\}$ such that $d_i = 0$ for $i \in \Sigma_0$ and $d_i > 0$ for $i \notin \Sigma_0$.

ii. $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, there is a constant $\boldsymbol{\beta} >> \mathbf{0}$ such that $\mathbf{f}(\boldsymbol{\beta}) = \mathbf{0}$ which is minimal.

iii. The system is cooperative; i.e., $f_i(\boldsymbol{\alpha})$ is nondecreasing in all components of $\boldsymbol{\alpha}$ with the possible exception of the ith one.

iv. $\mathbf{f}(\boldsymbol{\alpha})$ is uniformly Lipschitz continuous in $\boldsymbol{\alpha}$ so that there is $\rho > 0$ such that for any $\boldsymbol{\alpha}_i \geq \mathbf{0}$, $i = 1, 2$, $|\mathbf{f}(\boldsymbol{\alpha}_1) - \mathbf{f}(\boldsymbol{\alpha}_2)| \leq \rho |\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_2|$.

v. $\mathbf{f}$ has the Jacobian $\mathbf{f}'(\mathbf{0})$ at $\mathbf{0}$ with the property that $\mathbf{f}'(\mathbf{0})$ has a positive eigenvalue whose eigenvector has positive components.

Spreading speeds

Let Q denote the time one solution map of (1). Q is order-preserving. Define

$$\mathbf{a}_{n+1}(c;x) = \max\{\phi(x), [Q(\mathbf{a}_n(c;\cdot)](x+c)\}$$

where $\mathbf{a}_0(c;x) = \phi(x)$, and $\phi(x)$ is any nonincreasing continuous function with $\phi(x) = \mathbf{0}$ for $x \geq 0$ and $0 << \phi(-\infty) << \boldsymbol{\beta}$. $\mathbf{a}_n$ increases to a function $\mathbf{a}(c;x)$. $\mathbf{a}(c;-\infty) = \boldsymbol{\beta}$ with $\mathbf{a}(c;\infty)$ an equilibrium nondecreasing in c and independent of the choice of ϕ. Define

$$c^* := \sup\{c; \mathbf{a}(c;\infty) = \boldsymbol{\beta}\},$$

and

$$c_+^* := \sup\{c; \mathbf{a}(c;\infty) \neq \mathbf{0}\}.$$

Clearly $c_+^* \geq c^*$. If there are only two equilibria $\mathbf{0}$ and $\boldsymbol{\beta}$, $c_+^* = c^*$

Spreading speeds

Theorem

c^* is the slowest spreading speed and c^*_+ is an upper bound for all the spreading speeds for (1).

H. F. Weinberger, M. Lewis, and B. Li. J. Math. Biol. 45 (2002), 183-218.

Definition of spreading speed

Consider for example

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial x^2} + ru(1-u).$$

$c^* = 2\sqrt{rd}$ is the spreading speed in the following sense:
i. If $0 \leq u(x,0) < 1$ and $u(x,0) \equiv 0$ for large x, then for any $\epsilon > 0$

$$\lim_{x\to\infty}\left\{\max_{x\geq(c^*+\epsilon)t} u(x,t)\right\} = 0.$$

ii. For every positive number σ there exists a positive number r_σ such that if $0 \leq u(x,0) \leq 1$, and if $u(x,0) \geq \sigma$ on an interval of length r_σ, then for any positive ϵ,

$$\lim_{x\to\infty}\left\{\max_{|x|\leq(c^*-\epsilon)t}(1-u(x,t))\right\} = 0$$

Linear determinacy hypotheses

i. The matrix $\mathbf{f}'(\mathbf{0})$ is in Frobenius normal form. Let

$$C_\mu = \mu^2 D + \mu E + \mathbf{f}'(\mathbf{0}).$$

There is a positive entry to the left of each of the irreducible diagonal blocks other than the first (uppermost) one.

ii. Let $\gamma_\sigma(\mu)$ be the principal eigenvalue of the σth irreducible diagonal block of C_μ. Let $\boldsymbol{\xi}(\mu)$ be the eigenvector of C_μ which corresponds to $\lambda_1(\mu)$.

a. $\gamma_1(0) > 1$; and
b. $\gamma_1(0) > \gamma_\sigma(0)$ for all $\sigma > 0$.

Linear Determinacy Hypotheses

iv. Let

$$\bar{c} := \inf_{\mu>0} (1/\mu)\gamma_1(\mu) \qquad (2)$$

Either

(a) $\bar{\mu}$ is finite $\gamma_1(\bar{\mu}) > \gamma_\sigma(\bar{\mu})$, and

$$\mathbf{f}(\min\{\tau\boldsymbol{\xi}(\bar{\mu}), \boldsymbol{\beta}\}) - \mathbf{f}'(0)\tau\boldsymbol{\xi}(\bar{\mu}) \leq \mathbf{0}$$

for all positive τ; or

(b) there is a sequence $\mu_\nu \nearrow \bar{\mu}$ such that for each ν the above inequalities with $\bar{\mu}$ replaced by μ_ν are valid

Theorem

Assume that Hypotheses I and Linear determinacy Hypotheses are satisfied. Then $c^* = c^*_+ = \bar{c}$ where $\bar{c}$ is given by (2).

H. F. Weinberger, M. Lewis, and B. Li. J. Math. Biol. 45 (2002), 183-218.

Traveling waves in non-degenerate systems

Assume that Hypotheses I.ii-v are satisfied and all $d_i > 0$.

Theorem

i. *for* $c \geq c^*$ *, there is a nonincreasing traveling wave solution* $\mathbf{w}(x - ct)$ *connecting* $\boldsymbol{\beta}$ *and an equilibrium other than* $\boldsymbol{\beta}$*; and*
ii. *there is no nonincreasing traveling wave* $\mathbf{w}(x - ct)$ *with speed* $c < c^*$ *connecting* $\boldsymbol{\beta}$ *and an equilibrium other than* $\boldsymbol{\beta}$*; and*

Theorem

i. *for* $c \geq c^*_+$*, there is a nonincreasing traveling wave solution* $\mathbf{w}(x - ct)$ *connecting* $\mathbf{0}$ *and an equilibrium other than* $\mathbf{0}$*; and*
ii. *there is no nonincreasing traveling wave* $\mathbf{w}(x - ct)$ *with speed* $c < c^*_+$ *connecting* $\mathbf{0}$ *and* $\boldsymbol{\beta}$*.*

B. Li, H. F. Weinberger and M. Lewis. Math. Biosci. 196 (2005), 82-98
B. Li and L. Zhang. Nonlinearity 24 (2011), 1759-1776

Non-compactness

If $d_i = 0$ for some i, the time-t solution operator Q_t is not compact in general, and consequently the previously established traveling wave results cannot be used.

Weak compactness assumption is satisfied by some spatial evolution equations.

X. Liang, Yi, and X.-Q. Zhao. J. Diff. Eqs 231 (2006), 57-77.
X. Liang and X.-Q. Zhao. J. Funct. Anal. 259 (2010), 857-903.

Another example of noncompact differential system

$$\frac{\partial \mathbf{u}}{\partial t} = D \int_{-\infty}^{\infty} \mathbf{K}(x-y)\mathbf{u}(y)dy - D\mathbf{u}(x) + \mathbf{f}(\mathbf{u}).$$

Y. Jin and X.-Q. Zhao. Nonlinearity 22 (2009), 1167-1189
C. Hu and B. Li 2012 (preprint)

Integral system

Choose $\kappa > \rho$ where ρ is given in Hypothesis I iv. Define $\mathbf{H}(\mathbf{u}) = (\mathbf{f}(\mathbf{u}) + \kappa\mathbf{u})/\kappa$. Then $f(\boldsymbol{\alpha}) = 0$ if and only if $H(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$.
For $i \in \Sigma_0$, if $c - e_i > 0$, define

$$(\mathbf{m}_c)_i(x) = \begin{cases} 0 \text{ when } x > 0, \\ \frac{\kappa}{c-e_i} e^{\frac{\kappa}{c-e_i}x} \text{ when } x \leq 0, \end{cases}$$

$(\mathbf{m}_c)_i$ is defined in a similar way if $c - e_i < 0$,
For $i \notin \Sigma_0$, define

$$(\mathbf{m}_c)_i(x) = \frac{\kappa}{d_i(\lambda_{i1} - \lambda_{i2})} \begin{cases} e^{-\lambda_{i1}x} \text{ when } x \geq 0 \\ e^{-\lambda_{i2}x} \text{ when } x < 0, \end{cases}$$

where λ_{i1} and λ_{i2} are roots of $d_i z^2 - (c - e_i)z - \kappa = 0$.

J. Wu and X. Zou. J. Dyn. Diff. Eqs. 13 (2001), 651-686.
J. Fang and X. Q. Zhao. J. Dyn. Diff. Eqs 21 (2009), 663-680.

Integral system

Let

$$\mathbf{m}_c(x) = \text{diag}((\mathbf{m}_c)_1(x), ..., (\mathbf{m}_c)_k(x)).$$

We have that

$$\int_{-\infty}^{\infty} \mathbf{m}_c(x)dx = \mathbf{I}.$$

Theorem

Assume that $d_i \geq 0$ for all i and that Hypotheses I ii-v are satisfied. Let $c \neq e_i$ for all i with $d_i = 0$. Then $\mathbf{w}(x - ct)$ is a nonincreasing traveling wave solution of (1) connecting two different constant equilibria $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ if and only if $\mathbf{w}$ is a continuous nonincreasing function satisfying

$$\mathbf{w}(x) = \int_{-\infty}^{\infty} \mathbf{m}_c(x - y)\mathbf{H}(\mathbf{u})(y)dy.$$

B. Li. J. Diff Eqs. 252 (2012), 4842 - 4861.

Proof outline

If $d_i = 0$, $(c - e_i)w_i' - \kappa w_i = -(f_i(\mathbf{w}) + \kappa w_i)$. Assume that $c - e_i > 0$.

$$w_i(x) = w_i(x_0)e^{\frac{\kappa}{c-e_i}(x-x_0)} + \frac{\kappa}{(c-e_i)}\int_x^{x_0} e^{\frac{\kappa}{c-e_i}(x-y)} H_i(\mathbf{w})(y)dy.$$

We take the limit $x_0 \to \infty$ to obtain

$$w_i(x) = \frac{\kappa}{c-e_i}\int_x^{\infty} e^{(c-e_i)(x-y)} H_i(\mathbf{w})(y)dy$$

which is equivalent to

$$w_i(x) = \int_{-\infty}^{\infty} (\mathbf{m}_c)_i(x-y) H_i(\mathbf{w})(y)dy.$$

If $d_i > 0$, $d_i w_i'' + (c - e_i)w_i' - \kappa w_i = -(f_i(\mathbf{w}) + \kappa w_i)$. We solve the system, use integration by parts, and take appropriate limits to show that w_i satisfies the integral equation.

Approximation and equicontinuity

Let $D^{(\ell)} = D + (1/\ell)\mathbf{I}$ with $\ell \geq 1$. $D^{(\ell)}$ is a diagonal matrix with positive diagonal entries. The solution map operators for

$$\frac{\partial \mathbf{u}}{\partial t} = D^{(\ell)} \frac{\partial^2 \mathbf{u}}{\partial x^2} - E \frac{\partial \mathbf{u}}{\partial x} + \mathbf{f}(\mathbf{u}(t, x)), \tag{3}$$

are compact, and the existing theory on the existence of traveling wave solutions can be applied to (3).

Lemma

Assume that $\mathbf{w}^{(\ell)}(x - ct)$ is a nonincreasing traveling wave solution of (3) with speed $c \neq e_i$ for $i \in \Sigma_0$. Then the family $\mathbf{w}^{(\ell)}$ is an equicontinuous family of functions.

Idea: using

$$\mathbf{w}^{(\ell)}(x) = \int_{-\infty}^{\infty} \mathbf{m}_c^{(\ell)}(x - y)\mathbf{H}(\mathbf{w}^{(\ell)})(y)dy$$

Existence of traveling waves under Hypotheses I

Define $\tilde{c}^*$ and $\tilde{c}^*_+$ using the definitions for c^* and c^*_+ with Q replaced by the time one solution map of (3).

Theorem

i. *for* $c \geq \tilde{c}^*$ *and* $c \neq e_i$ *for* $i \in \Sigma_0$*, there is a nonincreasing traveling wave* $\mathbf{w}(x - ct)$ *connecting* $\boldsymbol{\beta}$ *and an equilibrium other than* $\boldsymbol{\beta}$*; and*

ii. *there is no nonincreasing traveling* $\mathbf{w}(x - ct)$ *with* $c < \tilde{c}^*$ *connecting* $\boldsymbol{\beta}$ *and an equilibrium other than* $\boldsymbol{\beta}$*.*

Theorem

i. *for* $c \geq \tilde{c}^*_+$ *and* $c \neq e_i$ *for* $i \in \Sigma_0$*, there is a nonincreasing traveling wave* $\mathbf{w}(x - ct)$ *connecting* $\mathbf{0}$ *and an equilibrium other than* $\mathbf{0}$*; and*

ii. *there is no nonincreasing traveling wave* $\mathbf{w}(x - ct)$ *with* $c < \tilde{c}^*_+$ *connecting* $\mathbf{0}$ *and* $\boldsymbol{\beta}$*.*

Proof outline of first theorem

1. For $c > \tilde{c}^*$, (3) has traveling wave $\mathbf{w}^{(\ell)}(x - ct)$ with $|\boldsymbol{\beta} - \mathbf{w}^{(\ell)}(0)| = \eta$, connecting $\boldsymbol{\beta}$, and an equilibrium other than $\boldsymbol{\beta}$.

2. $\lim_{\ell\to\infty} \int_{-\infty}^{\infty} |(\mathbf{m}_c^{(\ell)})_i(x) - (\mathbf{m}_c)_i(x)| dx = 0$.

3. $\mathbf{w}^{(\ell)}$ has a subsequence $\mathbf{w}^{(\ell_j)}$ such that $\mathbf{w}^{(\ell_j)}(x)$ converges to $\mathbf{w}(x)$ uniformly on every bounded interval.

4.$\mathbf{w}^{(\ell_j)}(x) = \int_{-\infty}^{\infty} \mathbf{m}_c(x-y)\mathbf{H}(\mathbf{w}^{(\ell_j)})(y)dy + \int_{-\infty}^{\infty} (\mathbf{m}_c^{(\ell_j)}(y) - \mathbf{m}_c(y))\mathbf{H}(\mathbf{w}^{(\ell_j)})(x-y)dy$. Take limits to obtain

$$\mathbf{w}(x) = \int_{-\infty}^{\infty} \mathbf{m}_c(x-y)\mathbf{H}(\mathbf{w})(y)dy$$

5. The existence of traveling wave with speed $\tilde{c}^*$ is obtained by taking an appropriate limit.

Linear determinacy

Lemma

Assume that Hypotheses I and and Linear Determinacy Hypotheses are satisfied. Then

$$c^* = c^*_+ = \tilde{c}^* = \tilde{c}^*_+ = \bar{c}$$

where $\bar{c}$ is given by (2).

Proof.

$C_\mu^{(\ell)} = \mu^2 D^{(\ell)} + \mu E + \mathbf{f}'(\mathbf{0}) = C_\mu^{(\ell)} = C_\mu + (\mu^2/\ell)\mathbf{I}$. Let $\gamma_1^{(\ell)}(\mu)$ be the principal eigenvalue of $C_\mu^{(\ell)}$.

$$\gamma_1^{(\ell)}(\mu) = \gamma_1(\mu) + \mu^2/\ell.$$

$$c^*(\ell) = c^*(\ell)_+ = \inf_{\mu>0}(1/\mu)(\gamma_1(\mu) + \mu^2/\ell).$$

It follows that

$$\tilde{c}^*_+ = \tilde{c}^* = \liminf_{\ell\to\infty} \inf_{\mu>0}(1/\mu)(\gamma_1(\mu) + \mu^2/\ell) = \inf_{\mu>0}(1/\mu)\gamma_1(\mu) = \bar{c}.$$

Applications to a Lotka-Volterra competition model

$$\frac{\partial p}{\partial t} = d_1 \frac{\partial^2 p}{\partial x^2} - e_1 \frac{\partial p}{\partial x} + r_1 p(1 - p - a_1 q),$$

$$\frac{\partial q}{\partial t} = -e_2 \frac{\partial q}{\partial x} + r_2 q(1 - q - a_2 p),$$

We assume that

$$a_1 < 1$$

so that equilibrium $(0, 1)$ is invadable. Let $u = p$, $v = 1 - q$. We have the cooperative system

$$\begin{aligned} \frac{\partial u}{\partial t} &= d_1 \frac{\partial^2 u}{\partial x^2} - e_1 \frac{\partial u}{\partial x} + r_1 u(1 - a_1 - u + a_1 v), \\ \frac{\partial v}{\partial t} &= -e_2 \frac{\partial v}{\partial x} + r_2 (1 - v)(a_2 u - v). \end{aligned} \tag{4}$$

For this model $\bar{c} = e_1 + 2\sqrt{d_1(1 - a_1)}$.

Result

Let

$$e_1 + 2\sqrt{d_1(1 - a_1)} \geq e_2 + r_2 \max\{a_1 a_2 - 1, 0\} \sqrt{d_1/(r_1(1 - a_1))}. \tag{5}$$

Theorem

Assume that (5) holds and $a_1 < 1$. Then the following statements hold for the system (4).

i. *If $\bar{c} > e_2$, or if $\bar{c} = e_2$ and $a_2 \leq 1$, then for $c \geq \bar{c}$ the system (4) has a nonincreasing traveling wave solution with speed c connecting $\mathbf{0}$ with $\boldsymbol{\beta}$;*

ii. *If $\bar{c} = e_2$ and $a_2 > 1$, then (4) has no classical nonincreasing traveling wave solution with speed $\bar{c} = e_2$ connecting $\mathbf{0}$ with $\boldsymbol{\beta}$; and*

iii. *(4) has no nonincreasing traveling wave solution with speed c connecting $\mathbf{0}$ with $\boldsymbol{\beta}$ if $c < \bar{c}$.*