ON THE REGULAR SET OF BMO SOLUTIONS TO STRONGLY COUPLED ELLIPTIC SYSTEM

Dung Le

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Outline

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1. Introduction
2. Function spaces and Assumptions
3. Main results
4. Sketch of proof

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The system

We consider the following strongly coupled p Laplacian system

$$\begin{cases} -\mathrm{div}(A(u, Du)) = f & \text{in } Q, \\ u = 0 & \text{on } \partial Q \end{cases} \tag{1}$$

Here:

1. Q is a bounded domain in $\mathbb{R}^n$ $(n > 1)$,
2. u, f have vector valued in $\mathbb{R}^m$ $(m > 1)$ (for simplicity f is bounded),
3. $A(u, Du)$ is a matrix $n \times m$.

We will characterize the regular set of a BMO weak solution u.
The matrix $A(u, Du)$ can be nonregular and singular.

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Historical note:

- Classical fact: bounded weak solutions to scalar eqns are Hölder continuous.
- Counterexamples exist for systems: bounded weak solutions may not be Hölder continuous.
- Partial regularity: bounded weak solutions for regular elliptic systems are Hölder continuous on a full measure and open set.
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Function spaces

a locally integrable vector valued function $u : Q \to \mathbb{R}^m$ is BMO if the seminorm

$$\|u\|_{BMO(Q)} = \sup_{B \subset Q} \fint_B |u - u_B| \, dz < \infty,$$

where the supremum is taken over all balls $B \subset Q$.

Let μ be a doubling measure on $\mathbb{R}^n$ and Ψ be a μ-measurable nonnegative function and $\gamma > 1$. We say that $\Psi \in A_\gamma(\mu)$ or Ψ is an $A_\gamma(d\mu)$ weight if the quantity

$$[\Psi]_\gamma = \sup_B \left(\fint_B \Psi \, d\mu \right) \left(\fint_B \Psi^{1-\gamma'} \, d\mu \right)^{\gamma - 1} < \infty. \tag{2}$$

Here, $\gamma' = \gamma/(\gamma - 1)$ and the supremum is taken over all balls B of $\mathbb{R}^n$. The $A_\infty(\mu)$ class is defined by $A_\infty(\mu) = \cup_{\gamma > 1} A_\gamma(\mu)$.

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Uniform ellipticity and Continuity

A.1) (Ellipticity) For any $u \in \mathbb{R}^m$ and $\zeta, \xi \in \mathbb{R}^{nm}$ there are nonnegative constants $\lambda(u), \Lambda(u)$ and $p > 1$ such that

$$\lambda(u)|\zeta|^p \le \langle A(u,\zeta), \zeta\rangle \le \Lambda(u)|\zeta|^p. \tag{3}$$

and

$$\langle A(u,\zeta) - A(u,\xi), \zeta - \xi\rangle \ge \lambda(u)|\zeta - \xi|^p. \tag{4}$$

Moreover, there are nonnegative constants λ_0, λ_1 such that

$$\lambda(u) \ge \lambda_0, \tag{5}$$

$$\Lambda(u) \le \lambda_1 \lambda(u). \tag{6}$$

A.2) (Continuity) For any $u, v \in \mathbb{R}^m$ and $\zeta \in \mathbb{R}^{nm}$ there is a function $\Delta(u, v)$ such that

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On the ellipticity constant

L.1) $\lambda(u)$ is quasi convex in the sense that there is a constant C such that for any vector valued function u and any ball $Q_R \subset \mathbb{R}^n$

$$\lambda(u_R) \le C(\lambda(u))_R. \quad (8)$$

L.2) $A(u,\zeta)$ is Hölder continuous in u. In fact, for the ellipticity and continuity constants $\lambda(u), \Delta$ in A.2) there are positive $\alpha, \beta, \theta_1, \alpha_0, \beta_0, \theta_0$ such that for any vectors $u, v \in \mathbb{R}^m$

$$|\Delta(u,v)| \le \max\{\lambda(u)^\alpha \lambda(v)^\beta, \lambda(v)^\alpha \lambda(u)^\beta\}|u-v|^{\theta_1}, \quad (9)$$

and

$$|\lambda(u)-\lambda(v)| \le C\max\{\lambda(u)^{\alpha_0}\lambda(v)^{\beta_0}, \lambda(v)^{\alpha_0}\lambda(u)^{\beta_0}\}|u-v|^{\theta_0}. \quad (10)$$

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Higher integrability result with weights

Using only the assumption that $\lambda(u)$ belongs to some $A_q(dz)$ class, with q being determined by p, n we will show that there exist positive numbers γ_*, C depending only on the A_q characterization of $\lambda(u)$ such that for any $\gamma \in (1, \gamma_*]$ and $\tau \in (0,1)$ the following higher integrability result holds

$$\left(\fint_{Q_{\tau R}} |Du|^{\gamma p}\, d\mu\right)^{\frac{1}{\gamma p}} \leq C\left(\fint_{Q_R} |Du|^{p}\, d\mu\right)^{\frac{1}{p}}, \quad d\mu = \lambda(u)dz. \quad (11)$$

On the parameters

With γ_* being described earlier, we will consider

P.1) For some $\gamma \in (1, \gamma_*]$

$$\alpha > \frac{1}{p'}, \quad \beta \le \frac{1}{p} - \frac{1}{p'\gamma'}, \quad \alpha \le \beta, \tag{12}$$

$$\beta_0 > \alpha_0, \quad \beta_0 \le 1/\gamma'. \tag{13}$$

P.2) For some $\gamma \in (1, \gamma_*]$

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P.3) There is $\gamma \in (1, \gamma_*]$ such that

$$\alpha+\beta-1 < \min\{\frac{1}{p'\gamma'}, p\theta_1\} \text{ and } \alpha_0+\beta_0-1 < \min\{\frac{1}{\gamma'}, \theta_0\}.$$

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First theorem

On the regular set of a BMO weak solution u to (1)

Theorem

Assume A.1), A.2), B), L.1), L.2), P.1) or P.2), and P.3). Assume also that $\lambda_0 > 0$. There exist positive constants C, ε_0, ν_0 depending only on $\|u\|_{BMO}, \|\lambda(u)\|_{BMO}$ such that for

$$\Sigma_0 = \{z_0 \in Q : \liminf_{R\to 0} \fint_{Q_R(z_0)} |u - u_R|^p \lambda(u)\, dz < \varepsilon_0\} \tag{16}$$

then for any balls Q_ρ, Q_R contained in Q and centered at $z_0 \in \Sigma_0$ the following holds

$$\fint_{Q_\rho} |u - u_\rho|^p \lambda(u)\, dz \le C \left(\frac{\rho}{R}\right)^{\nu_0} \fint_{Q_R} |u - u_R|^p \lambda(u)\, dz. \tag{17}$$

Moreover u is locally Hölder continuous in Σ_0.

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Quasi concave $\lambda(u)$

Our next result concerns the case where $\lambda(u)$ does not satisfy the quasi convexity condition L.1) but

L.1') $\lambda(u)$ is quasi concave in the sense that there is a constant C such that for any vector valued function u and any ball $Q_R \subset \mathbb{R}^n$

$$\lambda(u_R) \geq C(\lambda(u))_R. \tag{18}$$

In this case, we will assume that u locally has the *vanishing mean oscillation* (VMO) property. We say that a locally integrable function u has VMO property at a point z_0, or VMO at that point, if

$$\lim_{R\to 0} \fint_{Q_{z_0,R}} |u - u_{Q_{z_0,R}}|\, dz = 0.$$

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Second theorem

We have the following result.

Theorem

holderthm2 Assume that A.1), A.2), B), L.1') hold, $\lambda_0 > 0$, and that there are positive α, β, θ_1 such that for any vectors $u, v \in \mathbb{R}^m$

$$|\Delta(u,v)| \le \max\{\lambda(u)^\alpha \lambda(v)^\beta, \lambda(v)^\alpha \lambda(u)^\beta\}|u-v|^{\theta_1}, \tag{19}$$

with $\alpha+\beta \le 1/p$. Then a BMO weak solution u to (1) is Hölder at z_0 if u and $\lambda(u)$ have VMO property at z_0.

Of course, if $\lambda(u)$ is Hölder in u then $\|\lambda(u)\|_{BMO(Q_R)}$ is bounded by $\|u\|_{BMO(Q_R)}$, and thus the above theorem asserts that u is Hölder at a point if and only if it is VMO there.

The singular case

We can drop the assumption that $\lambda(u)$ is BMO in B) to have the following result concerning the singular case $\lambda_0 = 0$.

We only assume A.1), A.2), L.1), L.2) with $\lambda_0 = 0$. In addition, suppose that P.1)-P.3) are verified and

$$\alpha + \beta - 1 \text{ and } \alpha_0 + \beta_0 - 1 > 0, \tag{20}$$

$$\lambda(u) \in A_{\gamma_0} \text{ and } \lambda(u)^{-1} \in A_{\sigma_0} \tag{21}$$

$\gamma_0 \in (1, p(n-1)/n]$ and $\sigma_0 \in (1, \min\{2, n/(n - \alpha(p))\})$.

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The third theorem

Theorem

There are constants $C, \varepsilon_0 \nu_0$ depending on $\gamma_0, \gamma_0, [\lambda(u)]_{\gamma_0}$ and $[\lambda(u)^{-1}]_{\sigma_0}$ such that if we define

$$\Sigma_0 = \{z_0 \in Q \,:\, \liminf_{R\to 0} \fint_{Q_R(z_0)} |u - u_R|^p \lambda(u)\, dz < \varepsilon_0\} \tag{22}$$

then for any balls Q_ρ, Q_R in Q and centered at $z_0 \in \Sigma_0$ the following decay estimate holds

$$\fint_{Q_\rho} |u - u_\rho|^p \lambda(u)\, dz \leq C \left(\frac{\rho}{R}\right)^{\nu_0} \fint_{Q_R} |u - u_R|^p \lambda(u)\, dz. \tag{23}$$

Notes

Concerning the Hölder continuity of u we consider a point $z_0 \in \Sigma_0$ where $\lambda(u(z_0)) > 0$. Therefore, if $q \in (1, p)$ then

$$\fint_{Q_\rho} |u - u_\rho|^q \, dz \leq \left(\fint_{Q_\rho} (\lambda(u))^{-\frac{q}{p-q}} \, dz \right)^{\frac{p-q}{p}} \left(\fint_{Q_\rho} |u - u_\rho|^p \lambda(u) \, dz \right)^{\frac{q}{p}}.$$

By Lebesgue's theorem and the fact that $\lambda(u(z_0)) > 0$, the first factor on the right is bounded when ρ is sufficiently small. Combining these facts with (23) we assert that u is Hölder continuous at z_0.

Tools

Caccioppoli-type inequality:

$$\fint_{Q_R} |Du|^p \lambda(u)\, dz \le CR^{-p} \fint_{Q_{2R}} |u - \fint_{Q_{2R}} u\, dz|^p \lambda(u)\, dz. \qquad (24)$$

Weighted Sobolev-Poincaré inequality: Let $d\mu = \lambda(u)dz$ and $l = p\gamma n/(\gamma n + p)$. Then

$$\fint_{Q} |u - \fint_{Q} u\, dz|^p\, d\mu \le C([\lambda(u)]_{\gamma}) R^p \left(\fint_{Q} |Du|^l\, d\mu \right)^{\frac{p}{l}}. \qquad (25)$$

Higher integrability of gradients: There are $\gamma \in (1, \gamma_*)$ and $\tau \in (0, 1)$ such that

$$\left(\fint_{Q_{\tau R}} |Du|^{\gamma p}\, d\mu \right)^{\frac{1}{\gamma p}} \le C(\gamma_*, [\lambda(u)]_{\gamma_0}, \tau) \left(\fint_{Q_R} |Du|^p\, d\mu \right)^{\frac{1}{p}}. \qquad (26)$$

Here, $C(\gamma_*, [\lambda(u)]_{\gamma_0}, \tau)$ is bounded in τ^{-1}.

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$$\fint_{Q} |u - \fint_{Q} u\, dz|^p\, d\mu \leq C([\lambda(u)]_\gamma) R^p \left(\fint_{Q} |Du|^l\, d\mu \right)^{\frac{p}{l}}. \qquad (25)$$

Higher integrability of gradients: There are $\gamma \in (1, \gamma_*)$ and $\tau \in (0, 1)$ such that

$$\left(\fint_{Q_{\tau R}} |Du|^{\gamma p}\, d\mu \right)^{\frac{1}{\gamma p}} \leq C(\gamma_*, [\lambda(u)]_{\gamma_0}, \tau) \left(\fint_{Q_R} |Du|^p\, d\mu \right)^{\frac{1}{p}}. \qquad (26)$$

Here, $C(\gamma_*, [\lambda(u)]_{\gamma_0}, \tau)$ is bounded in τ^{-1}.

Tools

Caccioppoli-type inequality:

$$\fint_{Q_R} |Du|^p \lambda(u)\, dz \le CR^{-p} \fint_{Q_{2R}} \left|u - \fint_{Q_{2R}} u\, dz\right|^p \lambda(u)\, dz. \qquad (24)$$

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Approximation

For any ball Q_R contained in Q we will compare our solution u with that of

$$\begin{cases} -\mathrm{div}(A(u_R, DV)) = f & \text{in } Q_R, \\ V = u & \text{on } \partial Q_R. \end{cases} \tag{27}$$

Splitting domains: $\lambda(u), \lambda(u_R)$ are NOT comparable. Let Q_R^1 be any measurable subset of Q_R and $Q_R^2 = Q_R \setminus Q_R^1$. For $w = V - u$, we have

$$\int_{Q_R^1} \lambda(u_R)|Dw|^p\,dz + \int_{Q_R^2} \lambda(u)|Dw|^p\,dz \le$$
$$\int_{Q_R^1} |\Delta(u, u_R)||Du|^{p-1}|Dw|\,dz + \int_{Q_R^2} |\Delta(u, u_R)||DV|^{p-1}|Dw|\,dz.$$

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Decay estimate for DV

For some constant $\alpha(p) > 0$

$$\fint_{Q_{\tau R}} |DV|^p \, dz \leq C\tau^{-p+\alpha(p)} \fint_{Q_R} |DV|^p \, dz. \tag{28}$$

This usually gives Hölder continuity for V.

Can this be *transfered* to Du?

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$$(\tau R)^p \fint_{Q_{\tau R}} |Du|^p \lambda(u)\, dz \le C_*[\tau^{-n+p}\Phi(u,R)+\tau^{\nu}]R^p \fint_{Q_{2R}} |Du|^p \lambda(u)\, dz. \tag{29}$$

Here, ν depends on n, p and C_* depends on $n, p, C_0, [\lambda(u)]_{\gamma_0}, [\lambda(u)]_{\gamma'}$ and $[\lambda(u)^{-1}]_{\sigma_0}$.

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$\Phi(u,R)$

For any measurable subset Q_R^1 of Q_R and $Q_R^2 = Q_R \setminus Q_R^1$

$$\Phi(u,R) = \Phi_0(u,R) + \Phi_1(u,R) + \Phi_2(u,R)$$

$$\Phi_0(u,R) = \left(\frac{1}{|Q_R|} \int_{Q_R^1} \left| \frac{\lambda(u) - K_R}{\lambda(u)^{\frac{1}{\gamma}} (\lambda(u))_R^{\frac{1}{\gamma'}}} \right|^{\gamma'} dz \right)^{\frac{1}{\gamma'}}, \tag{30}$$

$$\Phi_1(u,R) = \left(\frac{1}{|Q_R|} \int_{Q_R^1} \left| \frac{|\Delta(u,u_R)|}{\lambda(u_R)^{\frac{1}{p}} \lambda(u)^{\frac{1}{p'\gamma}}} \right|^{p'\gamma'} dz \right)^{\frac{1}{\gamma'}} (\lambda(u))_R^{-\frac{1}{\gamma}}, \tag{31}$$

and

$$\Phi_2(u,R) = \left(\frac{1}{|Q_R|} \int_{Q_R^2} \left| \frac{|\Delta(u,u_R)|}{\lambda(u)^{\frac{1}{p}}} \right|^{p'\gamma'} dz \right)^{\frac{1}{\gamma'}} (\lambda(u))_R^{-1}. \tag{32}$$