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MATHEMATICS COMPETITION

XXXIV

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20 points (1) **Remainder Remainder Remainder**

For positive integers a and b, let $R_a(b)$ be the remainder of a divided by b. That is, $R_a(b)$ is an integer r such that $b > r \geq 0$ and $a = bq + r$ for some integer q.

If $R_a(b) > 0$, we can compose this function with itself to get $R_a^2(b) = R_a(R_a(b))$. Similarly for integers $n > 1$, as long as $R_a^{n-1}(b)$ is defined and positive, then $R_a^n(b)$ is defined. This gives us a sequence of integers we call the *remainder sequence* for a divided by b and denote $a /\!/ b$.

For example, $R_{10}(7) = 3$, $R_{10}^2(7) = R_{10}(3) = 1$, and $R_{10}^3(7) = 0$. However $R_{10}^4(7)$ is undefined. So 10 divided by 7 has the remainder sequence $10 /\!/ 7 = \{3, 1, 0\}$.

(a) Give three pairs of positive integers (a, b), other than $(10, 7)$, such that $a /\!/ b = \{3, 1, 0\}$.

(b) What are all the pairs (a, b) of positive integers a and b for which $a /\!/ b = \{3, 1, 0\}$?

(c) Show that for any integers $r > s > 0$, there exist integers a and b for which $a /\!/ b = \{r, s, 0\}$.

(d) Show that $\{6, 3, 1, 0\}$ is not a remainder sequence but $\{7, 3, 1, 0\}$ is.

(e) Show that if $\{r_k, r_{k-1}, \ldots, r_2, r_1, 0\}$ is a remainder sequence, then there exists a number r_{k+1} such that $\{r_{k+1}, r_k, r_{k-1}, \ldots, r_2, r_1, 0\}$ is a remainder sequence.

19 points (2) **Winning by Two**

Players A and B play a game of multiple rounds. They alternate starting the rounds, with player A going first. Each round, either A or B (but not both) scores one point. The game ends when one player's score totals two more than the other player's. The player with the highest score at the end of the game is the winner.

Regardless of what happened in prior rounds,

- the probability that A scores when A starts a round is p_A, and
- the probability that B scores when B starts a round is p_B.

Example: The probability that the game ends in exactly two rounds is

$$p_A(1 - p_B) + (1 - p_A)p_B.$$

(a) What is the probability that the game ends in exactly 4 rounds?

(b) What is the probability that the game ends with exactly n rounds played?

(c) What is the probability that A is the winner of a game with 10 or fewer rounds?

(d) What is the probability that at least n rounds must be played before the game ends?

Let $P(n, x)$ be the probability that the game lasts at least n rounds when $p_A = p_B = x$.

(e) Find all n such that $P(n, \frac{1}{2}) < 2^{-10}$.

(f) Given n, find all x such that $P(n, x) < 2^{-10}$.

18 points (3) **On-Off Functions**

Let $\mathbb{O}$ be the set "on-off" functions of real numbers whose range is in the two element set $\{0,1\}$. The constant functions $\mathbf{0}$ and $\mathbf{1}$ are in $\mathbb{O}$. They are defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all real numbers x. For another example, if

$$h(x) = \begin{cases} 1 & \text{if } x \text{ is a prime integer} \\ 0 & \text{otherwise} \end{cases}$$

then h belongs to $\mathbb{O}$ and we write $h \in \mathbb{O}$.

For $f, g \in \mathbb{O}$, define the operation $f * g$ by:

$$f * g = f + g - 2 \cdot f \cdot g$$

Here $+, -, \cdot$ are the usual addition, subtraction, and multiplication of functions.

(a) Show that if $f, g \in \mathbb{O}$ then $f \cdot g \in \mathbb{O}$ and $f * g \in \mathbb{O}$.

(b) Show that for $f \in \mathbb{O}$ then $f * f = \mathbf{0}$.

(c) Assume $f, g, h \in \mathbb{O}$. Prove or disprove the following:
(i) $(f * g) * h = f * (g * h)$ (ii) $f \cdot (g * h) = (f \cdot g) * (f \cdot h)$

(d) Consider functions $f, g, h \in \mathbb{O}$.
(i) If they satisfy $f \cdot h = g$, can the function h be determined in terms of f and g?
(ii) If they satisfy $f * h = g$, can the function h be determined in terms of f and g?

23 points (4) **Balls in Bins**

There are n bins in a line, and there are n bins in a circle. Consider the placements of balls in these bins so that

(A) no bin has more than one ball and
(B) no two balls are in adjacent bins.

Following these rules, let

- $L_n(b)$ be the number of ways of placing b balls in the *line* of n bins, and
- $K_n(b)$ be the number of ways of placing b balls in the *circle* of n bins.

(a) Find formulas for (i) $L_{2b}(b)$ and (ii) $L_{2b+1}(b)$.

(b) Find an expression for $L_n(b)$.

(c) For a fixed $n \geq 2$, what r maximizes $L_n(r)$?

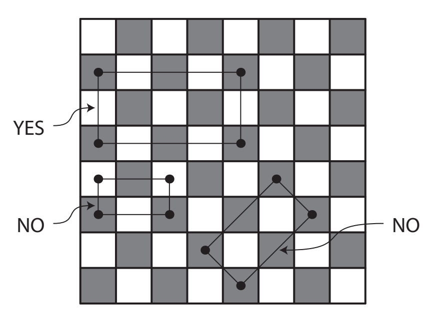
(d) Find formulas for (i) $K_{2b}(b)$ and (ii) $K_{2b+1}(b)$.

(e) Find an expression for $K_n(b)$.

25 points (5) **Monochrome Quads**

Let $G(a, b)$ be a $a \times b$ grid of unit squares where a, b are either positive integers or ∞. An n–coloring of $G(a, b)$ is a choice of one of n colors for each square. In this problem, a *quad* in a grid $G(a, b)$ is a set of four distinct squares that form the corners of a subgrid $G(a', b')$ with $a' \leq a$ and $b' \leq b$. If $G(a, b)$ is n–colored, then a quad is *monochome* if its four squares each have the same color.

For example, the standard checkerboard shown is a 2–coloring of $G(8, 8)$. Also shown is a monochrome quad and two sets of four squares that are not monochrome quads.



(a) How many 2–colorings does $G(8, 8)$ have?

(b) Does every 2–coloring of $G(8, 8)$ have a monochrome quad?

(c) What is the smallest area grid $G(a, b)$ such that every 2–coloring has a monochrome quad?

(d) Does every 3–coloring of $G(20, 14)$ have a monochrome quad?

(e) Does every 2014–coloring of $G(\infty, \infty)$ have a monochrome quad?