DAVID ESSNER
MATHEMATICS COMPETITION

XXXIII

October 19, 2013

19 points (1) **Isolated Zeros**

An n–sequence $a_1, a_2, a_3, \ldots, a_n$ of integers is *ternary* if each term a_i is $0, 1$, or 2. A sequence has *isolated zeros* if it has no adjacent 0's. For example, 0120101 is a ternary 7–sequence with isolated zeros that has exactly three zeros; 1221222 is a ternary 7–sequence with isolated zeros that has exactly zero zeros. The sequences 0100121 and 1200010102 do not have isolated zeros.

Denote the number of ternary n–sequences with isolated zeros by $s(n)$ and those that have exactly k zeros by $s(n, k)$.

(a) (i) Determine $s(3, k)$ for $k = 0, 1, 2, 3$. Also give $s(3)$.
(ii) Determine $s(4, k)$ for $k = 0, 1, 2, 3, 4$. Also give $s(4)$.
(iii) Determine $s(5, 2)$.

(b) Give explicit formulas for $s(n, 0)$, $s(n, 1)$, and $s(n, 2)$.

(c) Give a formula for $s(n, k)$, explicit or recursive.

(d) Give a formula for $s(n)$, explicit or recursive.

19 points (2) **Recycling Calendars**

This is the year 2013 and I'm using a Simpson's Calendar for the year 1991. Throughout the entire year each day of the month in 2013 is on the same day of the week as in 1991. Indeed, October 19, 1991 was a Saturday just as October 19, 2013 is a Saturday.

In the Gregorian calendar system each week has 7 days and most years have 365 days. Typically every year divisible by 4 is a leap year with the extra day February 29 giving it a total of 366 days. The exceptions to this are the years divisible by 100 that are not also divisible by 400. Last year, 2012, was a leap year and so was the year 2000. The years 1900 and 1991 were not leap years.

(a) Which day of the week was October 19, 1995?

(b) What is the next year that I can use my Simpson's Calendar?

(c) In the years 1991 through 2113, what years could I use my Simpson's Calendar?

(d) Could I use the Simpson's Calendar in 2991?

(e) Assume each week were to have only 6 days instead of 7, but with all else remaining the same. In what years from 1991 to 2091 would I be able to use my 1991 Simpson's calendar?

19 points (3) **Inequalities**

Suppose a, b, c, d are positive real numbers.

(a) (i) Prove that $(a+b)^2 \geq 4ab$ and equality holds if and only if $a = b$.

(ii) Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$ and equality holds if and only if $a = b = c$.

(b) Use a result in part (a) to show that$(a+b+c+d)^4 \geq 256abcd$ and equality holds if and only if $a = b = c = d$.

(c) Use the result in part (b) to show that $(a+b+c)^3 \geq 27abc$ and equality holds if and only if $a = b = c$.

(d) Use the result in part (c) to show that

$$(a+b+c)\left(\frac{1}{a+2b} + \frac{1}{b+2c} + \frac{1}{c+2a}\right) \geq 3$$

and equality holds if and only if $a = b = c$.

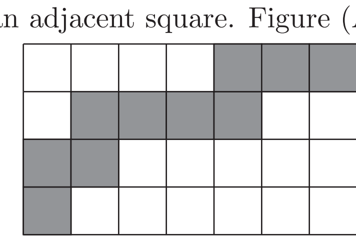
(e) Use the result in part (d) to show that

$$(a+b+c)^2x^2 - 6x + \left(\frac{1}{a+2b} + \frac{1}{b+2c} + \frac{1}{c+2a}\right)^2 \geq 0$$

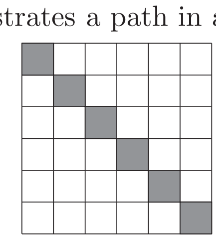
for all real numbers x. Find necessary and sufficient conditions for equality to hold.

19 points (4) **Corner to Corner**

Say a *path* in an $m \times n$ grid of squares is a sequence of squares beginning with the square in the lower left corner, ending in the upper right corner, and each step in the sequence moves either up or right to an adjacent square. Figure (A) below illustrates a path in a 4×7 grid.



(A) A *path* in a 4×7 grid



(B) The *diagonal* of a 6×6 grid

(a) In an $m \times n$ grid, what is the maximum and minimum number of squares that a path may visit (including the first)?

(b) (i) What is the total number of possible paths in a 4×7 grid?
(ii) What is the total number of paths in an $n \times m$ grid?

(c) In a $2k+1 \times 2k+1$ grid, how many paths visit the center square?

(d) (i) Figure (B) above shows the *diagonal* of shaded squares in a 6×6 grid. If two people were to choose paths at random in a 6×6 grid, what is the probability that these two paths visit the same square in the diagonal?
(ii) Repeat part (i) for an $n \times n$ grid.

24 points (5) **Circles of Circles**

For integers $n \geq 3$, a set A_n of n congruent circles in the plane is a *ring* if

(i) there is a regular n–gon P_n such that the vertices of P_n are the centers of the circles belonging to A_n and

(ii) the sides of P_n have the same lengths as the diameters of the circles in A_n.

We define the *radius of the ring* A_n to be the radius of the circle circumscribing P_n, that is, the radius of the circle containing the centers of the circles in A_n.

(a) If C is a circle in a ring A_n, how many other circles in A_n are tangent to C?

(b) Show that for a ring A_n, the ratio of the radius of a circle in A_n to the radius of the ring only depends on the integer n. Give a formula for this ratio in terms the integer n.

(c) Can there be two rings A_n and A_{2n} such that each circle of A_n is tangent to exactly two circles of A_{2n}? Either provide an example or prove that two such rings do not exist.

(d) (i) Show there is a pair of rings A_n and A'_n such that each circle of A_n is tangent to exactly one circle of A'_n and each circle of A'_n is tangent to exactly one circle of A_n.
(ii) For each integer n, what is the set of ratios of the radius of the ring A_n to the radius of the ring A'_n that may have this property?