Splitting Spacetime

University of Miami, December 2018

A Celebration of Mathematical Relativity in Miami

In Honor of:

Greg Galloway

Carlos Vega, Binghamton University

Orangutan mothers raise their young for 6 to 7 years.

Orangutan mothers raise their young for 6 to 7 years.

This is surpassed in the wild only by certain PhD advisors.

Orangutan mothers raise their young for 6 to 7 years.

This is surpassed in the wild only by certain PhD advisors.

Thank you, Dr. Galloway,

for so many years of patience, support, encouragement,

for so much **fun**, and so much **beautiful mathematics**.

Galloway has been at the heart of Lorentzian splitting geometry since the subject began.

Galloway has been at the heart of Lorentzian splitting geometry since the subject began.

My aim here will only be to give a flavor of some of our joint work in this field.

Hawking-Penrose Singularity Theorem (1970)

'Spatially closed GR spacetimes are generically singular.'

Hawking-Penrose Singularity Theorem (1970)

Let M^{n+1} be a spacetime with:

(1) compact Cauchy surfaces

(2) $Ric(X, X) \geq 0$, for all timelike $X \in TM$

(3) all causal geodesics satisfy the 'generic condition'

Then M has an incomplete causal geodesic.

Hawking-Penrose Singularity Theorem (1970)

Let M^{n+1} be a spacetime with:

(1) compact Cauchy surfaces

(2) $Ric(X,X) \geq 0$, for all timelike $X \in TM$

(3) all causal geodesics satisfy the 'generic condition'

Then M has an incomplete causal geodesic.

1970, Geroch: posed rough rigidity question in $3+1$

Hawking-Penrose Singularity Theorem (1970)

Let M^{n+1} be a spacetime with:

(1) compact Cauchy surfaces

(2) $Ric(X, X) \geq 0$, for all timelike $X \in TM$

(3) all causal geodesics satisfy the 'generic condition'

Then M has an incomplete causal geodesic.

1970, Geroch: posed rough rigidity question in $3 + 1$

1980's, Yau: proposed rigidity *in specific case of H-P* above
posed Lorentzian splitting problem in 1982

Hawking-Penrose Singularity Theorem (1970)

Let M^{n+1} be a spacetime with:

(1) compact Cauchy surfaces

(2) $Ric(X, X) \geq 0$, for all timelike $X \in TM$

(3) all causal geodesics satisfy the 'generic condition'

Then M has an incomplete causal geodesic.

1970, Geroch: posed rough rigidity question in $3 + 1$

1980's, Yau: proposed rigidity *in specific case of H-P* above
posed Lorentzian splitting problem in 1982

1993: Beem, Harris: the generic condition is generic!

Bartnik Splitting Conjecture (1988)

Let M^{n+1} be a spacetime with:

(1) compact Cauchy surfaces

(2) $Ric(X, X) \geq 0$, for all timelike $X \in TM$

(3) timelike geodesically complete

Then M splits isometrically as:

$$(M^{n+1}, g) \approx (\mathbb{R} \times \Sigma^n, -dt^2 + h)$$

Bartnik Splitting Conjecture (1988)

Let M^{n+1} be a spacetime with:

(1) compact Cauchy surfaces

(2) $Ric(X,X) \geq 0$, for all timelike $X \in TM$

(3) timelike geodesically complete

Then M splits isometrically as:

$$(M^{n+1}, g) \approx (\mathbb{R} \times \Sigma^n, -dt^2 + h)$$

1984, Galloway: splitting if no observer horizons

1988, Bartnik: splitting if *one* observer has no horizon

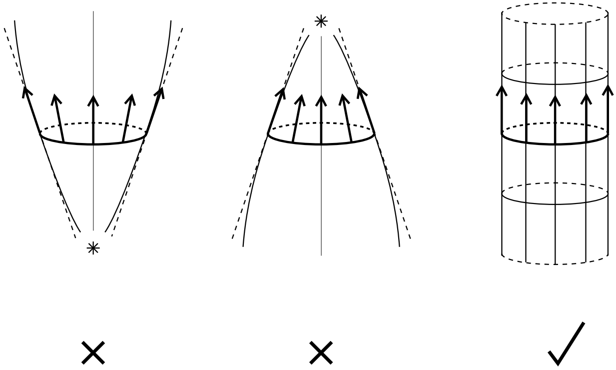
(Bartnik has referred to this as 'the Galloway Conjecture'.)

For a warped product $(M^{n+1}, g) = (\mathcal{I} \times \Sigma^n, -dt^2 + f^2(t)h)$,

$(1) \implies \quad \Sigma$ compact

$(2) \implies \quad f'' \leq 0$

$(3) \implies \quad \mathcal{I} = \mathbb{R}$



Two basic routes to splitting:

- construct a compact maximal spacelike hypersurface Σ

  barrier methods

- construct a timelike line α

  Busemann function b_γ from timelike ray γ
  $\{b_\gamma = 0\}$ = conventional Lorentzian 'horosphere'
  Lorentzian Splitting Theorem

Rough Idea:

Rough Idea:

want a *maximal hypersurface?*

try making a *big sphere!*

Rough Idea:

want a *maximal hypersurface?*

try making a *big sphere!*

$$M^{n+1} \ \sim \ \text{flat} \qquad\qquad (\mathrm{Ric}^T \sim 0)$$

Rough Idea:

want a *maximal hypersurface?*

try making a *big sphere!*

$$M^{n+1} \ \sim \ \text{flat} \qquad\qquad (\mathrm{Ric}^T \sim 0)$$

$$H(S_r) \ \sim \ \frac{n}{r} \qquad\qquad (\text{Raychaudhuri/Riccati comp})$$

Rough Idea:

want a *maximal hypersurface?*

try making a *big sphere!*

$$M^{n+1} \ \sim \ \text{flat} \qquad\qquad (\mathrm{Ric}^T \sim 0)$$

$$H(S_r) \ \sim \ \frac{n}{r} \qquad\qquad (\text{Raychaudhuri/Riccati comp})$$

$$H(S_\infty) \ \sim \ 0$$

In joint work with G. Galloway:

- new, geometric approach to Lorentzian horospheres

  ray horosphere, $S_\infty(\gamma)$, from timelike ray γ $\quad \{b_\gamma = 0\}$

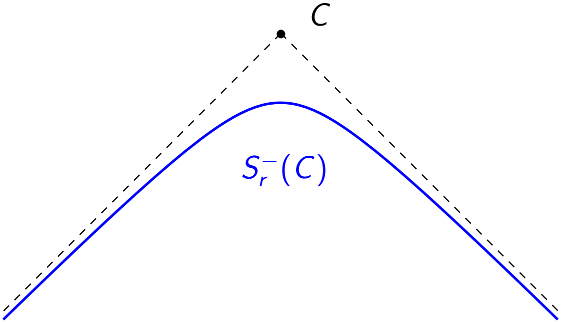
  Cauchy horosphere, $S_\infty(S)$, from Cauchy surface S

  generalized horospheres S_∞, and achronal limits A_∞

- convexity and rigidity

  maximum principle of Andersson, Galloway, Howard, 1993

  new splitting results, proofs

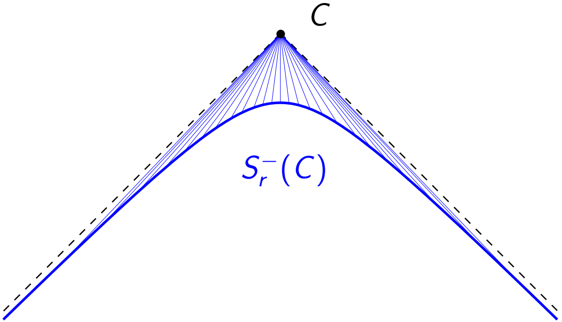


In Minkowski space, let $C = \{q\}$ and $S_r^- = S_r^-(C)$.

(1) $S_r^- = \partial I^-(S_r^-)$ is an acausal C^∞ hypersurface

(2) Each $x \in S_r^-$ joined to C by a maximal radial geodesic

(3) $H(S_r^-) = -\frac{n}{r}$

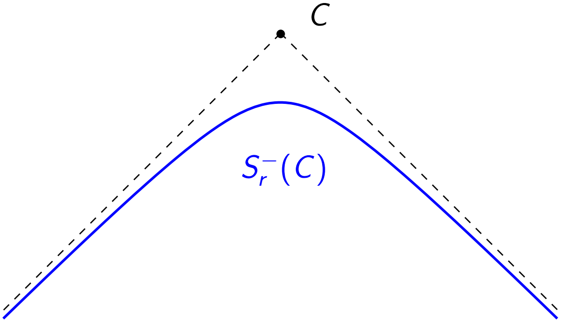


In Minkowski space, let $C = \{q\}$ and $S_r^- = S_r^-(C)$.

(1) $S_r^- = \partial I^-(S_r^-)$ is an acausal C^∞ hypersurface

(2) Each $x \in S_r^-$ joined to C by a maximal radial geodesic

(3) $H(S_r^-) = -\frac{n}{r}$

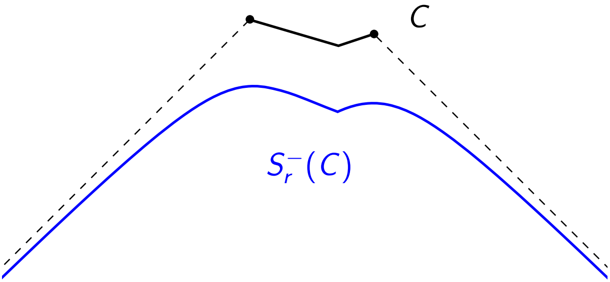


In Minkowski space, let $C = \{q\}$ and $S_r^- = S_r^-(C)$.

(1) $S_r^- = \partial I^-(S_r^-)$ is an acausal C^∞ hypersurface

(2) Each $x \in S_r^-$ joined to C by a maximal radial geodesic

(3) $H(S_r^-) = -\frac{n}{r}$

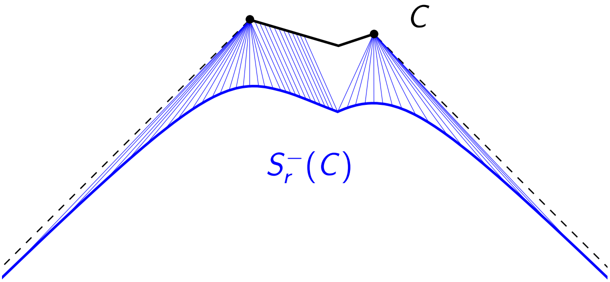


Let M be globally hyperbolic and C compact, let $S_r^- = S_r^-(C)$.

(1) $S_r^- = \partial I^-(S_r^-)$ is an acausal C^0 hypersurface

(2) Each $x \in S_r^-$ joined to C by a maximal radial geodesic

(3) $H(S_r^-) \geq -\frac{n}{r}$ in support sense, if $\mathrm{Ric}^T \geq 0$

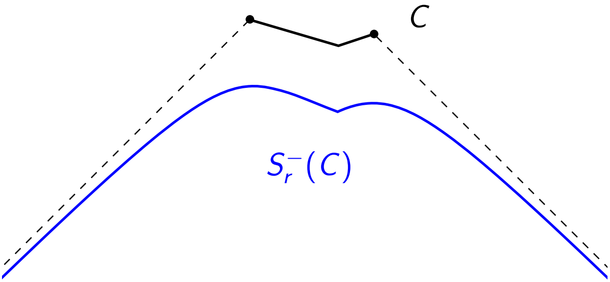


Let M be globally hyperbolic and C compact, let $S_r^- = S_r^-(C)$.

(1) $S_r^- = \partial I^-(S_r^-)$ is an acausal C^0 hypersurface

(2) Each $x \in S_r^-$ joined to C by a maximal radial geodesic

(3) $H(S_r^-) \geq -\frac{n}{r}$ in support sense, if $\mathrm{Ric}^T \geq 0$



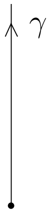
Let M be globally hyperbolic and C compact, let $S_r^- = S_r^-(C)$.

(1) $S_r^- = \partial I^-(S_r^-)$ is an acausal C^0 hypersurface

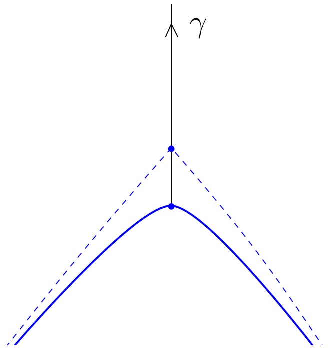
(2) Each $x \in S_r^-$ joined to C by a maximal radial geodesic

(3) $H(S_r^-) \geq -\frac{n}{r}$ in support sense, if $\mathrm{Ric}^T \geq 0$

Ray horosphere:

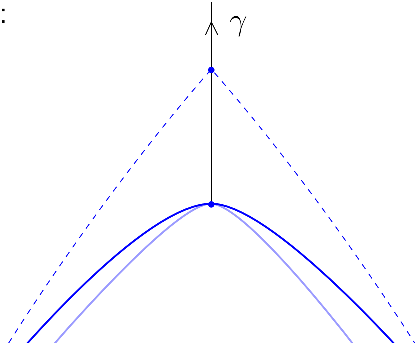


Ray horosphere:



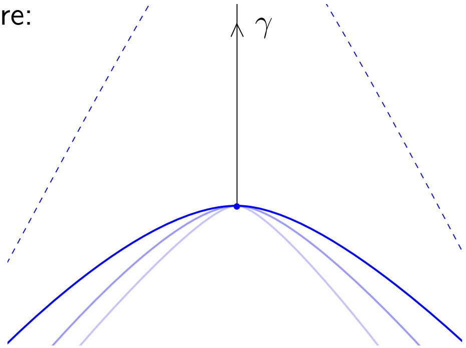
Look at past spheres $S_k^- := S_k^-(\gamma(k))$

Ray horosphere:



reverse triangle inequality $\implies I^-(S_k^-) \subset I^-(S_{k+1}^-)$

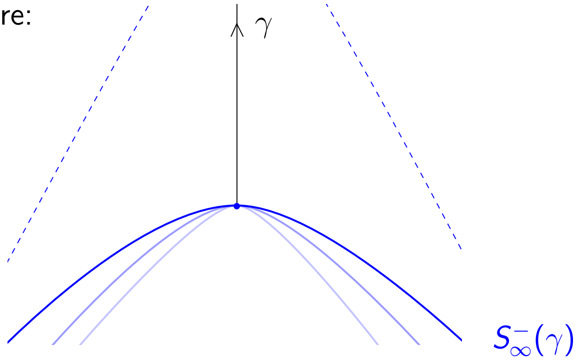
Ray horosphere:



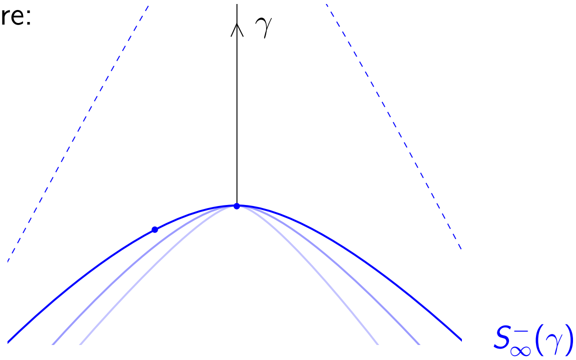
We define the *ray horosphere*

$$S_\infty^-(\gamma) := \partial\left(\bigcup_k I^-(S_k^-)\right)$$

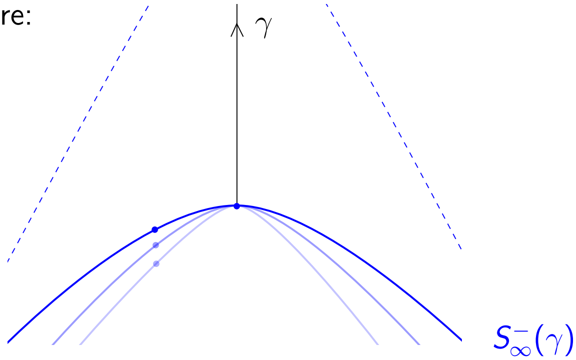
Ray horosphere:



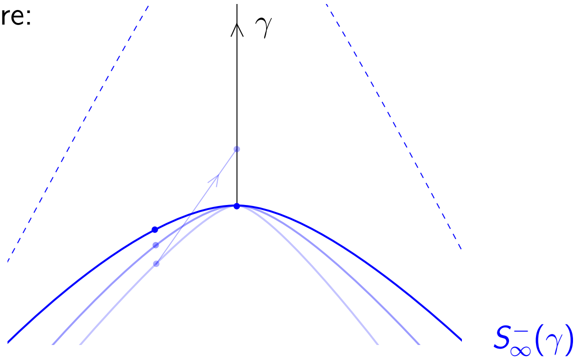
Ray horosphere:



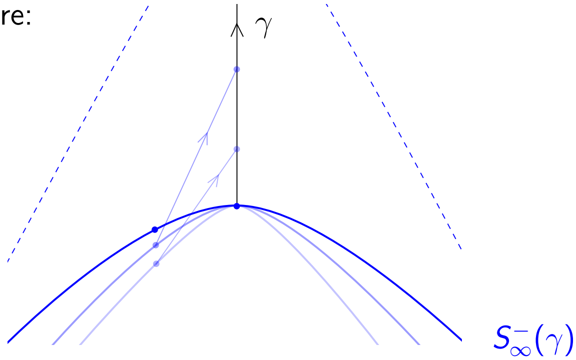
Ray horosphere:



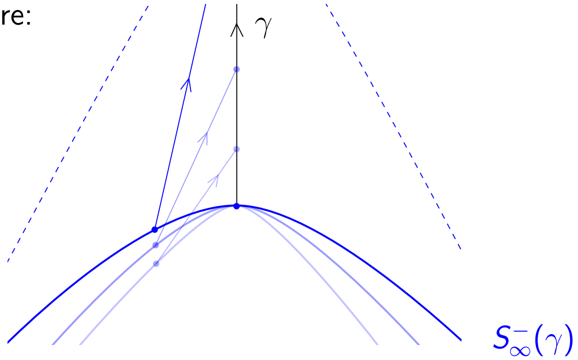
Ray horosphere:



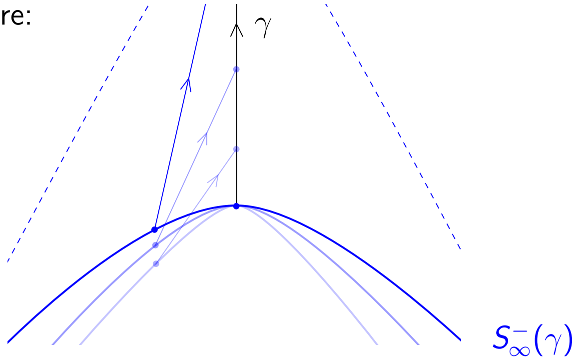
Ray horosphere:



Ray horosphere:

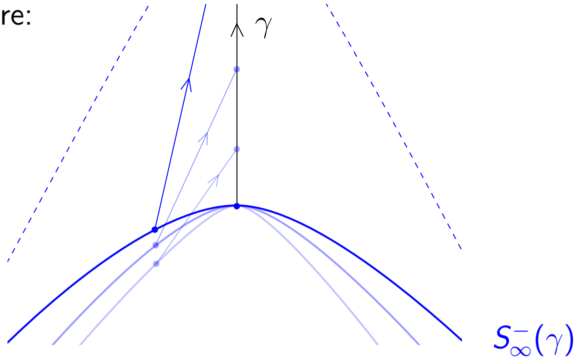


Ray horosphere:



$S^-_\infty(\gamma)$ has a maximal radial geodesic from each point x

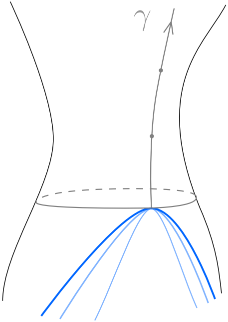
Ray horosphere:



$S^-_\infty(\gamma)$ has a maximal radial geodesic from each point x

joining $x \in S^-_\infty(\gamma)$ to the "center, $\gamma(\infty)$"

Let S be a compact Cauchy surface. Let γ be a future S-ray.

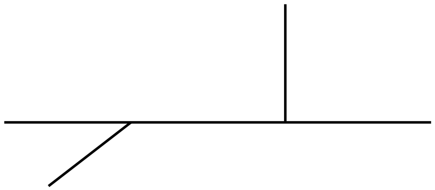


(1) $S^-_\infty(\gamma) = \partial I^-(S^-_\infty(\gamma))$ is an acausal C^0 hypersurface

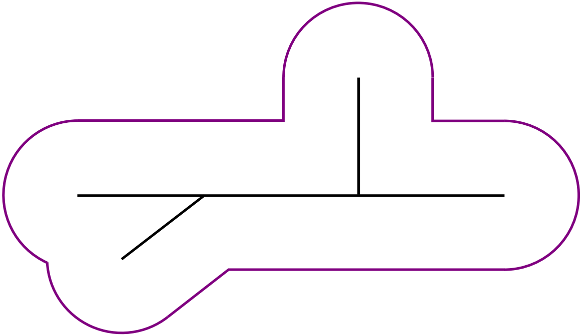
(2) $S^-_\infty(\gamma)$ has a future timelike $S^-_\infty(\gamma)$-ray from each point

(3) $H(S^-_\infty(\gamma)) \geq 0$ in support sense, if $\mathrm{Ric}^T \geq 0$

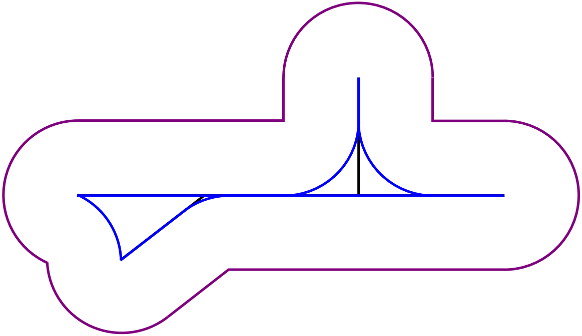
Rolling Pin Convexity



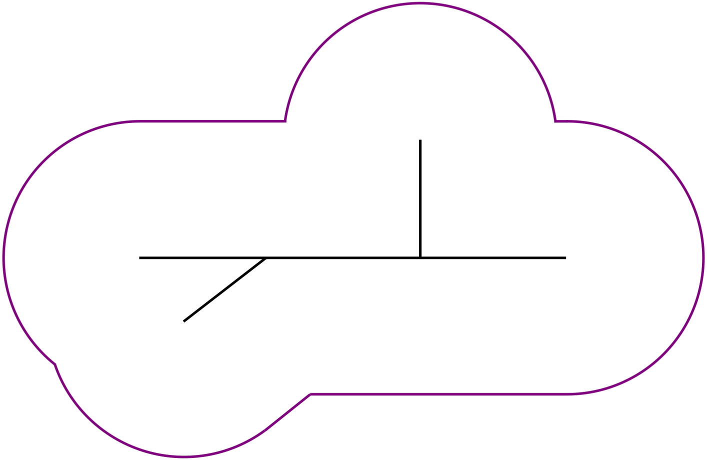
Rolling Pin Convexity



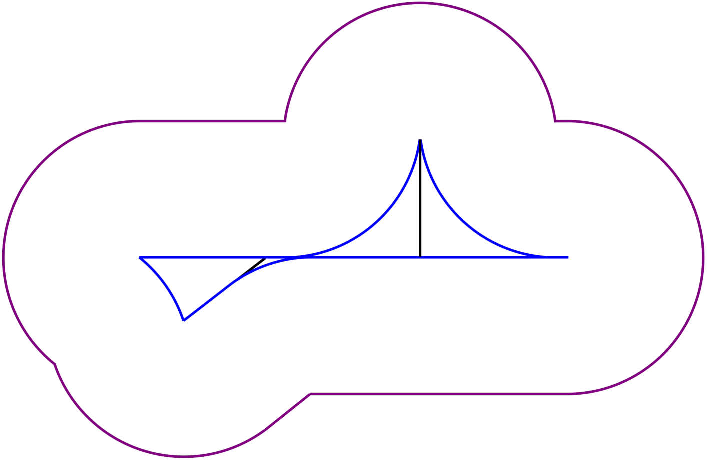
Rolling Pin Convexity



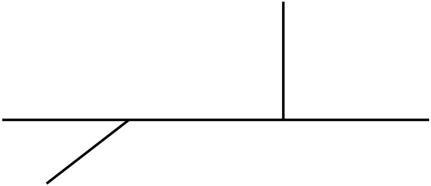
Rolling Pin Convexity



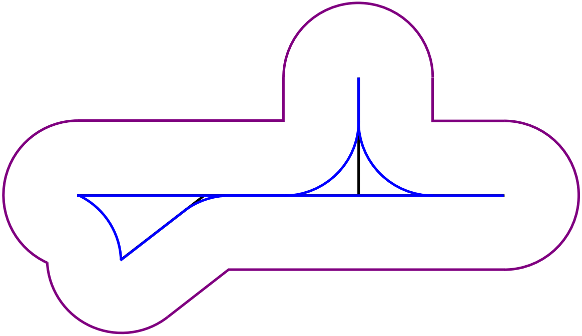
Rolling Pin Convexity



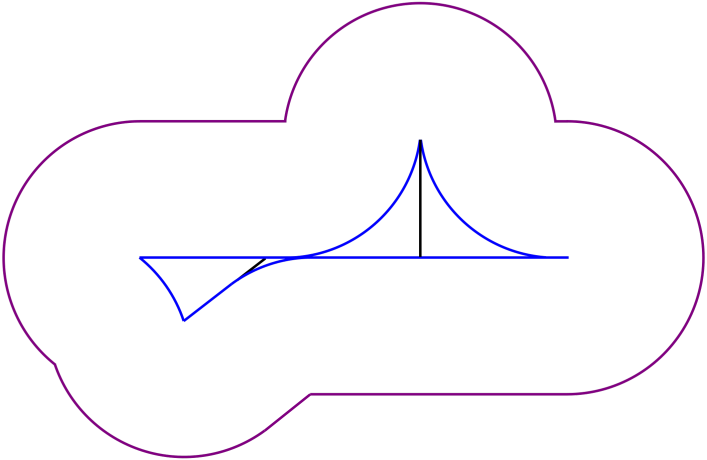
Rolling Pin Convexity



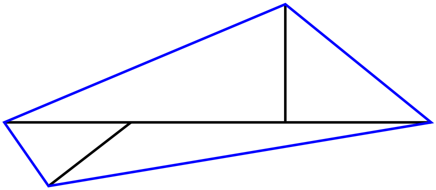
Rolling Pin Convexity



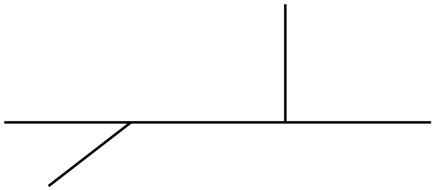
Rolling Pin Convexity



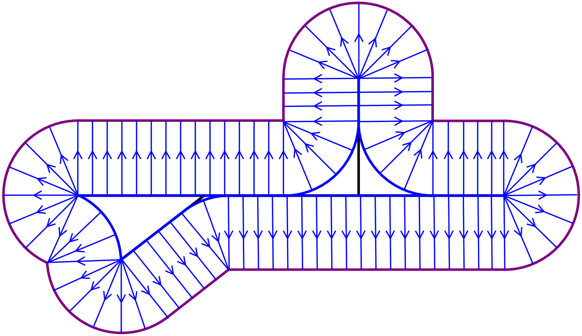
Rolling Pin Convexity



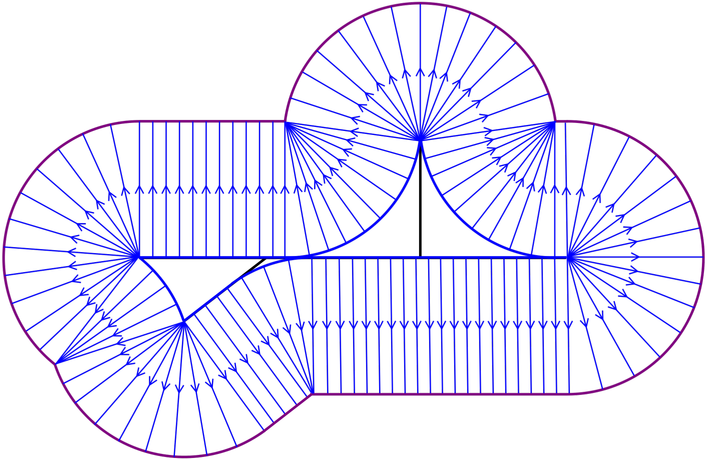
Rolling Pin Convexity



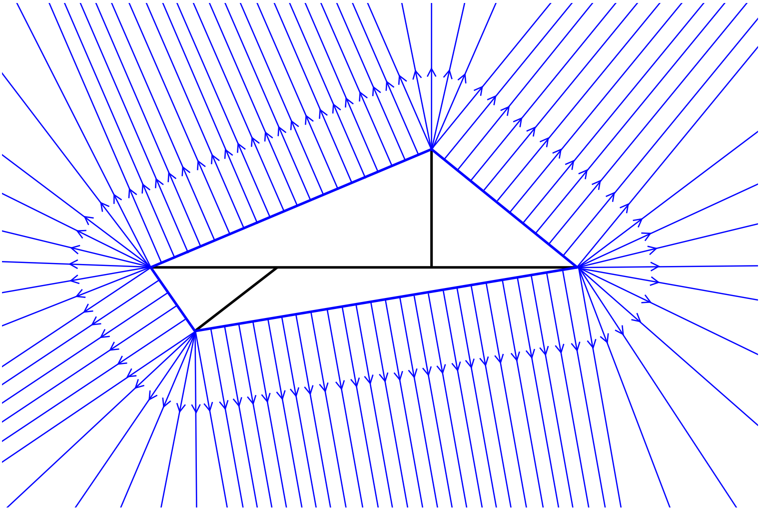
Rolling Pin Convexity



Rolling Pin Convexity



Rolling Pin Convexity



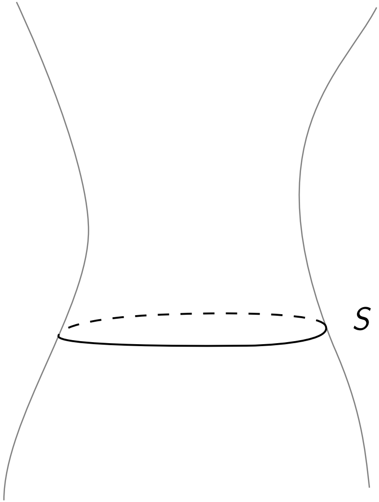
Cauchy horosphere:

S = a compact C^0 Cauchy surface

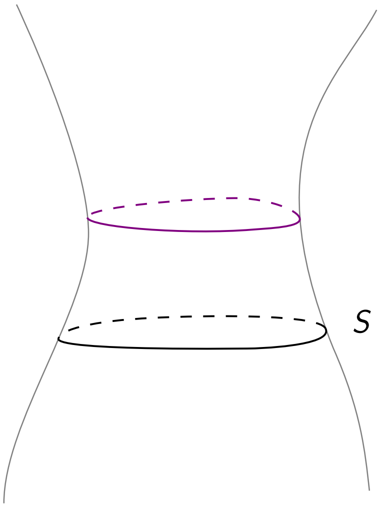
$S_k^+(S)$:= future sphere of radius k from S

$S_k^-(S_k^+(S))$:= past sphere of radius k from $S_k^+(S)$

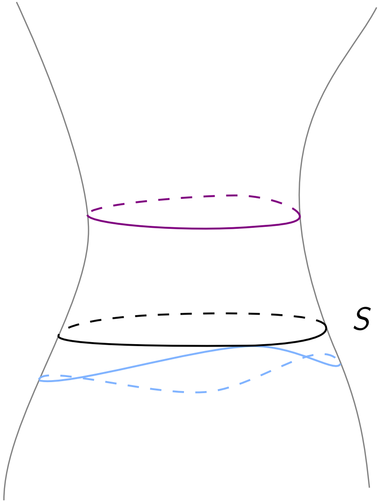
Cauchy horosphere:



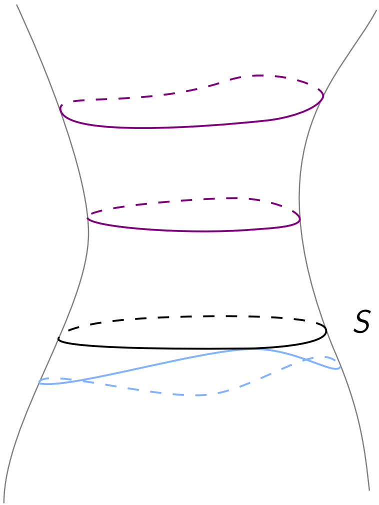
Cauchy horosphere:



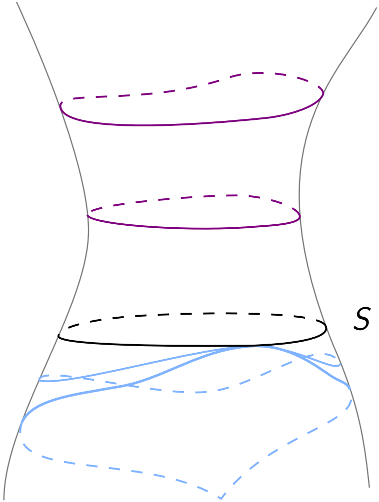
Cauchy horosphere:



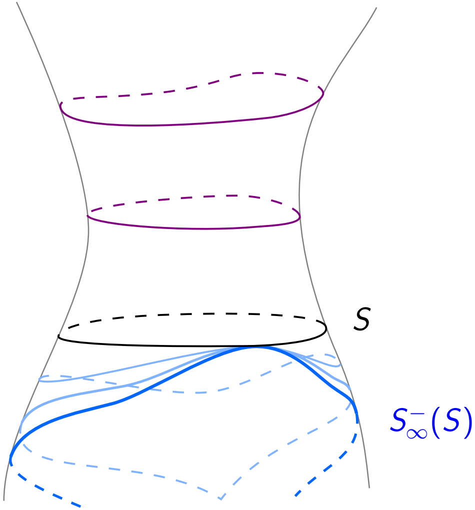
Cauchy horosphere:



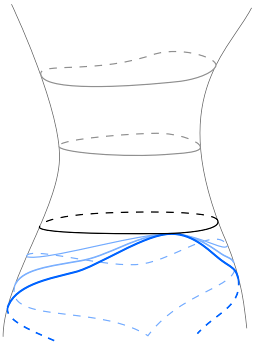
Cauchy horosphere:



Cauchy horosphere:



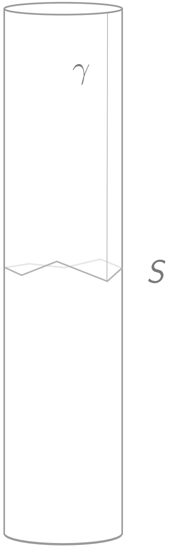
Let S be a compact Cauchy surface.



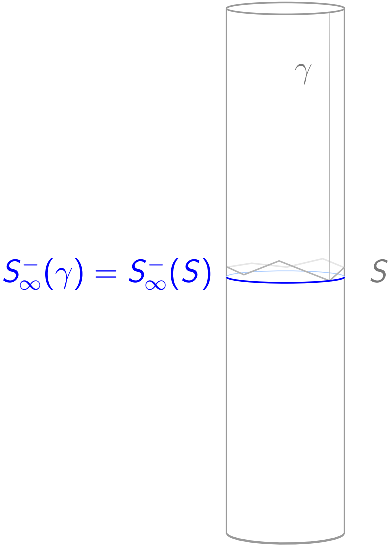
(1) $S^-_\infty(S) = \partial I^-(S^-_\infty(S))$ is an acausal C^0 hypersurface

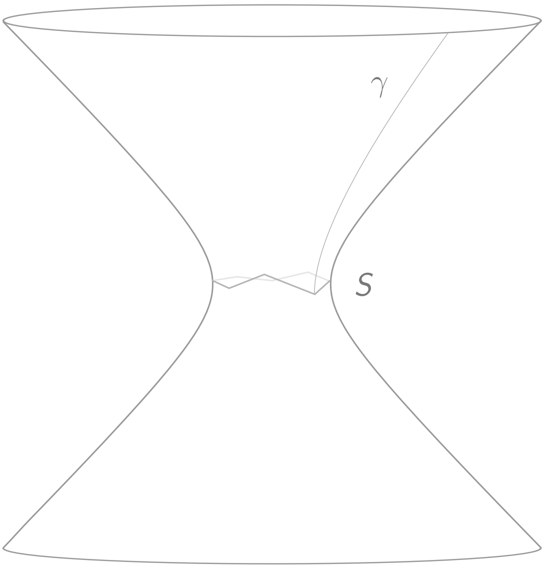
(2) $S^-_\infty(S)$ has a future timelike $S^-_\infty(S)$-ray from each point

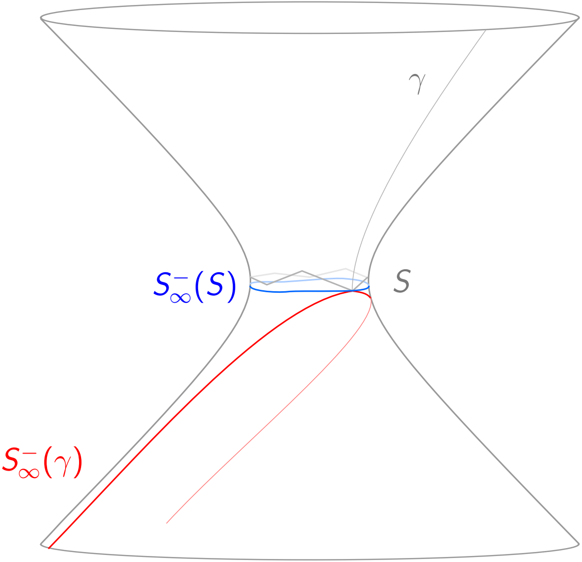
(3) $H(S^-_\infty(S)) \geq 0$ in support sense, if $\mathrm{Ric}^T \geq 0$



γ
S


γ
$S^-_\infty(\gamma) = S^-_\infty(S)$
S


γ
S



γ
S
$S^-_\infty(S)$
$S^-_\infty(\gamma)$

Theorem (Andersson, Galloway, Howard, 1998)

Suppose S_1 *and* S_2 *are* C^0 *spacelike hypersurfaces, with:*

- S_2 *is locally to the future of* S_1 *near* $p \in S_1 \cap S_2$
- S_1 *has support mean curvature* $\geq c$
- S_2 *has support mean curvature* $\leq c$

Then for some neighborhood U *of* p, $S_1 \cap U = S_2 \cap U$, *and this intersection is smooth with mean curvature* $H = c$.

Theorem (Andersson, Galloway, Howard, 1998)

Suppose S_1 and S_2 are C^0 spacelike hypersurfaces, with:

S_2 *is locally to the future of* S_1 *near* $p \in S_1 \cap S_2$

S_1 *has support mean curvature* $\geq c$

S_2 *has support mean curvature* $\leq c$

Then for some neighborhood U of p, $S_1 \cap U = S_2 \cap U$, and this intersection is smooth with mean curvature $H = c$.

In our context:

past horospheres S_∞^- have support mean curvature ≥ 0

future horospheres S_∞^+ have support mean curvature ≤ 0

Theorem (Basic Horosphere Rigidity, 2013)

Let M be a spacetime:

(1) globally hyperbolic

(2) $Ric(X, X) \geq 0$, for all timelike $X \in TM$

(3) timelike geodesically complete

Suppose that S_∞^- is a past horosphere which is future bounded, and admits a past S_∞^--ray, (e.g., if S_∞^- compact). Then S_∞^- is a smooth spacelike Cauchy surface, and

$$(M, g) \approx (\mathbb{R} \times S_\infty^-, -dt^2 + h)$$

Theorem (Bartnik under 'max-min', 2013)

Let M be as in the Bartnik conjecture, with

(1) compact Cauchy surface S

(2) $Ric(X, X) \geq 0$, for all timelike $X \in TM$

(3) timelike geodesically complete

If S satisfies the 'max-min' condition, then the past Cauchy horosphere $S_\infty^-(S)$ is a smooth spacelike Cauchy surface, and

$$(M, g) \approx (\mathbb{R} \times S_\infty^-(S), -dt^2 + h)$$

Theorem (Generalized Horosphere Rigidity, 2016)

Let M be a spacetime:

(1) globally hyperbolic

(2) $Ric(X, X) \geq 0$, for all timelike $X \in TM$

(3) timelike geodesically complete

If any two horospheres S_∞^- and S_∞^+ meet at a 'spacelike point' with $I^-(S_\infty^-) \cap I^+(S_\infty^+) = \emptyset$, then $S_\infty^- = S_\infty^+ =: S_\infty$ is a smooth geodesically complete spacelike hypersurface which splits M:

$$(M, g) \approx (\mathbb{R} \times S_\infty, -dt^2 + h)$$

Gives new proof of basic Lorentzian Splitting Theorem.

Theorem (Bartnik under 'horo-to-horo', 2016)

Let M be as in the Bartnik conjecture, with

(1) compact Cauchy surfaces

(2) $Ric(X,X) \geq 0$, for all timelike $X \in TM$

(3) timelike geodesically complete

If M has Cauchy surfaces S_1 and S_2 such that

$$J^-(S^-_\infty(S_1)) \cap J^+(S^+_\infty(S_2)) \neq \emptyset,$$

Then M splits;

$$(M^{n+1}, g) \approx (\mathbb{R} \times \Sigma^n, -dt^2 + h)$$

Subsumes previous partial results.

Theorem (Bartnik under conformal symmetry+vacuum, 2018)

Let M be a spacetime with

(1) compact Cauchy surfaces

(ii) $Ric(X, X) \equiv 0$

(3) timelike geodesically complete

If M admits a timelike conformal Killing field X, then M splits,

$$(M^{n+1}, g) \approx (\mathbb{R} \times \Sigma^n, -dt^2 + h)$$

and X must be Killing.

Theorem (Bartnik under conformal symmetry+vacuum, 2018)

Let M be a spacetime with

(1) compact Cauchy surfaces

(ii) $Ric(X, X) \equiv 0$

(3) timelike geodesically complete

If M admits a timelike conformal Killing field X, then M splits,

$$(M^{n+1}, g) \approx (\mathbb{R} \times \Sigma^n, -dt^2 + h)$$

and X must be Killing.

1986: Eardley, Isenberg, Marsden, Moncrief

2018: Costa e Silva, Flores, Hererra: relaxed to $\mathrm{Ric}^T \geq 0$, assuming completeness of X, (or other technical condition)

Theorem (Future Asymptotically dS Rigid Singularity, 2013)

Let M be a spacetime with

(1) compact Cauchy surfaces

(2) $Ric(X, X) \geq -n$, for all timelike unit $X \in TM$

(3) future timelike geodesically complete

Suppose S is a Cauchy surface such that $S_k^+(S)$ has support mean curvature $\geq a_k$, with $min\{n, a_k\} = n + o(e^{-2k})$. If $S_\infty^-(S)$ admits a past $S_\infty^-(S)$-ray, then either $S_\infty^-(S)$ admits a past incomplete $S_\infty^-(S)$-ray, or $S_\infty^-(S)$ is a smooth, compact spacelike Cauchy surface with mean curvature $H = n$, and

$$(M, g) \approx (\mathbb{R} \times S_\infty^-(S), -dt^2 + e^{2t} h)$$

Theorem (Future Asymptotically dS Rigid Singularity, 2013)

Let M be a spacetime with

(1) compact Cauchy surfaces

(2) $Ric(X,X) \geq -n$, for all timelike unit $X \in TM$

(3) future timelike geodesically complete

Suppose S is a Cauchy surface such that $S_k^+(S)$ has support mean curvature $\geq a_k$, with $\min\{n, a_k\} = n + o(e^{-2k})$. If $S_\infty^-(S)$ admits a past $S_\infty^-(S)$-ray, then either $S_\infty^-(S)$ admits a past incomplete $S_\infty^-(S)$-ray, or $S_\infty^-(S)$ is a smooth, compact spacelike Cauchy surface with mean curvature $H = n$, and

$$(M, g) \approx (\mathbb{R} \times S_\infty^-(S), -dt^2 + e^{2t}h)$$

2000: Cai, Galloway

2002: Andersson, Galloway

Thank You!

speaker as a grad student at U Miami