Rigidity of asymptotically $AdS_2 \times S^2$ spacetimes

(joint with G. Galloway)

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- First remarks
  - $AdS_2 =$ universal cover, in particular causally simple (i.e. causal and $J^{\pm}(p)$ closed)
  - Whenever we assume the null energy condition, i.e., $\mathrm{Ric}(X, X) \geq 0$ for all null vectors X, it would actually be sufficient to assume

$$\int_0^\infty \mathrm{Ric}(\eta'(s), \eta'(s)) ds \geq 0$$

    for all future or past complete null rays η.

Properties of exact $AdS_2 \times S^2$

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- Family of wider/narrower metrics $\mathring{g}_\alpha$ $(\alpha \in \mathbb{R})$ with $\mathring{g}_\alpha = -\alpha \cosh(x)^2 dt^2 + dx^2 + d\Omega^2$

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For future use: Define $M(r) := (M_1 \cup M_2) \cap \{|x| \geq r\}$

Main results

Theorem (Galloway, G., 2018)

Let (M, g) be an asymptotically $AdS_2 \times S^2$ spacetime satisfying the null energy condition (NEC). Then

1. *(M, g) possesses two continuous transverse foliations by smooth totally geodesic null hypersurfaces $\{N_u\}_{u \in \mathbb{R}}, \{\hat{N}_v\}_{v \in \mathbb{R}}$ and $N_u, \hat{N}_v \approx \mathbb{R} \times S^2$*
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Theorem (Galloway, G., 2018)

Let (M, g) be an asymptotically $AdS_2 \times S^2$ spacetime satisfying the NEC. If $\nabla \mathrm{Ric} = 0$, then (M, g) is globally isometric to $AdS_2 \times S^2$.

Construction of the foliation by null hypersurfaces

- Get control over asymptotics: $\forall r \in [a, \infty)\ \exists \alpha_r < 1, \beta_r > 1$ st. $\mathring{g}_{\alpha_r} \prec g \prec \mathring{g}_{\beta_r}$ on $M(r)$ and $\alpha_r, \beta_r \to 1$

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- This is a continuous codimension one foliation!

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- the $S_{u,v}$ move outward as $v \searrow u$
- So, since $K_{u,v}$ is constant in v, it must equal 1 and $S_{u,v} \cong (S^2, d\Omega^2)$

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- N_u totally geodesic $\iff$ null second fundamental form vanishes $\implies g|_{S_{u,v_1}} = g|_{S_{u,v_2}}$
- Lemma: $\forall \varepsilon > 0 \exists r$ st. $\forall S_{u,v} \subset M(r)$: $1 - \varepsilon \leq K_{u,v} \leq 1 + \varepsilon$ and
- the $S_{u,v}$ move outward as $v \searrow u$
- So, since $K_{u,v}$ is constant in v, it must equal 1 and $S_{u,v} \cong (S^2, d\Omega^2)$
- $\rightsquigarrow$ continuous co-dimension two foliation by totally geodesic round 2-spheres.

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- Question: Is the above true in general? How to prove it?

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- See where this leads! (Examples, completenes of null $\bar{g}$-geodesics ending at $\mathcal{J}$, topological structure/rigidity, relation to coordinate-asymptotics-definition, ...)

Happy Gregfest!