Spacetime Topology and Singularities

Daniel Pollack

University of Washington

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In honor of Greg Galloway
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The aim of this talk is to weave together a number of intersecting threads connecting the topology of spacetime with the presence or absence of Marginally Outer Trapped Surfaces (MOTS) in initial data sets. This is an area of research which has been guided by the results of Greg Galloway and his collaborators.

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- Introduction: Perspectives on Spacetime topology as offered by the Cauchy problem.
- Topology can be arbitrary but nontrivial topology leads to singularities (geodesic incompleteness).
- Topological Censorship from both the spacetime and initial data perspectives, MOTS and singularity theorems.
- The topology of black hole horizons (outermost MOTS) in higher dimensions.
- The topology of the exterior region (outside any black holes) of initial data sets in higher dimensions.
- Recent results on cosmological spacetimes and CMC Cauchy surfaces

Global hyperbolicity and Cauchy Surfaces

Focus on globally hyperbolic spacetimes (connected, time-oriented Lorentz manifolds).

Definition

1. (M, g) *is* **globally hyperbolic** *if it is* **strongly causal** *and for every pair* $p < q$ *the "causal diamonds"* $J(p, q) = J^+(p) \cap J^-(q)$ *are compact.*
2. *A* **Cauchy surface** S *is an achronal subset of* M *which is met by every inextendible causal curve in* M*.*

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The existence of Cauchy surfaces and global hyperbolicity are closely connected.

Theorem (Geroch)

Let M be a spacetime.

1. *If M is globally hyperbolic if and only if it admits a Cauchy surface.*
2. *If V is a Cauchy surface for M then M is homeomorphic to $\mathbb{R} \times V$.*

In particular, for globally hyperbolic spacetimes, the topology of a Cauchy surface V determines the topology of the entire spacetime.

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Are there restrictions on the topology of V? What properties of V lead to M having singularities (causal geodesic incompleteness)?

The Einstein Field Equations

Regard these as an (evolutionary) PDE system for an spacetime (M, g):

$$Ric(g) - \frac{1}{2}R(g)g + \Lambda g = 8\pi T \tag{1}$$

Here

- $T = T_{ab}$ is the stress-energy tensor, which encodes the non-gravitational physics (e.g. electromagnetic fields, scalar fields, etc). In practice this leads to other equations coupled to gravity (e.g. Maxwell's equations, etc)
- This is in units where $G = c = 1$.
- Λ is the cosmological constant ($\Lambda = 0$ in this talk).

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At this level, physics is imposed (in the most naive sense) via "**energy conditions**".

- **Null energy condition (NEC)**: $Ric(X, X) \geq 0$ for all **null** vectors X. (Note: For null X, $Ric(X, X) = T(X, X)$.)
- **Strong energy condition**: $Ric(X, X) \geq 0$ for all **timelike** vectors X
- **Dominant energy condition (DEC)** $G(X, Y) \geq 0$ for all future directed causal vectors X, Y where $G = Ric - \frac{1}{2}R(g)g$ is the Einstein tensor .

The Einstein Constraint Equations

In order for us to view the Einstein equations as an evolutionary system of PDE, we need to specify initial data on a hypersurface. The classical approach to this comes from considering the induced data on a spacelike hypersurface $V \hookrightarrow M$.

If (M, g) is a spacetime satisfying the Einstein field equations and $V \hookrightarrow M$ is a spacelike hypersurface with induced Riemannian metric h and second fundamental form K, then the Gauss and Codazzi equations tell us that

$$\begin{aligned} R(h) - |K|_h^2 + (\mathrm{tr}_h K)^2 &= 16\pi T_{00} = 2\rho && \text{(Hamiltonian constraint)} \qquad (2) \\ \mathrm{div} K - \nabla(\mathrm{tr}_h K) &= 8\pi T_{0i} = J. && \text{(Momentum constraint)} \qquad (3) \end{aligned}$$

We regard the triple (V, h, K) as an **initial data set**. The scalar function ρ is the *local mass density* and the vector field J is the *local current density*.

The Dominant Energy Condition (DEC) can be expressed at this initial data level as the inequality:

$$\rho \geq |J|.$$

The Cauchy Problem

In 1952, Yvonne Choquet-Bruhat established the existence of a local in time solution of the vacuum Einstein equations, $Ric(g) = 0$, thus showing that the constraint equations are sufficient as well as necessary conditions for a solution to exist. A "global" evolution result was obtained almost 17 years later.

Theorem (Choquet-Bruhat & Geroch, 1969)

Given an initial data set $(V; h, K)$ *satisfying the vacuum constraint equations there exists a unique, globally hyperbolic, maximal, spacetime* (M, g) *satisfying the vacuum Einstein equations* $Ric(g) = 0$ *where* $V \hookrightarrow M$ *is a Cauchy surface with induced metric* h *and second fundamental form* K*. (The solution is maximal in the sense that any other such solution is a subset of* (M, g)*.)*

This still leaves open the many central and fundamentally important questions concerning global existence (e.g. stability results and the formation of singularities).

Nontrivial topology and singularities

In the context of (physcially reasonable) globally hyperbolic spacetimes $M \cong \mathbb{R} \times V$, it is worth first noting that the topology of V can be arbitrary.

When V is an arbitrary closed manifold there are vacuum solutions to the Einstein Constraint equations with $\tau = \mathrm{tr} K$ an arbitrary positive constant:

> *Let h be a metric constant scalar curvature $R = -n(n-1)$ (Kazdan-Warner, 1975) and set $K = h$, this yields $\tau = n$, now rescale as desired.* $\square$

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Using this and a gluing construction, we showed

Theorem (Isenberg, Mazzeo, P., 2003)

Let V be any closed n-dimensional manifold, and $p \in V$ arbitrary. Then $V \setminus \{p\}$ admits an asymptotically Euclidean initial data set satisfying the vacuum constraint equations.

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On the other hand, in the mid-70's it was observed that, for isolated systems, nontrivial fundamental group leads to singularities (in the sense of Penrose).

Theorem (Gannon 1975, Lee 1976)

Let (M, g) be a globally hyperbolic spacetime which satisfies the NEC, and which contains a Cauchy surface V which is regular near infinity. If V is not simply connected then M is future null geodesically incomplete.

Topological Censorship

We now turn to specifically to isolated systems, i.e. Asymptotically flat spacetimes.

Weak Cosmic Censorship Conjecture: generically the process of gravitational collapse leads to the formation of an event horizon which hides the singularities from view.

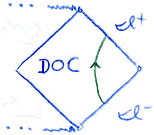
Topological Censorship: nontrivial topology is hidden behind the event horizon, and the DOC - the region exterior to all black holes (and white holes) - has simple topology.

These both concern black hole spacetimes. We consider spacetimes admitting a regular null infinity

$$\mathscr{I} = \mathscr{I}^+ \cup \mathscr{I}^-, \qquad \mathscr{I}^\pm \approx \mathbb{R} \times S^2$$

a neighborhood U of which is simply connected. The Domain of Outer Communications is then defined in terms of the past and future of null infinity.

$$D = DOC = I^-(\mathscr{I}^+) \cap I^+(\mathscr{I}^-)$$



Definition

(M, g) is a **black hole spacetime** if $BH = M \setminus D$ is nonempty. This is the black hole region, and it's boundary is the event horizon.

Topological Censorship and the topology of the DOC

Theorem (Friedman, Schleich and Witt 1993)

Let (M, g) be a globally hyperbolic, AF spacetime satisfying the NEC. Then every causal curve from $\mathscr{I}^-$ to $\mathscr{I}^+$ can be deformed (with endpoints fixed) to a curve lying in the simply connected neighborhood U of $\mathscr{I}$.

The FSW Topological Censorship theorem does not give direct information about the topology of the DOC.

In 1994 Chruściel and Wald used FSW to prove that for stationary black hole spacetimes, the DOC is simply connected. Subsequent to that Galloway showed that the simple connectivity of the DOC holds in general.

Theorem (Galloway, 1995)

Let (M, g) be an asymptotically flat spacetime such that a neighborhood of $\mathscr{I}$ is simply connected, Suppose that the DOC is globally hyperbolic and satisfies the NEC. Then the DOC is simply connected.

Thus, topological censorship can be taken as the statement that the Domain of Outer Communications is simple connected.

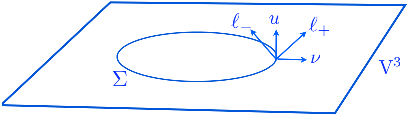
In 1999, Galloway, Schleich, Witt and Woolgar extended these ideas to the case of a negative cosmological constant.

Singularities & Topological Censorship via initial data

Penrose Singularity Theorem: In certain settings, conditions on an initial data set that imply the existence of a **trapped surface** lead to geodesic incompleteness (singularities).

$$(M^4, g) = \text{4-dim spacetime}$$
$$(V, h, K) = \text{3-dim initial data set in } (M^4, g)$$
$$\Sigma = \text{closed 2-sided surface in } V$$

Σ admits a smooth unit normal field ν in V.



$\ell_+ = u + \nu$ future directed outward null normal

$\ell_- = u - \nu$ future directed inward null normal

Trapped and marginally outer trapped surfaces (MOTS)

With respect to ℓ_+ and ℓ_- we define the null expansion scalars, θ_+, θ_-:

$$\theta_\pm = \mathrm{tr}\chi_\pm = \mathrm{div}_\Sigma \ell_\pm$$

Σ is a **trapped surface** if both $\theta_- < 0$ and $\theta_+ < 0$. This signals the presence of a strong gravitational field leading to collapse and the formation of a black hole.

Focusing on the outward null normal, Σ is called a **marginally outer trapped surface** (MOTS) if $\theta_+ = 0$. In terms of initial data (V, h, K),

$$\theta_\pm = \mathrm{tr}_\Sigma K \pm H \,,$$

where $H =$ mean curvature of Σ within V. Thus, we see that in the time-symmetric (Riemannian) case, when $K \equiv 0$, MOTS correspond to minimal surfaces.

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There is an even more general object in an initial data set that gives rise to a Penrose-type singularity theorem, which we refer to as an immersed MOTS.

Definition

A subset $\Sigma \subset V$, in an initial data set (V, h, K), is an immersed MOTS if there exists a finite cover $p : \tilde{V} \to V$ and a closed, embedded MOTS $\tilde{\Sigma}$ in $(\tilde{V}, p^*h, p^*K)$ such that $p(\tilde{\Sigma}) = \Sigma$.

Singularity theorems

Theorem (Penrose, 1965)

Suppose (M, g) *is globally hyperbolic with non-compact Cauchy surface* V *and*

(i) M *obeys the Null Energy Condition (NEC):* $\mathrm{Ric}(X, X) \geq 0$ *for all null vectors* X*.*

(ii) V *contains a trapped surface* Σ*.*

Then at least one of the future directed null normal geodesics to Σ *must be incomplete.*

With Eichmair and Galloway, we showed that this theorem remains true if, instead of a trapped surface, we only have a MOTS (or immersed MOTS) provided we assume that the *generic condition* holds on each future and past inextendible null normal geodesic, i.e there is a point p on each null geodesic η and a vector X at p orthogonal to η' such that $R(X, \eta', \eta', X) \neq 0$.

Singularity theorems

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(i) *M obeys the Null Energy Condition (NEC): $\mathrm{Ric}(X, X) \geq 0$ for all null vectors X.*

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Chruściel & Galloway (2013) improved this by replacing the generic condition with either

(a) Σ is a *strictly stable* MOTS,

(b) the null second fundamental form χ_+ of Σ is not identically zero.

More generally, they showed that (for a noncompact Cauchy surface) the presence of a MOTS leads to null geodesic incompleteness if not for the given initial data, then for an arbitrarly small perturbation of the data near S which makes S outer trapped, provided the initial data set has no local Killing Initial Data (KIDs) near S. This later condition asserts that there are no local solutions to the adjoint of the linearization of the constraints map near S. This was shown to be generically satisfied by Bieg, Chruściel & Schoen in 2004.

Initial data version of Gannon-Lee singularity theorem

Theorem (Eichmair, Galloway and P., 2013)

Let (V, h, K) be a 3-dimensional asymptotically flat initial data set. If V is not diffeomorphic to $\mathbb{R}^3$ then V contains an immersed MOTS.

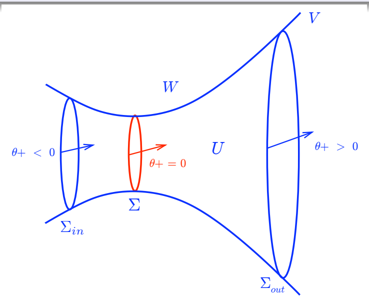
Thus, if V is not $\mathbb{R}^3$, spacetime is singular, from the initial data point of view.

- This theorem may be viewed as a non-time-symmetric version of a theorem of Meeks-Simon-Yau (1982), which implies that an asymptotically flat 3-manifold (no curvature conditions!) that is not diffeomorphic to $\mathbb{R}^3$ contains an *embedded* stable minimal sphere or projective plane.
- Note that there are no energy conditions imposed here. If one assumes that the DEC holds on (V, h, K) then the conclusion can be refined to: If V is not diffeomorphic to $\mathbb{R}^3$ then V contains an immersed *spherical* MOTS. (We will return to the question of the topology of the horizon shortly.)
- The proof relies on existence results for MOTSs together with our understanding of the topology of three-manifolds.

Existence of MOTSs (Née: Miami Waves Conference 2004)

Theorem (Schoen; Andersson & Metzger; Eichmair)

Let W be a connected compact manifold-with-boundary in an initial data set (V, h, K). Suppose, $\partial W = \Sigma_{in} \cup \Sigma_{out}$, such that Σ_{in} is outer trapped ($\theta_+ < 0$) and Σ_{out} is outer untrapped ($\theta_+ > 0$). Then there exists a smooth compact MOTS, Σ, in W that separates Σ_{in} from Σ_{out}.



The proof is based on inducing blow-up of Jang's equation, cf., survey article by Andersson, Eichmair, Metzger,

Initial data version of Gannon-Lee singularity theorem

Sketch of proof: Assume V contains no immersed MOTSs and show it is diffeomorphic to $\mathbb{R}^3$. A key point here is to exploit the fact that the large coordinate spheres in an asymptotically flat end are either outer trapped or outer untrapped, and therefore produce the barriers needed for the existence of MOTS.

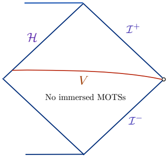
- V must have only one end (else it would contain a MOTS ※).
- V must be orientable (else, by working in the oriented double cover, it would contain an immersed MOTS ※).

  Hence, $V = \mathbb{R}^3 \# N$, where N is a compact orientable 3-manifold.

- N must be simply connected.
  - If N were not simply connected, since $\pi_1(N)$ is residually finite, N admits a finite nontrivial cover. Hence V would admit a finite cover $\tilde{V}$ with more than one end, again leading to the presence of an immersed MOTS in V ※.
- Thus, by the resolution of the Poincaré conjecture, N is diffeomorphic to S^3. Consequently, V is diffeomorphic to $\mathbb{R}^3$. □

Initial data version of topological censorship

Consider the initial data setting for topological censorship:



- The initial data manifold V represents an asymptotically flat spacelike slice in the DOC whose boundary ∂V corresponds to a cross section of the event horizon.
- This cross section is assumed to be represented by a MOTS.
- We assume further that there are no immersed MOTSs in $V \setminus \partial V$.

Theorem (Eichmair, Galloway & P., 2013)

Let (V, h, K) be a 3-dimensional asymptotically flat initial data set such that V is a manifold-with-boundary, whose boundary ∂V is a compact MOTS. If all components of ∂V are spherical and if there are no immersed MOTS in $V \setminus \partial V$, then V is diffeomorphic to $\mathbb{R}^3$ minus a finite number of open balls.

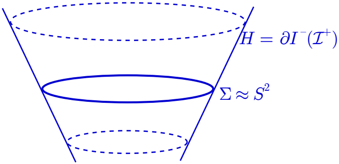
The proof is very similar to the proof of the initial data version of Gannon-Lee, but added care is needed in dealing with the presence of the MOTS boundary components.

Horizon Topology & Black holes in $3+1$ dimensions

A basic step in the proof of the uniqueness of the Kerr solution is Hawking's theorem on the topology of black holes in $3+1$ dimensions.

Theorem (Hawking's black hole topology theorem)

Suppose (M, g) is a $(3+1)$-dimensional asymptotically flat, stationary, black hole spacetime obeying the dominant energy theorem. Then cross sections of the event horizon are topologically 2-spheres.



Black holes in higher dimensional spacetimes

Does black hole uniqueness hold in higher dimensions?

- Inspired by developments in string theory, in 1986, Myers and Perry constructed natural higher dimensional generalizations of the Kerr solution, which, in particular, have spherical horizon topology.
- In 2002, Emparan and Reall discovered a remarkable example of a $4 + 1$ dimensional asymptotically flat stationary vacuum black hole spacetime with horizon topology $S^2 \times S^1$ (the black ring).

Thus in higher dimensions, black hole **uniqueness does not hold** and **horizon topology need not be spherical**.

This result lead to a great deal of interest in the topology of higher dimensional black holes, and the efforts to produce other examples.

Basic Question: What horizon topologies are allowed in higher dimensions?

Topology of MOTS in higher dimensions

A MOTS Σ in (V, h, K) is said to be outermost provided there are no outer trapped ($\theta_+ < 0$) or marginally outer trapped ($\theta_+ = 0$) surfaces outside of and homologous to Σ.

Theorem (Galloway & Schoen, 2006)

Let (V^n, h, K), $n \geq 3$, be an initial data set in a spacetime obeying the DEC. If Σ^{n-1} is an outermost MOTS in V^n then (apart from certain exceptional circumstances) Σ^{n-1} must admit a metric of positive scalar curvature.

Note that in 2001, Cai-Galloway obtained similar results in the time-symmetric case. In the Galloway-Schoen result there are no restrictions (other than the DEC) on the extrinsic curvature.

In the "exceptional circumstances" various geometric quantities vanish, e.g. $\chi_+ = 0$, $T(u, \ell_+)|_\Sigma = 0$, $\mathrm{Ric}_\Sigma = 0$. In particular, the theorem does not rule out the possibility of a vacuum black hole spacetime with toroidal horizon topology.

Theorem (Galloway, 2008)

Let (V^n, h, K), $n \geq 3$, be an initial data set in a spacetime obeying the DEC. If a MOTS Σ^{n-1} does not admit a metric of positive scalar curvature, then it cannot be outermost.

Topology of MOTS in higher dimensions, II

This 2008 result requires that the DEC hold in the spacetime, not just on the initial data set. More recently Galloway showed how to obtain the optimal, purely initial data, result

Theorem (Galloway, 2017)

Let (V^n, h, K), $n \geq 3$, be an initial data set obeying the DEC, $\mu \geq |J|$. If Σ^{n-1} is an outermost MOTS in (V^n, h, K) then Σ^{n-1} admits a metric of positive scalar curvature.

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Recall that the existence of a metric of positive scalar curvature on a smooth, closed orientable manifold has topological implications.

- In dimension 2, if Σ has positive scalar curvature than $\Sigma = S^2$ by Gauss-Bonnet.
- Schoen-Yau (1979) & Gromov-Lawson (1980) showed that if Σ^3 admits a metric of positive scalar curvature, then Σ^3 is diffeomorphic to a finite connected sum of spherical space-forms (finite quotients of S^3) or $S^2 \times S^1$ (après Poincaré).
- They also showed that the existence of a metric of positive scalar curvature on a closed orientable manifold is preserved under surgeries of codimension $q \geq 3$.
- Simply connected manifolds Σ^n, $n \geq 5$ which admit metrics of positive scalar curvature are understood by the work of Stolz (1990). Here the problem becomes a question of homotopy theory, via the vanishing of a certain characteristic number.
- Schoen-Yau (2017) If Σ^n is any closed manifold, then $\Sigma^n \# T^n$ does not have a metric of positive scalar curvature (previously known only when $n \leq 8$ or Σ^n spin).

Topology of MOTS in higher dimensions, III

Question: Is the existence of a metric of positive scalar curvature the only topological requirement for a manifold to appear as an outermost MOTS in an initial data set obeying the DEC?

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This question is largely unresolved. Nonetheless, Dahl & Larsson have recently used geometric surgery techniques to constructed outermost MOTSs in time-symmetric initial data sets, with very general topologies.

Theorem (Dahl & Larsson, 2016)

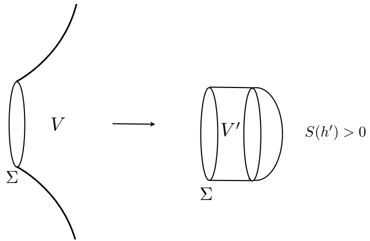
Let $\Sigma \subset \mathbb{R}^n$ be a compact embedded smooth submanifold of codimension at least 3. Then there is an asymptotically Euclidean metric on $\mathbb{R}^n$ with non-negative scalar curvature for which the outermost apparent horizon is diffeomorphic to the unit normal bundle.

Initial data sets with horizons - Topology of the exterior region in higher dimensions

Theorem (Andersson, Dahl, Galloway, P., 2015)

Let (V^n, h, K) be an n-dimensional initial data set, $3 \leq n \leq 7$, satisfying the DEC, $\rho \leq |J|$, having MOTS boundary $\Sigma = \partial V$, and assume there are no MOTS in the "exterior region" $V \setminus \partial V$. Then the compactification V' admits a metric h' of positive scalar curvature, $S(h') > 0$, such that

- *the metric on Σ induced from h' is conformal to (a small perturbation of) the metric on Σ induced from h,*
- *h' is a product metric in a neighborhood of Σ.*



Topology of Initial data sets with horizons

- As in the horizon case, one can then apply certain index theory obstructions, and minimal surface theory obstructions, to obtain restrictions on the topology of V . In particular, since h' is a product metric in a neighborhood of Σ you can double the compactification V' along Σ and conclude that this compact manifold admits a metric of positive scalar curvature.
- One can regard this result as a generalization of the Galloway-Schoen result on the topology of black holes to the topology of the exterior region. (Note the restriction on dimension here, which comes from our use of existence results for smooth solutions of Jang's equation.)
- This construction is built upon the work of a number of people (many of whom are here) Chruściel, Eichmair, Huang, Lee, Mazzeo, Metzger, Schoen & Yau.

Cosmological Spacetimes and CMC Cauchy surfaces

Galloway has done recent work in a number of different aspects of the question of geodesic incompleteness with multiple collaborators (Ling, Vega, Woolgar). Here we focus briefly on some recent work of Galloway & Ling on the existence of constant mean curvature Cauchy surfaces in cosmological spacetimes. The main result of their paper is to establish an existence result for CMC Cauchy surfaces under a spacetime curvature condition. This led them to make to following conjecture

Conjecture (Galloway & Ling, 2018)

Let (M, g) be a spacetime with compact Cauchy surfaces. If (M, g) is future timelike geodesically complete and satisfies the strong energy condition, i.e. $Ric(U, U) \geq 0$ for all timelike U, then (M, g) contains a CMC Cauchy surface.

Cosmological Spacetimes and CMC Cauchy surfaces

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Recall that in 1988 Bartnik (motivated by a construction of spacetimes without maximal surfaces due to Brill) provided an example of a cosmological spacetime, with dust satisfying the strong energy energy condition, which did not admit any CMC slices. The existence of a vacuum example of this type was resolved in 2005 by Chruściel, Isenberg and myself (CIP) via a gluing construction. This example was built on previous gluing constructions of Isenberg, Mazzeo and myself, together with the localized gluing techniques of Corvino, Corvino-Schoen and Chruściel-Delay. Galloway & Ling's conjecture thus raised the natural question of the timelike completeness or incompleteness of the CIP no-CMC slice examples.

Cosmological Spacetimes and CMC Cauchy surfaces II

The gluing construction used in CIP has a free parameter, m and the result is that for m sufficiently small the resulting spacetime has no CMC Cauchy surfaces. Recently M. Burkhart and M. Lesourd, were able to show that for m sufficiently small the CIP spacetimes are null geodesically incomplete.

Theorem (Burkhart-Lesourd, 2018)

There exists an $m_0 > 0$ such that for the $m \leq m_0$ the CIP examples (M, g_m) of spacetimes with no CMC Cauchy surfaces are future null geodesically incomplete.

Note that in these examples $M = T^3 \# T^3$ and the parameter m can be heuristically thought of as the size of a Schwarzschild like neck connecting the two T^3's. Burkhart & Lesourd use the symmetry of the construction to see that there is a MOTS in the center of neck, and further analysis to show stability. A covering argument together with the Chruściel-Galloway extension of Penrose then leads to the geodesic incompleteness.

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The geometry of the IMP and CIP gluing has previously been exploited in work of Chruściel-Mazzeo (2003) and Corvino, Eichmair & Miao (2013). Together with Burkhart, we show that the existence of outer trapped and outer untrapped surfaces are present, provided m is sufficiently small, whenever the IMP gluing method is employed, regardless of symmetry considerations. In many circumstances (e.g. when the initial data set has a noncompact cover) this leads to geodesic incompleteness for the resulting spacetimes.