Vacuum Spacetimes with Compact Cauchy Horizons with Non-Closed Generators

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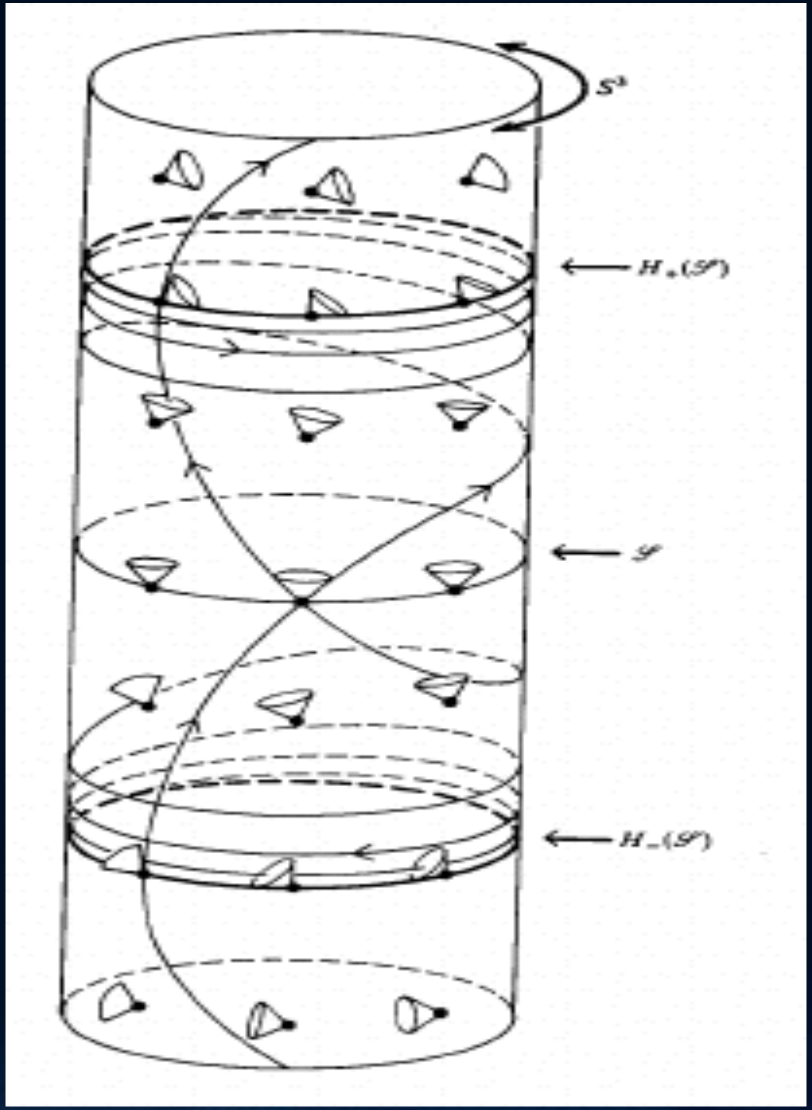
Taub-NUT Spacetimes

- Misner: Taub-NUT Spacetimes as the counter-example to almost everything
- Globally hyperbolic region extends across Cauchy horizon
- Cauchy horizon indicates breakdown of determinism
- Taub-NUT Cauchy horizons are compact

Diagram of Taub-NUT Space-Time



Strong Cosmic Censorship Conjecture

- Generic solutions of Einstein's equations do not contain Cauchy horizons
- There are an infinite number of space-times which do have Cauchy horizons
- ? Do Cauchy horizons occur in generic solutions of Einstein's equations?
- ?Do *compact* Cauchy horizons occur in generic solutions of Einstein's equations?
- This is the focus of my talk here

THEOREM (Isenberg and Moncrief)

- Space-time solutions of Einstein's Equation with compact Cauchy horizons with non-degenerate generators must contain a Killing vector field
- Previous results required generators of the horizon to be closed.
- Our new results allow the generators to be non-closed (but not ergodic).
- Also, if the generators are not closed, there must be a two-dimensional isometry group: i.e. two independent Killing vector fields

Key Ingredients of the Proof

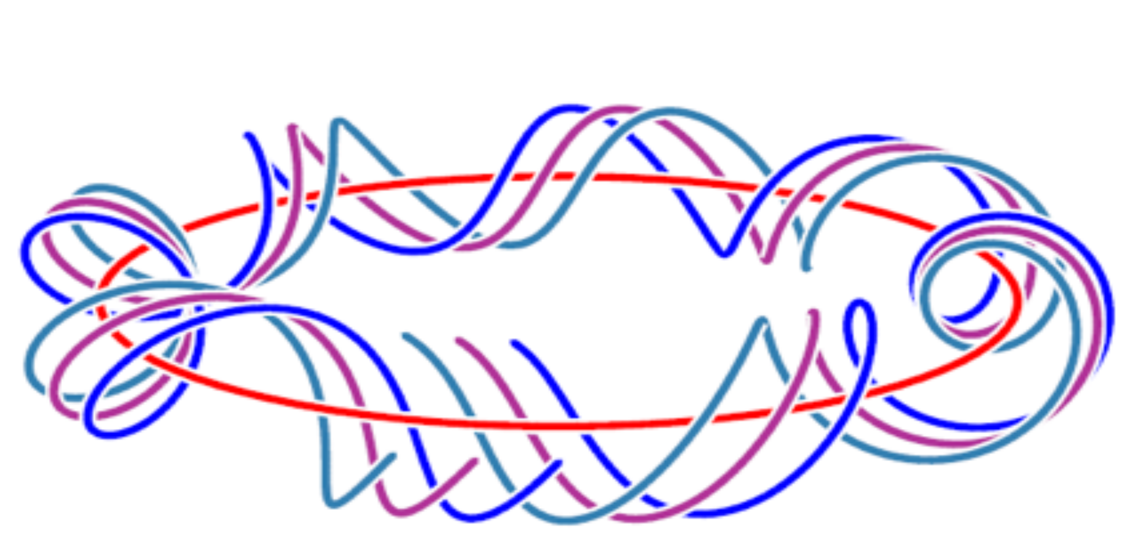
- 1) Topology of the Compact Cauchy Horizon and its Non-Closed Generators
- 2) Adapted Geodesic Null Coordinates
- 3) Analyticity
- 4) Einstein Equations
- 5) Poincaré Recurrence

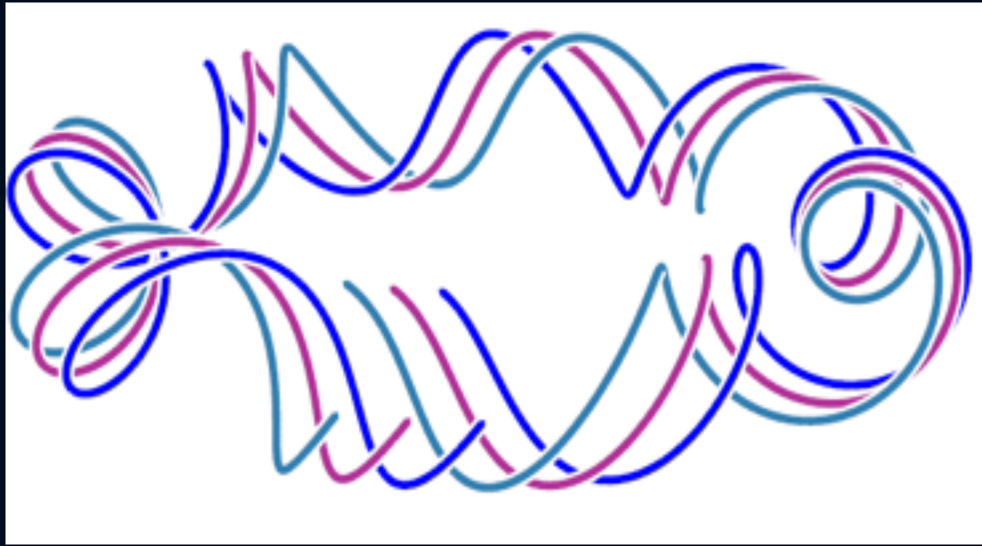
Key Ingredients of the Proof (continued)

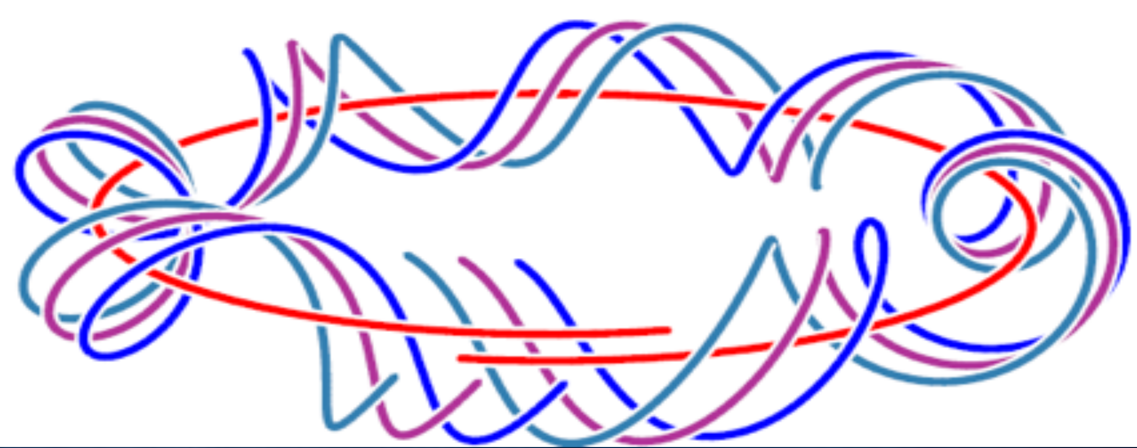
- 6) Invariance of Geometry of Discs Transverse to Generators
- 7) "Ribbon Arguments" using Stokes' Theorem
- 8) Kolmogorov–Kontsevich Theorem on Transverse Foliation of a 2-Torus
- 9) Grauert Tubes and the Banach Space of Complex Analytic Functions
- 10) Cauchy-Kowaleski Theorem

Basic Steps of the Proof

- 1) Topology of the Cauchy Horizon and its Non-Closed Generators







Basic Steps of the Proof (continued)

- 2) Choose Adapted Geodesic Null Coordinates for Null Surface N Corresponding to Zero Level Set of Analytic Function t and Null Vector Field K

$$
\begin{aligned}
g &= dt \otimes dx^3 + dx^3 \otimes dt \\
&\quad + \varphi\, dx^3 \otimes dx^3 + \beta_a (dx^a \otimes dx^3 + dx^3 \otimes dx^a) \\
&\quad + \mu_{ab}\, dx^a \otimes dx^b.
\end{aligned}
$$

$$
\varphi|_{t=0} = \beta_a|_{t=0} = 0
$$

Basic Steps of the Proof (continued)

- 3) Use Hawking & Ellis Result Showing that the Volume Element of the Transverse Discs is Invariant Along the Generator Flow.

$$(\det \mu_{ab})_{,3}\big|_{t=0} = 0$$

Basic Steps of the Proof (continued)

- 4) Use Einstein's equations to Show that the Metric on the Transverse Discs is Invariant along the Generator Flow

$$R_{tt} = -\tfrac{1}{2}\mu^{ab}\mu_{ab,tt} + \tfrac{1}{4}\mu^{ac}\mu^{bd}\mu_{ab,t}\mu_{cd,t},$$

$$R_{t3} = \frac{1}{\sqrt{\mu}}\left[\sqrt{\mu}\left(\frac{1}{2}\phi_{,t} - \frac{1}{2}\beta^a\beta_{a,t}\right)\right]_{,t} + \frac{1}{\sqrt{\mu}}\left[\sqrt{\mu}\left(\frac{\mu^{ab}}{2}\beta_{a,t}\right)\right]_{,b}$$
$$-\tfrac{1}{2}\mu^{ab}\mu_{ab,t3} + \tfrac{1}{4}\mu^{ac}\mu^{bd}\mu_{ab,t}\mu_{cd,3},$$

$$R_{tb} = \frac{1}{\sqrt{\mu}}\left[\sqrt{\mu}\left(\frac{1}{2}\beta_{b,t} - \frac{1}{2}\beta^a\mu_{ab,t}\right)\right]_{,t} + {}^{(2)}\nabla_c\left(\frac{1}{2}\mu^{ac}\mu_{ab,t}\right) - {}^{(2)}\nabla_b\left(\frac{1}{2}\mu^{cd}\mu_{cd,t}\right),$$

$$
\begin{aligned}
R_{ab} = {} & \frac{1}{\sqrt{\mu}}\left[\sqrt{\mu}\left(-\frac{1}{2}\mu_{ab,3} + \frac{1}{2}\left({}^{(2)}\nabla_b\beta_a + {}^{(2)}\nabla_a\beta_b\right) + \frac{\phi}{2}\mu_{ab,t} - \frac{1}{2}\beta^c\beta_c\mu_{ab,t}\right)\right]_{,t} \\
& + {}^{(2)}\nabla_c\left[\frac{\beta^c}{2}\mu_{ab,t}\right] - \frac{1}{\sqrt{\mu}}\left[\sqrt{\mu}\,\frac{\mu_{ab,t}}{2}\right]_{,3} \\
& + {}^{(2)}R_{ab} - \{\tfrac{1}{2}(\beta_{a,t} - \beta^c\mu_{ac,t})(\beta_{b,t} - \beta^d\mu_{bd,t}) \\
& + \tfrac{1}{4}\mu^{df}\mu_{fb,t}(-2\mu_{ad,3} + 2\,{}^{(2)}\nabla_d\beta_a + (\phi - \beta^g\beta_g)\mu_{ad,t}) \\
& + \tfrac{1}{4}\mu^{df}\mu_{fa,t}(-2\mu_{bd,3} + 2\,{}^{(2)}\nabla_d\beta_b + (\phi - \beta^g\beta_g)\mu_{bd,t})\}\,,
\end{aligned}
$$

$$R_{3b} = \frac{1}{\sqrt{\mu}}\left[\sqrt{\mu}\left(\frac{1}{2}\phi_{,b} + \frac{1}{2}\phi\beta_{b,t} - \frac{1}{2}\beta^a(\beta_{a,b} - \beta_{b,a}) - \frac{1}{2}\beta^a\beta_a\beta_{b,t} - \frac{1}{2}\beta^a\mu_{ab,3}\right)\right]_{,t}$$

$$+ {}^{(2)}\nabla_c\left[\frac{\mu^{ac}}{2}(\beta_{a,b} - \beta_{b,a}) + \frac{1}{2}\beta^c\beta_{b,t} + \frac{1}{2}\mu^{ac}\mu_{ab,3}\right]$$

$$- \frac{1}{\sqrt{\mu}}\left[\sqrt{\mu}\frac{\beta_{b,t}}{2}\right]_{,3} - \frac{1}{2}\mu^{cd}\,{}^{(2)}\nabla_b(\mu_{cd,3})$$

$$- \{\tfrac{1}{2}(\beta_{b,t} - \beta^a\mu_{ab,t})(\phi_{,t} - \beta^c\beta_{c,t})$$

$$+ \tfrac{1}{2}\mu^{dc}\mu_{cb,t}(\phi_{,d} + \phi\beta_{d,t} - \beta^a(\beta_{a,d} - \beta_{d,a}) - \beta_{d,3} - \beta^a\beta_a\beta_{d,t})$$

$$+ \tfrac{1}{2}\mu^{ac}\beta_{a,t}(-\mu_{bc,3} + {}^{(2)}\nabla_c\beta_b)\},$$

$$
\begin{aligned}
R_{33} = {} & \frac{1}{\sqrt{\mu}}\left[\sqrt{\mu}\left(\frac{1}{2}\phi_{,3} + \frac{1}{2}\phi\phi_{,t} + \frac{1}{2}\beta^a\phi_{,a} - \beta^a\beta_{a,3} - \frac{1}{2}\beta^a\beta_a\phi_{,t}\right)\right]_{,t} \\
& + \frac{1}{\sqrt{\mu}}\left[\sqrt{\mu}\,\mu^{ac}\left(-\frac{\phi_{,a}}{2} + \beta_{a,3} + \frac{\beta_a}{2}\phi_{,t}\right)\right]_{,c} \\
& - \frac{1}{\sqrt{\mu}}\left[\sqrt{\mu}\,\frac{\phi_{,t}}{2}\right]_{,3} - \frac{1}{2}\mu^{ab}\mu_{ab,33} + \frac{1}{4}\mu^{ac}\mu^{bd}\mu_{ab,3}\mu_{cd,3} \\
& - \{\tfrac{1}{2}(\phi_{,t} - \beta^a\beta_{a,t})^2 + \tfrac{1}{4}\mu^{ac}\mu^{bd}(\beta_{a,b} - \beta_{b,a})(\beta_{d,c} - \beta_{c,d}) \\
& + \tfrac{1}{2}\mu^{dc}\beta_{c,t}(2\phi_{,d} + \phi\beta_{d,t} - 2\beta^a(\beta_{a,d} - \beta_{d,a}) - 2\beta_{d,3} - \beta^a\beta_a\beta_{d,t})\}\,,
\end{aligned}
$$

R33 = 0, Restricted to N

$$\mathring{R}_{33} = 0 = \Bigg[\left(\ell n \sqrt{\det \mu}\right)_{,33} + \frac{1}{2}\varphi_{,t} \left(\ell n \sqrt{\det \mu}\right)_{,3} + \frac{1}{4}\mu^{ac}\mu^{bd}\mu_{ab,3}\mu_{cd,3} \Bigg]\Bigg|_{t=0}$$

Basic Steps of the Proof (continued)

- 5) Construction of Fiducial Riemannian Metric on the Space-Time

$$g'(Y,Z) = g(Y,Z) + 2g(Y,V)g(Z,V)$$

Basic Steps of the Proof (continued)

- 6) Restrictions on the Geometry of the Transverse Discs following from Poincaré Recurrence and Invariance of Geometry along the Flow
- Scalar curvature is constant on the horizon N.

Basic Steps of the Proof (continued)

- 7) Closures of the Orbits of the Geodesics Form 2-Tori

Basic Steps of the Proof (continued)

- 8) Kolmogorov-Kontsevich Theorem Guarantees Existence of Foliations of Geodesic 2-Tori by Transverse Circles

Basic Steps of the Proof (continued)

- 9) Define Privileged One-form ω Based on Space-Time Connection
- Expressions for ω and its Exterior Derivative can be Expressed in Terms of the Metric Coefficients and their Derivatives:

$$\omega_X = -\frac{1}{2}\mathring{\varphi}_{,t}dx^3 - \frac{1}{2}\mathring{\beta}_{a,t}dx^a.$$

$$\begin{aligned} d\omega_X = &-\frac{1}{2}(\mathring{\varphi}_{,ta} - \mathring{\beta}_{a,t3})dx^a \wedge dx^3 \\ &-\frac{1}{2}\mathring{\beta}_{a,tb}dx^b \wedge dx^a. \end{aligned}$$

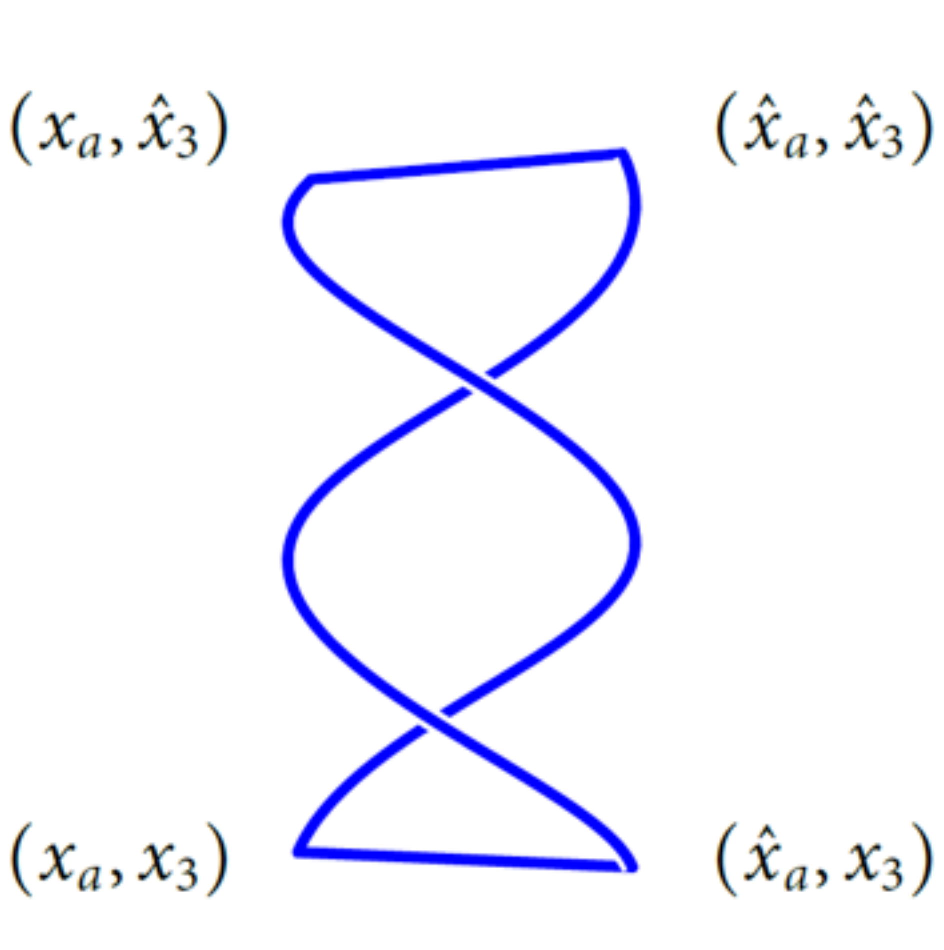
Basic Steps of the Proof (continued)

- 10) Use Einstein's Equations to Simplify Formula for dω:

$$d\omega_X = -\frac{1}{2}\mathring{\beta}_{a,tb}\,dx^b \wedge dx^a.$$

Basic Steps of the Proof (continued)

- 11) Construct Ribbon from Path on N Flowed along Geodesics through the Ends of the Path



Basic Steps of the Proof (continued)

- 12) Use Stokes' Theorem to Show that the Integral over the Boundary of the Ribbon of the One-form **ω** vanishes.

$$\int_{\partial r} \omega_X = \int_r d\omega_X = 0$$

Basic Steps of the Proof (continued)

- 13) Use Ribbon Argument to Correlate Integrals of ω along Neighboring Generators

Basic Steps of the Proof (continued)

- 14) Use Ribbon Argument to Show that all Generators are either Complete or Incomplete:
- We restrict to the Case that the Generators are All Complete in One Direction and Incomplete in the Other Direction
- We call this the Non-Degenerate Case

Basic Steps of the Proof (continued)

- 15) Candidate Vector Field Defined as Function U times Vector Field Tangent to Generators
- The formula is given by

$$u(x^3, x^a) = \frac{k}{2}\int_{x^3}^{\infty} d\rho \exp\left[-\int_{x^3}^{\rho} \frac{\overset{\circ}{\varphi}_{,t}}{2}(\xi, x^a)d\xi\right].$$

Basic Steps of the Proof (continued)

- 16) A Cauchy Sequence for U:

$$u_i(x^3, x^a) = \frac{k}{2} \int_{x^3}^{x^3+is^*} d\rho \exp\left[-\int_{x^3}^{\rho} \frac{\mathring{\varphi}_{,t}}{2}(\xi, x^a) d\xi\right]$$

Basic Steps of the Proof (continued)

- 17) Complexify the Space Using Grauert Tubes and Show that the Cauchy Sequence Converges:
- The Limit is an Analytic Function U

Basic Steps of the Proof (continued)

- 18) Initial Data for Cauchy-Kowaleski on the Cauchy Horizon N for Expansion into the Space-Time of the Killing Vector Field.

$$\mu_{a'b',t'x^{3'}}\Big|_{t'=0} = \left(\frac{\partial x^c}{\partial x^{a'}}\frac{\partial x^d}{\partial x^{b'}}\mu_{cd,t3}\right)\Big|_{t=0}$$

Basic Steps of the Proof (continued)

- 19) Patch Together Space-Time Geometry with Killing Vector Field

Basic Steps of the Proof (continued)

- 20) Use Isenberg-Moncrief Results to Show that a Killing Field with Non-Closed Orbits on a Compact Manifold Must Generate a Two-Dimensional Isometry Group

Extensions of this work?

- What about Non-Analytic Space-Times?
  - We don't know
- What about Degenerate Cauchy Horizons?
  - We don't think Space-Times with Degenerate Cauchy Horizons Exist
- What about Cauchy Horizons with Ergodic Generators?
  - We think that these are all Kasner Space-Times with Nonrational Identifications