A GEOMETRIC DESCRIPTION OF TWO-DIMENSIONAL SIMPLY CONNECTED SOBOLEV EXTENSION DOMAINS

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Abstract: Consider a Sobolev space of functions on an open domain in $\mathbf{R}^2$:

$$L_p^m(\Omega) = \{f : \Omega \to \mathbf{R} : \forall\, \alpha \in \mathbf{Z}^2,\ |\alpha| = m,\ f^{(\alpha)} \in L_p(\Omega)\}.$$

Obviously, there is a natural restriction operator $R_\Omega :\ L_p^m(\mathbf{R}^2) \to L_p^m(\Omega)$. A natural question arises: under what conditions on Ω will this operator be surjective? In other words, when every function from $L_p^m(\Omega)$ can be extended to a function from $L_p^m(\mathbf{R}^2)$?

This question has a very long history going back to seminal papers of Hassler Whitney in 1934, who essentially found a nice geometric condition on Ω, later called by M. Gromov "quasi-convexity", which is sufficient for every function from $L_\infty^m(\Omega)$ to be extendable to a function from $L_\infty^m(\mathbf{R}^2)$. The question whether this condition is actually also necessary was discussed until 1990s, when the speaker was able to prove that this is indeed the case for a finitely connected bounded connected planar domain, but also gave examples of infinitely connected bounded connected planar domains for which the quasi-convexity is not necessary for extendability of functions from $L_\infty^m(\Omega)$.

The case of $p < \infty$ was studied by a number of mathematicians, who were proving weaker and weaker sufficient conditions for such extendability (A. Calderon, E. Stein, P. Jones, V. Mazya, among others). In mid-1990s P. Koskela and his collaborators developed approaches which allowed to fully resolve the case when $p > 2$, and the domain is simply connected bounded and connected, but only for smoothness $m = 1$. Their description involves a special subhyperbolic metric on Ω, and the related domains are called p-quasiconvex. However, their methods could not be generalized to higher smoothnesses.

Pavel Shvartsman and the speaker developed another approach based on a new geometric tool we called the Square Separation Theorem, which allowed us to show that the condition of p-quasiconvexity is necessary and sufficient for any simply connected bounded connected domain to have the L_p^m extension property for any $m \geq 1$, $p > 2$.

I will discuss main geometric ideas involved in our proof and explore the remaining situations.