UNIMODALITY OF EULERIAN QUASISYMMETRIC FUNCTIONS

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Abstract. We prove two conjectures of Shareshian and Wachs about Eulerian quasisymmetric functions and polynomials. The first states that the cycle type Eulerian quasisymmetric function $Q_{\lambda,j}$ is Schur-positive, and moreover that the sequence $Q_{\lambda,j}$ as j varies is Schur-unimodal. The second conjecture, which we prove using the first, states that the cycle type (q,p)-Eulerian polynomial $A_\lambda^{\mathrm{maj},\mathrm{des},\mathrm{exc}}(q,p,q^{-1}t)$ is t-unimodal.

1. Introduction

The Eulerian polynomial $A_n(t) = \sum_{j=0}^{n-1} a_{n,j}t^j$ is the enumerator of permutations in the symmetric group $\mathfrak{S}_n$ by their number of descents or their number of excedances. Two well-known and important properties of the Eulerian polynomials are symmetry and unimodality (see [3, p. 292]). That is, the sequence of coefficients $(a_{n,j})_{0\le j\le n-1}$ satisfies

$$a_{n,j} = a_{n,n-1-j} \tag{1.1}$$

and

$$a_{n,0} \le a_{n,1} \le \cdots \le a_{n,\lfloor\frac{n-1}{2}\rfloor} = a_{n,\lfloor\frac{n}{2}\rfloor} \ge \cdots \ge a_{n,n-2} \ge a_{n,n-1}. \tag{1.2}$$

Brenti [1, Theorem 3.2] showed that the cycle type Eulerian polynomial $A_\lambda^{\mathrm{exc}}(t)$, which enumerates permutations of fixed cycle type λ by their number of excedances, is also symmetric and unimodal. More recently, Shareshian and Wachs [10] proved that the q-Eulerian polynomial $A_n^{\mathrm{maj},\mathrm{exc}}(q,q^{-1}t)$, which is the enumerator for the joint distribution of the major index and excedance number over permutations in $\mathfrak{S}_n$, is symmetric and unimodal when viewed as a polynomial in t with coefficients in $\mathbb{N}[q]$. They showed that symmetry holds for the cycle type (q,p)-analog $A_\lambda^{\mathrm{maj},\mathrm{des},\mathrm{exc}}(q,p,q^{-1}t)$ as a polynomial in t with coefficients

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2000 *Mathematics Subject Classification.* Primary 05E05; Secondary 05A15, 05A30.

[1]Supported in part by Australian Research Council grant DP0985184.

[2]Supported in part by National Science Foundation grant DMS 0902323.

in $\mathbb{N}[q,p]$ and conjectured that unimodality holds as well. (Symmetry fails for the less refined (q,p)-analog $A_n^{\mathrm{maj,des,exc}}(q,p,q^{-1}t)$, see [10].)

In this paper we prove the unimodality conjecture of Shareshian and Wachs, by first establishing a symmetric function analog, also conjectured in [10], and then using Gessel's theory of quasisymmetric functions to deduce the unimodality of $A_\lambda^{\mathrm{maj,des,exc}}(q,p,q^{-1}t)$.

The symmetric function analog of the unimodality conjecture involves the cycle type refinements $Q_{\lambda,j}$ of the Eulerian quasisymmetric functions $Q_{n,j}$, which were introduced by Shareshian and Wachs [10] as a tool for studying the q-Eulerian polynomials and the (q,p)-Eulerian polynomials. Both $Q_{n,j}$ and $Q_{\lambda,j}$ were shown to be symmetric functions in [10]. Moreover such properties as p-positivity, Schur-positivity, and Schur-unimodality were established for $Q_{n,j}$ and conjectured for $Q_{\lambda,j}$.

In subsequent work, Sagan, Shareshian and Wachs [9] established p-positivity of $Q_{\lambda,j}$ by proving [10, Conjecture 6.5], which gives the expansion of $Q_{\lambda,j}$ in the power-sum symmetric function basis. This was used to obtain a cyclic sieving result for the q-Eulerian polynomials refined by cycle type. Here we continue the study of Eulerian quasisymmetric functions by establishing Schur-positivity of $Q_{\lambda,j}$ and Schur-unimodality of the sequence $(Q_{\lambda,j})_{0\le j\le n-1}$.

We briefly recall the main concepts involved, referring the reader to [10] for the background and standard notation. The Eulerian quasisymmetric functions $Q_{n,j}$ in $\mathbf{x}=(x_1,x_2,x_3,\cdots)$, for $n,j\in\mathbb{N}$, are defined in [10] by

$$Q_{n,j}(\mathbf{x}) := \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \mathrm{exc}(\sigma)=j}} F_{\mathrm{DEX}(\sigma),n}(\mathbf{x}), \tag{1.3}$$

where $\mathrm{DEX}(\sigma)$ is the subset of $[n-1]:=\{1,2,\cdots,n-1\}$ defined in [10, Section 2], and $F_{S,n}(\mathbf{x})$ is the fundamental quasisymmetric function of degree n associated to $S\subseteq[n-1]$. It is immediate from this definition that $Q_{n,j}=0$ unless $j\le n-1$.

The apparently quasisymmetric functions $Q_{n,j}$ are symmetric functions by [10, Theorem 5.1(1)], and form a symmetric sequence, in the sense that $Q_{n,j}=Q_{n,n-1-j}$, by [10, (5.3)]. Moreover, [10, Theorem 1.2] shows that they have the following generating series:

$$\sum_{n,j} Q_{n,j}\, t^j z^n = \frac{(1-t)H(z)}{H(zt)-tH(z)}, \tag{1.4}$$

where as usual $H(z)=\sum_{n\ge0} h_n z^n$ is the generating series of the complete homogeneous symmetric functions.

Symmetry and Schur-unimodality of the sequence $(Q_{n,j})_{0\leq j\leq n-1}$ are consequences of (1.4). Various ways to see this are given in [10]: one way involves symmetric function manipulations of Stembridge [13] and another involves geometric considerations based on work of Procesi [7] and Stanley [11]. Indeed, (1.4) implies that $Q_{n,j}$ is the Frobenius characteristic of the representation of $\mathfrak{S}_n$ on the degree-$2j$ cohomology of the toric variety associated with the Coxeter complex of $\mathfrak{S}_n$. Schur-unimodality then follows from the hard Lefschetz theorem, see [11]. See [10, Section 7] for other occurrences of $Q_{n,j}$.

The cycle type Eulerian quasisymmetric functions $Q_{\lambda,j}$, for λ a partition of $n \in \mathbb{N}$ and $j \in \mathbb{N}$, are a refinement of the above symmetric functions in the sense that $Q_{n,j} = \sum_{\lambda\vdash n} Q_{\lambda,j}$. The definition in [10] is

$$Q_{\lambda,j}(\mathbf{x}) := \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{exc}(\sigma)=j \\ \lambda(\sigma)=\lambda}} F_{\mathrm{DEX}(\sigma),n}(\mathbf{x}), \tag{1.5}$$

where $\lambda(\sigma)$ denotes the cycle type of σ. It is immediate from this definition that $Q_{\lambda,j} = 0$ unless $j \leq n-k$, where k is the multiplicity of 1 as a part of λ.

The quasisymmetric functions $Q_{\lambda,j}$ are symmetric functions by [10, Theorem 5.8], and satisfy

$$Q_{\lambda,j} = Q_{\lambda,n-k-j} \tag{1.6}$$

by [10, Theorem 5.9]. These functions may all be obtained from those where λ has a single part, using the operation of plethysm which we denote by [ ]. Explicitly, [10, Corollary 6.1] states that if m_i denotes the multiplicity of i as a part of λ, then

$$\sum_j Q_{\lambda,j}\, t^j = \prod_{i\geq 1} h_{m_i}[\sum_j Q_{(i),j}\, t^j]. \tag{1.7}$$

The following consequence of (1.7) is also part of [10, Corollary 6.1]:

$$\sum_{n,j} Q_{n,j}\, t^j z^n = \sum_n h_n[\sum_{i,j} Q_{(i),j}\, t^j z^i]. \tag{1.8}$$

Note that (1.4),(1.7),(1.8) effectively provide an alternative definition of $Q_{\lambda,j}$. In this paper we will use only these equations, not the definition of $Q_{\lambda,j}$ in terms of quasisymmetric functions.

The first result of this paper appeared as [10, Conjecture 5.11].

Theorem 1.1. *The symmetric function $Q_{\lambda,j}$ is Schur-positive. Moreover, if k is the multiplicity of* 1 *in λ then the symmetric sequence*

$$Q_{\lambda,0}, Q_{\lambda,1}, \cdots, Q_{\lambda,n-k-1}, Q_{\lambda,n-k}$$

is Schur-unimodal in the sense that $Q_{\lambda,j} - Q_{\lambda,j-1}$ *is Schur-positive for* $1 \le j \le \frac{n-k}{2}$.

The proof will be given in Section 2; it involves constructing an explicit $\mathfrak{S}_n$-representation $V_{\lambda,j}$ whose Frobenius characteristic is $Q_{\lambda,j}$.

We recall some basic permutation statistics. Let $\sigma \in \mathfrak{S}_n$. The excedance number of σ is given by

$$\operatorname{exc}(\sigma) := |\{i \in [n-1] : \sigma(i) > i\}|.$$

The descent set of σ is given by

$$\operatorname{DES}(\sigma) := \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$$

and the descent number and major index are

$$\operatorname{des}(\sigma) := |\operatorname{DES}(\sigma)| \text{ and } \operatorname{maj}(\sigma) := \sum_{i \in \operatorname{DES}(\sigma)} i.$$

The cycle type (q,p)-Eulerian polynomial is defined in [10] by

$$A_\lambda^{\operatorname{maj},\operatorname{des},\operatorname{exc}}(q,p,q^{-1}t) := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \lambda(\sigma)=\lambda}} q^{\operatorname{maj}(\sigma)-\operatorname{exc}(\sigma)} p^{\operatorname{des}(\sigma)} t^{\operatorname{exc}(\sigma)}.$$

This records the joint distribution of the statistics $(\operatorname{maj}, \operatorname{des}, \operatorname{exc})$ over permutations of cycle type λ. We write $a_{\lambda,j}^{\operatorname{maj}',\operatorname{des}}(q,p)$ for the coefficient of t^j, which is an element of $\mathbb{N}[q,p]$.

The polynomial $a_{\lambda,j}^{\operatorname{maj}',\operatorname{des}}(q,p)$ may be obtained from the cycle type Eulerian quasisymmetric functions by a suitable specialization. Explicitly, [10, Lemma 2.4] shows that if λ has the form $(\mu, 1^k)$, where μ is a partition of $n-k$ with no parts equal to 1, then

$$a_{\lambda,j}^{\operatorname{maj}',\operatorname{des}}(q,p) = (p;q)_{n+1} \sum_{m \ge 0} p^m \sum_{i=0}^{k} q^{im}\, \mathbf{ps}_m(Q_{(\mu,1^{k-i}),j}), \tag{1.9}$$

where as usual $(p;q)_i$ denotes $(1-p)(1-pq)\cdots(1-pq^{i-1})$, and $\mathbf{ps}_m$ is the principal specialization of order m. In [10, Theorem 5.13], this is used to show that $A_\lambda^{\operatorname{maj},\operatorname{des},\operatorname{exc}}(q,p,q^{-1}t)$ is t-symmetric with center of symmetry $\frac{n-k}{2}$, in the sense that

$$a_{\lambda,j}^{\operatorname{maj}',\operatorname{des}}(q,p) = a_{\lambda,n-k-j}^{\operatorname{maj}',\operatorname{des}}(q,p). \tag{1.10}$$

The second result of this paper appeared as [10, Conjecture 5.14].

Theorem 1.2. *The t-symmetric polynomial $A_\lambda^{\operatorname{maj},\operatorname{des},\operatorname{exc}}(q,p,q^{-1}t)$ is t-unimodal in the sense that*

$$a_{\lambda,j}^{\operatorname{maj}',\operatorname{des}}(q,p) - a_{\lambda,j-1}^{\operatorname{maj}',\operatorname{des}}(q,p) \in \mathbb{N}[q,p]$$

for $1 \leq j \leq \frac{n-k}{2}$*, where* k *is the multiplicity of* 1 *in the partition* λ*.*

The proof will be given in Section 3; it makes use of Theorem 1.1 and (1.9).

2. Proof of Theorem 1.1

For any positive integer n, we define a symmetric function ℓ_n by

$$\ell_n = \frac{1}{n}\sum_{d|n} \mu(d) p_d^{n/d}, \tag{2.1}$$

where $\mu(d)$ is the usual Möbius function. It is well known [6, Ch. 4, Proposition 4] that ℓ_n is the Frobenius characteristic of the Lie representation Lie_n of $\mathfrak{S}_n$, which is by definition the degree-$(1, 1, \cdots, 1)$ multihomogeneous component of the free Lie algebra on n generators. Here and subsequently, all representations and other vector spaces are over $\mathbb{C}$ (any field of characteristic 0 would do equally well).

For us, a convenient construction of Lie_n is as the vector space generated by binary trees with leaf set $[n]$, subject to relations which correspond to the skew-symmetry and Jacobi identity of the Lie bracket. These relations are

$$\begin{aligned}&(T_1 \wedge T_2) + (T_2 \wedge T_1) = 0 \text{ and}\\ &((T_1 \wedge T_2) \wedge T_3) + ((T_2 \wedge T_3) \wedge T_1) + ((T_3 \wedge T_1) \wedge T_2) = 0,\end{aligned} \tag{2.2}$$

where $A \wedge B$ denotes the binary tree whose left subtree is A and right subtree is B, and in both cases the relation applies not just to the tree as a whole but to the subtree descending from any vertex (it being understood that the other parts of the tree are the same in all terms). The $\mathfrak{S}_n$-action is the obvious one by permuting the labels of the leaves. It is well known that Lie_n has a basis given by the trees of the form $(\cdots((s_1 \wedge s_2) \wedge s_3) \cdots \wedge s_n)$ where $s_1, s_2, \cdots, s_n$ is a permutation of $[n]$ such that $s_1 = 1$.

A famous result of Cadogan [2] is that the plethystic inverse of $\sum_{n\geq 1} h_n$ is $\sum_{n\geq 1}(-1)^{n-1}\omega(\ell_n)$. A slight variant of this result is the following (compare [6, Ch. 4, Proposition 1]).

Lemma 2.1. *We have an equality of symmetric functions:*

$$\sum_{n\geq 0} h_n[\sum_{m\geq 1} \ell_m] = (1 - h_1)^{-1}.$$

Proof. Using the well-known identity

$$\sum_{n\geq 0} h_n = \exp(\sum_{i\geq 1} \frac{1}{i} p_i), \tag{2.3}$$

the left-hand side of our desired equality becomes

$$\begin{aligned}\exp(\sum_{i\geq 1}\frac{1}{i}p_i[\sum_{m\geq 1}\ell_m]) &= \exp(\sum_{i\geq 1}\frac{1}{i}p_i[\sum_{d,e\geq 1}\frac{1}{de}\mu(d)p_d^e])\\ &= \exp(\sum_{i,d,e\geq 1}\frac{1}{ide}\mu(d)p_{id}^e)\\ &= \exp(\sum_{j,e\geq 1}\sum_{d|j}\frac{1}{je}\mu(d)p_j^e)\\ &= \exp(\sum_{e\geq 1}\frac{1}{e}p_1^e)\\ &= \exp(-\log(1-p_1)),\end{aligned}$$

which equals the right-hand side. □

We deduce a new expression for the symmetric functions $Q_{(n),j}$.

Proposition 2.2. *The symmetric functions $Q_{(n),j}$ have the generating series:*

$$\sum_{n\geq 1, j\geq 0} Q_{(n),j}\, t^j z^n = h_1 z + \sum_{m\geq 1}\ell_m[\sum_{r\geq 2}(t+t^2+\cdots+t^{r-1})\, h_r z^r].$$

Proof. Let A and B denote the left-hand and right-hand sides of the equation. We know from (1.4) and (1.8) that

$$\sum_{n\geq 0} h_n[A] = \frac{(1-t)H(z)}{H(zt)-tH(z)}.$$

Using the well-known fact

$$\sum_{n\geq 0} h_n[X+Y] = (\sum_{n\geq 0} h_n[X])(\sum_{n\geq 0} h_n[Y]) \tag{2.4}$$

as well as Lemma 2.1, we calculate

$$\begin{aligned}\sum_{n\geq 0} h_n[B] &= (\sum_{n\geq 0} h_n[h_1 z]) \sum_{n\geq 0} h_n[\sum_{m\geq 1} \ell_m[\sum_{r\geq 2}(t+t^2+\cdots+t^{r-1})\, h_r z^r]] \\ &= (\sum_{n\geq 0} h_n\, z^n)(\sum_{n\geq 0} h_n[\sum_{m\geq 1}\ell_m])[\sum_{r\geq 2}(t+t^2+\cdots+t^{r-1})\, h_r z^r] \\ &= (\sum_{n\geq 0} h_n\, z^n)(1-\sum_{r\geq 2}(t+t^2+\cdots+t^{r-1})\, h_r z^r)^{-1} \\ &= H(z)(1+\sum_{r\geq 1}\frac{t^r-t}{1-t}\, h_r z^r)^{-1} \\ &= H(z)\left(\frac{H(zt)-tH(z)}{1-t}\right)^{-1} \\ &= \frac{(1-t)H(z)}{H(zt)-tH(z)}.\end{aligned}$$

We conclude that $\sum_{n\geq 1} h_n[A] = \sum_{n\geq 1} h_n[B]$. By applying the plethystic inverse of $\sum_{n\geq 1} h_n$ to both sides of this equation we obtain $A = B$ as claimed. □

Proposition 2.2 allows us to construct an $\mathfrak{S}_n$-representation $V_{(n),j}$ whose Frobenius characteristic is $Q_{(n),j}$. We define a *marked set* to be a finite set S such that $|S| \geq 2$, together with an integer $j \in [|S|-1]$ called the *mark* (cf. [13]). For $n \geq 2$, let $V_{(n),j}$ be the vector space generated by binary trees whose leaves are marked sets which form a partition of $[n]$ (when the marks are ignored) and whose marks add up to j, subject to the relations (2.2). The $\mathfrak{S}_n$-action is by permuting the letters in the leaves.

Example 2.3. $V_{(6),3}$ is spanned by the following trees and their $\mathfrak{S}_6$-translates, where the superscript on a leaf indicates the mark:

$$\begin{aligned}&\{1,2,3,4,5,6\}^{(3)}, \\ &(\{1,2,3,4\}^{(2)} \wedge \{5,6\}^{(1)}), \\ &(\{1,2,3\}^{(2)} \wedge \{4,5,6\}^{(1)}), \\ &((\{1,2\}^{(1)} \wedge \{3,4\}^{(1)}) \wedge \{5,6\}^{(1)}).\end{aligned}$$

The resulting expression for $V_{(6),3}$ as a representation of $\mathfrak{S}_6$ is

$$\mathbf{1} \oplus \mathrm{Ind}_{\mathfrak{S}_4\times\mathfrak{S}_2}^{\mathfrak{S}_6}(\mathbf{1}) \oplus \mathrm{Ind}_{\mathfrak{S}_3\times\mathfrak{S}_3}^{\mathfrak{S}_6}(\mathbf{1}) \oplus \mathrm{Ind}_{\mathfrak{S}_2\wr\mathfrak{S}_3}^{\mathfrak{S}_6}(\mathrm{Lie}_3),$$

where $\mathbf{1}$ denotes the trivial representation of a group, and Lie_3 is regarded as a representation of the wreath product $\mathfrak{S}_2\wr\mathfrak{S}_3$ via the natural homomorphism to $\mathfrak{S}_3$.

Proposition 2.4. *For $n \geq 2$ and any j, $Q_{(n),j} = \operatorname{ch} V_{(n),j}$.*

Proof. We want to apply to Proposition 2.2 the representation-theoretic interpretation of plethysm given by Joyal in [6]. If we take the definition of Lie_n in terms of binary trees and replace the set $[n]$ with an arbitrary finite set I, we obtain a vector space $\mathrm{Lie}(I)$. This defines a functor Lie from the category of finite sets, with bijections as the morphisms, to the category of vector spaces; such a functor is called an $\mathfrak{S}$-module (or a tensor species, in the terminology of [6, Ch. 4]). The character $\mathrm{ch}(\mathrm{Lie})$ is by definition $\sum_{m\geq 1} \operatorname{ch} \mathrm{Lie}_m = \sum_{m \geq 1} \ell_m$.

We also define a graded $\mathfrak{S}$-module W (that is, a functor from the category of finite sets with bijections to the category of $\mathbb{N}$-graded vector spaces) by letting $W(I)$ be the graded vector space with

$$W(I)_a = \begin{cases} \mathbb{C}, & \text{if } 1 \leq a < |I|, \\ 0, & \text{otherwise,} \end{cases}$$

where the grading-preserving linear map $W(I) \to W(J)$ induced by a bijection $I \to J$ is the trivial one using only the identity map $\mathbb{C} \to \mathbb{C}$. The character $\mathrm{ch}_t(W)$, where we use the indeterminate t to keep track of the grading in the obvious way, is clearly $\sum_{r\geq 2}(t+t^2+\cdots+t^{r-1})h_r$.

We can then define a graded $\mathfrak{S}$-module $\mathrm{Lie} \circ W$, the partitional composition of Lie and W, by

$$(\mathrm{Lie} \circ W)(I) := \bigoplus_{\pi \in \Pi(I)} \mathrm{Lie}(\pi) \otimes \bigotimes_{J \in \pi} W(J),$$

where $\Pi(I)$ denotes the set of partitions of the set I, and we identify a partition π with its set of blocks. The grading on the tensor product of graded vector spaces is as usual, with $\mathrm{Lie}(\pi)$ considered as being homogeneous of degree zero. By [5, Corollary 7.6], which is an extension of Joyal's result [6, 4.4] to the graded setting, this operation of partitional composition corresponds to plethysm of the characters. So we have

$$\mathrm{ch}_t(\mathrm{Lie} \circ W) = \sum_{m \geq 1} \ell_m[\sum_{r \geq 2} (t+t^2+\cdots+t^{r-1})\, h_r]. \tag{2.5}$$

Comparing this equation with Proposition 2.2, we see that for $n \geq 2$, $Q_{(n),j}$ is the Frobenius characteristic of the representation of $\mathfrak{S}_n$ on the degree-j homogeneous component of $(\mathrm{Lie} \circ W)[n]$. It is easy to see that this is equivalent to the representation $V_{(n),j}$ defined above. $\square$

Remark 2.5. Equation (2.5) can also be obtained from an easy modification of [14, Theorem 5.5].

Remark 2.6. Proposition 2.4 is analogous to Stembridge's result [13, Proposition 4.1], which realizes $Q_{n,j}$ as the Frobenius characteristic of the permutation representation of $\mathfrak{S}_n$ on what he calls codes of length n and index j. In our terminology, these codes are the (possibly empty) sequences $(S_1, S_2, \cdots, S_m)$ of marked sets, whose underlying sets are disjoint subsets of $[n]$, and whose marks add up to j. So the marked sets appear in both contexts, but his result for $Q_{n,j}$ uses a representation with a basis consisting of sequences of marked sets, whereas our result for $Q_{(n),j}$ uses a representation spanned by binary trees of marked sets, which are subject to linear relations. The difference springs from the fact that the generating function (1.4) for $Q_{n,j}$ effectively has h_1^m in place of the ℓ_m in Proposition 2.2. Since $Q_{(n),j}$ is not h-positive (see [10, (5.4)]), it cannot be the Frobenius characteristic of a permutation representation.

We now define an $\mathfrak{S}_n$-representation $V_{\lambda,j}$ for any partition $\lambda \vdash n$. This is the vector space generated by forests $\{T_1, \cdots, T_m\}$, where each T_i is either a binary tree whose leaves are marked sets, or a single-vertex tree whose leaf is a singleton set with no mark. There are further conditions: for each tree T_i, the leaves (ignoring the marks) must form a partition of a set L_i, and in turn, $L_1, \cdots, L_m$ must form a partition of $[n]$; the sizes $|L_1|, \cdots, |L_m|$ must be the parts of the partition λ, in some order; and the sum of the marks must be j. These forests are once again subject to the relations (2.2). Note that if $n \geq 2$ and $\lambda = (n)$, this agrees with our earlier definition of $V_{(n),j}$.

Example 2.7. $V_{(4,3,3,1),4}$ is spanned by the following three forests and their $\mathfrak{S}_{11}$-translates:

$$\begin{array}{rccc} \{1,2,3,4\}^{(1)} & \{5,6,7\}^{(2)} & \{8,9,10\}^{(1)} & \{11\}, \\ \{1,2,3,4\}^{(2)} & \{5,6,7\}^{(1)} & \{8,9,10\}^{(1)} & \{11\}, \\ (\{1,2\}^{(1)} \wedge \{3,4\}^{(1)}) & \{5,6,7\}^{(1)} & \{8,9,10\}^{(1)} & \{11\}. \end{array}$$

Proposition 2.8. *For any λ and j, $Q_{\lambda,j} = \operatorname{ch} V_{\lambda,j}$.*

Proof. This follows by interpreting (1.7) along the lines of the proof of Proposition 2.4, using the result of Proposition 2.4 and the fact that $Q_{(1),0} = h_1$. □

From this description of $Q_{\lambda,j}$, Schur-positivity is immediate. We can also deduce the stronger Schur-unimodality statement of Theorem 1.1.

Proof of Theorem 1.1. For $1 \leq j \leq \frac{n-k}{2}$, define a linear map $\phi : V_{\lambda,j-1} \to V_{\lambda,j}$ which takes a forest $F = \{T_1, \cdots, T_m\}$ to the sum of all forests obtained from F by adding 1 to the mark of one of the marked sets (for this to give an allowable forest, the original mark must be at most the size of its set minus 2). It is clear that ϕ is still well-defined when one takes (2.2) into account, and that ϕ commutes with the action of $\mathfrak{S}_n$. By Proposition 2.8, in order to prove that $Q_{\lambda,j} - Q_{\lambda,j-1}$ is Schur-positive, we need only show that ϕ is injective (since then $Q_{\lambda,j} - Q_{\lambda,j-1}$ is the Frobenius characteristic of the cokernel of ϕ).

Now there is some collection $\mathcal{F}$ of unmarked forests, depending on λ but not on j, such that the marked forests as defined above whose underlying unmarked forest lies in $\mathcal{F}$ form a basis of $V_{\lambda,j}$. For example, if $\lambda = (n)$, we can take $\mathcal{F}$ to consist of all binary trees of the form $(\cdots((S_1 \wedge S_2) \wedge S_3) \cdots \wedge S_t)$ where $S_1, S_2, \cdots, S_t$ form a partition of $[n]$ and $1 \in S_1$. Since ϕ only changes the marking, it is enough to prove the injectivity when we have fixed the underlying unmarked forest to be some element F of $\mathcal{F}$.

We are now in a familiar situation. We have a collection of disjoint sets $A_1, A_2, \cdots, A_s$ (the nonsingleton leaves of F) such that $|A_i| \geq 2$ for all i and $|A_1| + |A_2| + \cdots + |A_s| = n - k$. We are considering a vector space V with basis $[|A_1| - 1] \times \cdots \times [|A_s| - 1]$, to which we give a grading $V = \bigoplus_j V_j$ by the rule that $(b_1, \cdots, b_s) \in V_{b_1+\cdots+b_s}$ for any $b_i \in [|A_i| - 1]$. We must show that for any j such that $1 \leq j \leq \frac{n-k}{2}$, the linear map $\phi : V_{j-1} \to V_j$ defined by

$$\phi(b_1, \cdots, b_s) = \sum_{\substack{1 \leq i \leq s \\ b_i \leq |A_i| - 2}} (b_1, \cdots, b_i + 1, \cdots, b_s)$$

is injective. This is a well-known fact, a special case of a far more general result on raising operators in posets [8]. □

3. Proof of Theorem 1.2

The $p = 1$ case of Theorem 1.2 follows immediately from Theorem 1.1 and the observation from [10, eq. (2.13)] that $a_{\lambda,j}^{\mathrm{maj}',\mathrm{des}}(q, 1) = (q; q)_n \mathbf{ps}(Q_{\lambda,j})$, where $\mathbf{ps}$ denotes the stable principal specialization (see [10, Lemma 5.2]). The proof for general p makes use of the (nonstable) principal specialization as in (1.9) and is much more involved.

For permutations $\alpha \in \mathfrak{S}_S$ and $\beta \in \mathfrak{S}_T$ on disjoint sets S, T, let $\mathrm{sh}(\alpha, \beta)$ denote the set of shuffles of α and β. That is,

$$\mathrm{sh}(\alpha, \beta) := \{\sigma \in \mathfrak{S}_{S \cup T} : \alpha \text{ and } \beta \text{ are subwords of } \sigma\}.$$

We define

$$\mathrm{sh}^*(\alpha, \beta) := \{\sigma \in \mathrm{sh}(\alpha, \beta) : \sigma_1 = \alpha_1\}.$$

Some care is needed with this definition in the case that S is empty, when α is the empty word $\varnothing$ and α_1 is not defined. We have $\mathrm{sh}(\varnothing, \beta) =$

$\{\beta\}$, and we declare that $\mathrm{sh}^*(\varnothing,\beta)$ is empty unless $\beta=\varnothing$ also, in which case we set $\mathrm{sh}^*(\varnothing,\varnothing)=\{\varnothing\}$.

For $i,j\in\mathbb{N}$ with $i\le j+1$, let ϵ_i^j denote the word $i,i+1,\cdots,j$ (which means the empty word $\varnothing$ if $i=j+1$).

Lemma 3.1. *Let $m,k,r\in\mathbb{N}$ with $r\le k$. Then for all $\alpha\in\mathfrak{S}_m$,*

$$\sum_{i=r}^{k}(pq^r;q)_{i-r}\sum_{\sigma\in\mathrm{sh}(\alpha,\epsilon_{m+1}^{m+k-i})}(pq^i)^{\mathrm{des}(\sigma)+1}q^{\mathrm{maj}(\sigma)}$$
$$=\sum_{i=r}^{k}\sum_{\sigma\in\mathrm{sh}^*(\alpha,\epsilon_{m+1}^{m+k-i})}(pq^i)^{\mathrm{des}(\sigma)+1}q^{\mathrm{maj}(\sigma)}.$$

Proof. We use induction on $k-r$. The case $r=k$ is trivial, because both sides have only one term, namely $(pq^k)^{\mathrm{des}(\alpha)+1}q^{\mathrm{maj}(\alpha)}$.

Now suppose $r<k$, and that we know the result when r is replaced by $r+1$. Then the left-hand side of our desired equation equals

$$\sum_{\sigma\in\mathrm{sh}(\alpha,\epsilon_{m+1}^{m+k-r})}(pq^r)^{\mathrm{des}(\sigma)+1}q^{\mathrm{maj}(\sigma)}$$
$$+(1-pq^r)\sum_{i=r+1}^{k}(pq^{r+1};q)_{i-r-1}\sum_{\sigma\in\mathrm{sh}(\alpha,\epsilon_{m+1}^{m+k-i})}(pq^i)^{\mathrm{des}(\sigma)+1}q^{\mathrm{maj}(\sigma)}$$
$$=\sum_{\sigma\in\mathrm{sh}(\alpha,\epsilon_{m+1}^{m+k-r})}(pq^r)^{\mathrm{des}(\sigma)+1}q^{\mathrm{maj}(\sigma)}$$
$$+(1-pq^r)\sum_{i=r+1}^{k}\sum_{\sigma\in\mathrm{sh}^*(\alpha,\epsilon_{m+1}^{m+k-i})}(pq^i)^{\mathrm{des}(\sigma)+1}q^{\mathrm{maj}(\sigma)}.$$

To complete the proof we need only show that

$$\sum_{\tau\in\mathrm{sh}(\alpha,\epsilon_{m+1}^{m+k-r})\setminus\mathrm{sh}^*(\alpha,\epsilon_{m+1}^{m+k-r})}(pq^r)^{\mathrm{des}(\tau)+1}q^{\mathrm{maj}(\tau)}$$
$$=pq^r\sum_{i=r+1}^{k}\sum_{\sigma\in\mathrm{sh}^*(\alpha,\epsilon_{m+1}^{m+k-i})}(pq^i)^{\mathrm{des}(\sigma)+1}q^{\mathrm{maj}(\sigma)}.\tag{3.1}$$

Now every $\tau\in\mathrm{sh}(\alpha,\epsilon_{m+1}^{m+k-r})\setminus\mathrm{sh}^*(\alpha,\epsilon_{m+1}^{m+k-r})$ can be written uniquely in the form $\epsilon_{m+1}^{m+i-r}\sigma'$ where $r<i\le k$ and $\sigma'\in\mathrm{sh}^*(\alpha,\epsilon_{m+i-r+1}^{m+k-r})$. Subtracting $i-r$ from every letter of σ' which exceeds m, we obtain

an element $\sigma \in \mathrm{sh}^*(\alpha, \epsilon_{m+1}^{m+k-i})$. This gives a bijection

$$\mathrm{sh}(\alpha, \epsilon_{m+1}^{m+k-r}) \setminus \mathrm{sh}^*(\alpha, \epsilon_{m+1}^{m+k-r}) \leftrightarrow \biguplus_{i=r+1}^{k} \mathrm{sh}^*(\alpha, \epsilon_{m+1}^{m+k-i})$$
$$\tau \mapsto \sigma.$$

It is easy to see that

$$\begin{aligned} \mathrm{des}(\tau) &= \mathrm{des}(\sigma) + 1, \\ \mathrm{maj}(\tau) &= \mathrm{maj}(\sigma) + (i-r)(\mathrm{des}(\sigma)+1). \end{aligned}$$

We thus have

$$\begin{aligned} &\sum_{\tau \in \mathrm{sh}(\alpha, \epsilon_{m+1}^{m+k-r}) \setminus \mathrm{sh}^*(\alpha, \epsilon_{m+1}^{m+k-r})} (pq^r)^{\mathrm{des}(\tau)+1} q^{\mathrm{maj}(\tau)} \\ &= \sum_{i=r+1}^{k} \sum_{\sigma \in \mathrm{sh}^*(\alpha, \epsilon_{m+1}^{m+k-i})} (pq^r)^{\mathrm{des}(\sigma)+2} q^{\mathrm{maj}(\sigma)+(i-r)(\mathrm{des}(\sigma)+1)} \\ &= pq^r \sum_{i=r+1}^{k} \sum_{\sigma \in \mathrm{sh}^*(\alpha, \epsilon_{m+1}^{m+k-i})} (pq^i)^{\mathrm{des}(\sigma)+1} q^{\mathrm{maj}(\sigma)}, \end{aligned}$$

which establishes (3.1). □

We deduce a result about the principal specialization of order m.

Proposition 3.2. *Let $k, n \in \mathbb{N}$ with $k \le n$. For any subset S of $[n-k-1]$,*

$$(p;q)_{n+1} \sum_{m \ge 0} p^m \sum_{i=0}^{k} q^{im} \mathbf{ps}_m(F_{S,n-k} h_{k-i}) \in \mathbb{N}[q,p].$$

More precisely, this expression equals

$$\sum_{i=0}^{k} \sum_{\sigma \in \mathrm{sh}^*(\alpha, \epsilon_{n-k+1}^{n-i})} (pq^i)^{\mathrm{des}(\sigma)+1} q^{\mathrm{maj}(\sigma)},$$

where α is any fixed permutation in $\mathfrak{S}_{n-k}$ with descent set S.

Proof. Let $\alpha \in \mathfrak{S}_{n-k}$ have descent set S. Note that $h_{k-i} = F_{\emptyset, k-i}$. As a special case of the general rule for multiplying fundamental quasisymmetric functions (see [12, Exercise 7.93]), we have

$$F_{S,n-k} h_{k-i} = \sum_{\sigma \in \mathrm{sh}(\alpha, \epsilon_{n-k+1}^{n-i})} F_{\mathrm{DES}(\sigma), n-i}. \tag{3.2}$$

Hence

$$
\begin{aligned}
(p;q)_{n+1} \sum_{m\geq 0} p^m \sum_{i=0}^{k} q^{im} \mathbf{ps}_m(F_{S,n-k} h_{k-i}) \\
&= (p;q)_{n+1} \sum_{i=0}^{k} \sum_{\sigma\in \mathrm{sh}(\alpha,\epsilon_{n-k+1}^{n-i})} \sum_{m\geq 0} (pq^i)^m \mathbf{ps}_m(F_{\mathrm{DES}(\sigma),n-i}) \\
&= (p;q)_{n+1} \sum_{i=0}^{k} \sum_{\sigma\in \mathrm{sh}(\alpha,\epsilon_{n-k+1}^{n-i})} \frac{(pq^i)^{\mathrm{des}(\sigma)+1} q^{\mathrm{maj}(\sigma)}}{(pq^i;q)_{n-i+1}} \\
&= \sum_{i=0}^{k} (p;q)_i \sum_{\sigma\in \mathrm{sh}(\alpha,\epsilon_{n-k+1}^{n-i})} (pq^i)^{\mathrm{des}(\sigma)+1} q^{\mathrm{maj}(\sigma)} \\
&= \sum_{i=0}^{k} \sum_{\sigma\in \mathrm{sh}^*(\alpha,\epsilon_{n-k+1}^{n-i})} (pq^i)^{\mathrm{des}(\sigma)+1} q^{\mathrm{maj}(\sigma)},
\end{aligned}
$$

with the second equation following from [4, Lemma 5.2] and the fourth equation following from the $r = 0, m = n - k$ case of Lemma 3.1. □

We can now deduce Theorem 1.2.

Proof of Theorem 1.2. Recall (1.9) that we can express $a_{\lambda,j}^{\mathrm{maj}',\mathrm{des}}(q,p)$ in terms of $\mathbf{ps}_m(Q_{(\mu,1^{k-i}),j})$, where $\lambda = (\mu, 1^k)$ and $\mu \vdash n-k$ has no parts equal to 1. It is clear from (1.7) that $Q_{(\mu,1^{k-i}),j} = Q_{\mu,j} h_{k-i}$. So (1.9) can be rewritten

$$
a_{\lambda,j}^{\mathrm{maj}',\mathrm{des}}(q,p) = (p;q)_{n+1} \sum_{m\geq 0} p^m \sum_{i=0}^{k} q^{im}\, \mathbf{ps}_m(Q_{\mu,j} h_{k-i}). \tag{3.3}
$$

For any j such that $1 \leq j \leq \frac{n-k}{2}$, we therefore have

$$
\begin{aligned}
&a_{\lambda,j}^{\mathrm{maj}',\mathrm{des}}(q,p) - a_{\lambda,j-1}^{\mathrm{maj}',\mathrm{des}}(q,p) \\
&\qquad = (p;q)_{n+1} \sum_{m\geq 0} p^m \sum_{i=0}^{k} q^{im}\, \mathbf{ps}_m((Q_{\mu,j} - Q_{\mu,j-1}) h_{k-i}).
\end{aligned} \tag{3.4}
$$

Now by Theorem 1.1, $Q_{\mu,j} - Q_{\mu,j-1}$ is a nonnegative integer linear combination of Schur functions s_ρ for $\rho \vdash n-k$. By [12, Theorem 7.19.7], each s_ρ is in turn a nonnegative integer linear combination of fundamental quasisymmetric functions $F_{S,n-k}$ for $S \subseteq [n-k-1]$. So (3.4) belongs to $\mathbb{N}[q,p]$ by Proposition 3.2. □

Acknowledgements

The research presented here began while both authors were visiting the Centro di Ricerca Matematica Ennio De Giorgi at the Scuola Normale Superiore in Pisa as participants in the program 'Configuration Spaces: Geometry, Combinatorics and Topology', May–June 2010. We thank the members of the scientific committee, A. Björner, F. Cohen, C. De Concini, C. Procesi, and M. Salvetti, for inviting us to participate, and the center for its support.

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