# TOP HOMOLOGY OF HYPERGRAPH MATCHING COMPLEXES, $p$-CYCLE COMPLEXES AND QUILLEN COMPLEXES OF SYMMETRIC GROUPS

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Abstract. We investigate the representation of the symmetric group on the homology of the Quillen complex of the symmetric group, extending the earlier work of Bouc, Ksontini, and Shareshian. We derive an explicit formula for the case of top (respectively, near top) dimensional homology in terms of the representation on the top (respectively, near top) homology of hypergraph matching complexes. A third complex called the $p$-cycle complex plays a key role in connecting these two complexes. We also conjecture an explicit formula for the representation of the symmetric group on the top homology of the $p$-regular hypergraph matching complex and prove our conjecture for $p = 3$ by deriving a nice decomposition of the representation into irreducibles.

## 1. Introduction

Our purpose is to obtain information on the homology of Quillen complexes of symmetric groups (at odd primes) by studying hypergraph matching complexes. This method was first investigated (for the prime two) by S. Bouc in [Bo2] and developed further by R. Ksontini in [Ks1, Ks2, Ks3] and later by Shareshian in [Sh]. We concentrate here on the top (and near top) homology of the two complexes just named, along with that of a complex called the $p$-cycle complex, which is essential for the transfer of information that we obtain. We work throughout this paper with complex coefficients.

Let us now define the objects with which we are concerned (further basic definitions appear in Section 2) and provide some more history and motivation along with a description of our (somewhat technical) results. Let $G$ be a finite group and let $p$ be a prime. The *Quillen complex* $\Delta\mathcal{A}_p(G)$ is the order complex of the partially ordered set $\mathcal{A}_p(G)$

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of all nontrivial elementary abelian $p$-subgroups of $G$. Interest in these complexes was sparked by the paper [Qu] of D. Quillen. (Earlier work on the closely related Brown complex $\Delta\mathcal{S}_p(G)$, which is the order complex of the poset of all nontrivial $p$-subgroups of a not necessarily finite group $G$, was done by K. S. Brown in [Br1, Br2].) Among many other things, it is shown in [Qu] that if $G$ is a group of Lie type in characteristic $p$ then $\Delta\mathcal{A}_p(G)$ is homotopy equivalent to the building for $G$. Thus in this case the (reduced) homology of $\Delta\mathcal{A}_p(G)$ is concentrated in a single dimension. Moreover, as noted in [We], it can be shown that the representation of $G$ on this unique nontrivial homology group obtained from the natural action of $G$ on $\mathcal{A}_p(G)$ is the same as that of $G$ on the unique nontrivial homology group of the building, namely, the Steinberg representation.

Given the results just mentioned, it is natural to investigate the homology of Quillen complexes of symmetric and alternating groups. (Note that if $p$ is odd then $\mathcal{A}_p(S_n) = \mathcal{A}_p(A_n)$.) The homology of $\Delta\mathcal{A}_p(S_n)$ need not be concentrated in a single dimension (see [Ks1, Section 15]). It seems quite difficult to determine the representation of $S_n$ on each nontrivial homology group of its Quillen complex, or even the Betti numbers of the complex. Two alternatives suggest themselves in the search for results analogous to those for the Lie type groups. Namely, one could investigate the Lefschetz virtual character (that is, the alternating sum of the characters for the representations on homology groups), as has been done successfully in various combinatorial settings (see [Bo1] for results along this line for Quillen complexes of symmetric groups), or one could investigate the top homology group. As mentioned above, we make the second choice.

For a prime $p$ and an integer $n$, the *p-cycle complex* $\mathsf{C}_p(n)$ is a simplicial complex with one vertex for each subgroup of $S_n$ that is generated by a $p$-cycle. A collection of such vertices forms a face of $\mathsf{C}_p(n)$ if and only if the subgroups in question together generate an abelian group. For any integers $p, n$ the *hypergraph matching complex* $\mathsf{M}_p(n)$ is the simplicial complex with one vertex for each subset of size $p$ from the $n$-set $[n]$, with a collection of such vertices forming a face of $\mathsf{M}_p(n)$ if and only if the sets in question are pairwise disjoint. One can view the vertices of $\mathsf{M}_p(n)$ as hyperedges of the complete $p$-regular hypergraph on $[n]$ and the faces of $\mathsf{M}_p(n)$ as $p$-matchings on $[n]$. For $p = 2, 3$ the complexes $\mathsf{C}_p(n)$ and $\mathsf{M}_p(n)$ are isomorphic, as for each $X \subseteq [n]$ of size $p$, there is unique cyclic group of order $p$ in $S_n$ having support $X$. Matching complexes and related complexes have been studied in the literature for their intrinsic combinatorial interest and in connection with applications in various fields of mathematics, see [Wa] for a survey and see [Jo1, Jo2, Jo3, SW] for more recent developments.

As mentioned above, the idea of using $\mathsf{M}_p(n)$ to study $\Delta\mathcal{A}_p(S_n)$ is originally due to Bouc, who studied the case $p = 2$. Various interesting results on $\mathsf{M}_2(n)$ appear in [Bo2], including a complete description of the representation of $S_n$ on its homology groups (see (14)). However, it seems quite difficult to use information about $\mathsf{M}_2(n)$ to obtain results on $\Delta\mathcal{A}_2(S_n)$. Such a transfer of information is easier when the prime in question is odd. The first evidence of this appears in the thesis [Ks1] of Ksontini, where a relationship between $\mathsf{C}_p(n)$ and $\Delta\mathcal{A}_p(S_n)$ is discussed. There is an obvious simplicial map from $\mathsf{C}_p(n)$ to $\mathsf{M}_p(n)$, induced by the map on vertices which sends a cyclic group generated by a $p$-cycle to its support, and it is natural to try to use this map to find useful relationships between the topology of $\mathsf{M}_p(n)$ and that of $\mathsf{C}_p(n)$. This is also done quite successfully in [Ks1]. In [Sh], a result of A. Björner, Wachs and V. Welker [BjWaWe] is used to make a precise statement showing how the homotopy type of $\mathsf{C}_p(n)$ is completely determined by that of the complexes $\mathsf{M}_p(m)$ for all $m \leq n$ satisfying $m \equiv n \bmod p$. Since we are interested here in representations on homology, we need an $S_n$-equivariant homology version of this result. Such a result was already proved in [BjWaWe].

It is straightforward to show that, for given $n, p$, the complexes $\Delta\mathcal{A}_p(S_n)$, $\mathsf{C}_p(n)$ and $\mathsf{M}_p(n)$ all have dimension $t := t(n,p) := \lfloor \frac{n}{p} \rfloor - 1$. The first key idea for our work is an equivariant version of a result of Ksontini [Ks1, Proposition 8.1] stating that if $i$ is within $p-3$ of $t$ then

$$\widetilde{H}_i(\Delta\mathcal{A}_p(S_n)) \cong_{S_n} \widetilde{H}_i(\mathsf{C}_p(n)),$$

where $\cong_{S_n}$ denotes isomorphism of $\mathbb{C}[S_n]$-modules. Since this result appears only in the nonequivariant form in the thesis of Ksontini [Ks1], we provide a proof in Section 3 (see Theorem 3.2). In Section 4 we apply the result from [BjWaWe] in order to obtain a complicated but explicit formula expressing the Frobenius characteristic of the $\mathbb{C}[S_n]$-module $\widetilde{H}_*(\mathsf{C}_p(n))$ in terms of the characteristics of the $\mathbb{C}[S_m]$-modules $\widetilde{H}_*(\mathsf{M}_p(m))$ for all $m \leq n$ such that $m \equiv n \bmod p$ (see Theorem 4.7).

This leaves us with the problem of determining the representation of $S_n$ on $\widetilde{H}_{t-i}(\mathsf{M}_p(n))$ when $i \leq p-3$. In Section 5 we give a conjectural description of this representation when $i = 0$ and $n = kp+1$ for some $k$ (so $t = k-1$). Namely, Conjecture 5.1 says that $\widetilde{H}_{k-1}(\mathsf{M}_p(kp+1))$ is the $\mathbb{C}[S_{kp+1}]$-submodule generated by all simple submodules of the chain space $C_{k-1}(\mathsf{M}_p(kp+1))$ that are isomorphic to Specht modules

$\mathcal{S}^\lambda$ where $\lambda$ has $k+1$ parts. (As shown in Section 5, if $C_{k-1}(\mathsf{M}_p(kp+1))$ has a submodule isomorphic to $\mathcal{S}^\mu$ then $\mu$ has at most $k+1$ parts.) It follows from the work of Bouc mentioned above that Conjecture 5.1 is true when $p=2$, and from work of Ksontini [Ks3] that it is true when $k \leq 2$. We prove the conjecture in the case $p=3$ by showing that the representation is isomorphic to the direct sum of Specht modules indexed by partitions of $3k+1$ into $k+1$ odd parts. As a corollary we have that the representation of $S_{3k+1}$ on the top homology of the Quillen complex $\Delta\mathcal{A}_3(S_{3k+1})$ also has this nice decomposition.

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## 2. Preliminaries: Definitions, notation and basic results

For a finite poset $P$, $\Delta P$ will denote the abstract simplicial complex whose $k$-simplices are chains of length $k$ from $P$. For an abstract simplicial complex $\Delta$, $|\Delta|$ will denote a geometric realization of $\Delta$ (note that all such realizations are homeomorphic). We will not distinguish between $\Delta$ and $|\Delta|$. Also, $\mathsf{P}\Delta$ will denote the poset of nonempty faces of a complex $\Delta$ and $\widetilde{H}_i(\Delta)$ will denote the $i^{th}$ reduced simplicial homology group of $\Delta$, with complex coefficients. A simplicial action of a group $G$ on a complex $\Delta$ determines representations of $G$ on the chain spaces and homology groups of $\Delta$. (Note that an order preserving action of $G$ on a poset $P$ determines a simplicial action of $G$ on $\Delta P$.) We note here that the natural action of $S_n$ on $[n]$ induces a simplicial action on $\mathsf{M}_p(n)$ and that the action of $S_n$ by conjugation on the set of its $p$-subgroups induces simplicial actions on both $\mathsf{C}_p(n)$ and $\Delta\mathcal{A}_p(S_n)$.

It is well known that for any complex $\Delta$, $\Delta\mathsf{P}\Delta$ is the barycentric subdivision of $\Delta$, so if $G$ is a group acting (simplicially) on $\Delta$ then there is a $G$-equivariant homeomorphism from $\Delta$ to $\Delta\mathsf{P}\Delta$. The next result follows, where $\cong_G$ denotes isomorphism of $\mathbb{C}[G]$-modules.

**Lemma 2.1.** *Let $\Delta$ be a finite simplicial and let $G$ be a group acting simplicially on $\Delta$. Then for any integer $i$ we have*

$$\widetilde{H}_i(\Delta) \cong_G \widetilde{H}_i(\Delta\mathsf{P}\Delta).$$

For a finite lattice $L$ with minimum element $\hat{0}$ and maximum element $\hat{1}$, let $\bar{L}$ denote the poset obtained from $L$ by removing $\hat{0}$ and $\hat{1}$ and let $\tilde{L}$ denote the subposet of $\bar{L}$ consisting of those elements which can be obtained by taking the meet (in $L$) of some set of maximal elements of $\bar{L}$. The next result is a well known equivariant version of Rota's Cross-cut Theorem ([Ro]).

**Lemma 2.2.** *Let $G$ be a group acting on a finite lattice $L$. Then for every integer $i$, we have*

$$\widetilde{H}_i(\Delta\bar{L}) \cong_G \widetilde{H}_i(\Delta\tilde{L}).$$

*Proof.* Let $\iota : \tilde{L} \to \bar{L}$ be the identity embedding. Since $\iota$ is $G$-equivariant, our claim will follow from the equivariant version of the Quillen fiber theorem, see [BjWaWe, Corollary 9.3], once we show that $\Delta\iota_{\geq y}$ is acyclic for all $y \in \bar{L}$, where $\iota_{\geq y}$ is the induced subposet $\{x \in \tilde{L} : x \geq y\}$ of $\tilde{L}$. For any such $y$, $\iota_{\geq y}$ contains a unique minimum element, namely, the meet of all maximal elements of $\bar{L}$ which lie above $y$. Hence $\Delta\iota_{\geq y}$ is a cone and is therefore acyclic. □

Let us now review some standard material on symmetric functions. See [Mac, St] for the relevant results. We will use the standard notation $\mathsf{e}, \mathsf{h}, \mathsf{p}, \mathsf{s}$ for the elementary, complete homogeneous, power sum, and Schur symmetric functions, respectively. We hope that the use of $\mathsf{p}$ for a symmetric function and $p$ for a prime will not confuse the reader. The irreducible representation of $S_n$ corresponding to the partition $\lambda$ of $n$ will be denoted by $\mathcal{S}^\lambda$. The Frobenius characteristic is the isomorphism from the graded ring whose $n$ component consists of all (virtual and actual) representations of $S_n$ to the graded ring of symmetric functions over the integers that sends $\mathcal{S}^\lambda$ to $\mathsf{s}_\lambda$ for each partition $\lambda$. In particular, the Frobenius characteristics of the trivial and alternating representations of $S_n$ are $\mathsf{h}_n$ and $\mathsf{e}_n$, respectively. The Frobenius characteristic of a representation on the space $V$ will be denoted by $\mathrm{ch}V$

The plethysm of symmetric functions $\mathsf{f}, \mathsf{g}$ will be denoted by $\mathsf{f}[\mathsf{g}]$ rather than $\mathsf{f} \circ \mathsf{g}$ (which is used in [Mac]). Plethysm is used here to manipulate representations of symmetric groups induced from stabilizers of set partitions. Say $n = kl$ with $1 < k, l < n$. The stabilizer $H$ in $S_n$ of a partition of $[n]$ into $k$ blocks $X_1, \dots, X_k$ of size $l$ is isomorphic to the wreath product $S_k[S_l]$. That is, $H$ contains the normal subgroup $K = \prod_{i=1}^k S_{X_i} \cong S_l^k$ (which we will call the *kernel* of $H$), and $H/K \cong S_k$. We get a complement $C$ to $K$ in $H$ as follows. Fix for each $(i,j) \in [k] \times [k]$ a bijection $\psi_{ij} : X_i \to X_j$ such that

- $\psi_{ij}\psi_{jm} = \psi_{im}$ for all $i, j, m$, and
- $\psi_{ji} = \psi_{ij}^{-1}$ for all $i, j$.

Define $\alpha : S_k \to S_n$ by $x\alpha(\tau) := x\psi_{i,i\tau}$ for $\tau \in S_k$ and $x \in X_i$. Then $\alpha$ is an injective homomorphism, and it is straightforward to check that $C = Image(\alpha)$ is a complement to $K$ in $H$. All such $C$ are conjugate in $H$, and we call such a $C$ a *standard complement* to $K$. If $\phi : S_k \to GL(V)$ and $\rho : S_l \to GL(W)$ are representations, we get (after identifying $C$ with $S_k$ and each $S_{X_i}$ with $S_l$) a representation $\phi[\rho] : H \to GL(V \otimes \bigotimes^k W)$ such that for $g \in S_{X_i}$ we have

$$(v \otimes w_1 \otimes \ldots \otimes w_i \otimes \ldots \otimes w_k)\phi[\rho](g) = v \otimes w_1 \otimes \ldots \otimes w_i\rho(g) \otimes \ldots \otimes w_k,$$

and for $c \in C$ we have

$$(v \otimes w_1 \otimes \ldots \otimes w_k)\phi[\rho](c) = v\phi(c) \otimes w_{1c} \ldots \otimes w_{kc}.$$

If $\phi, \psi$ have Frobenius characteristics $\mathsf{f}, \mathsf{g}$, respectively, then the Frobenius characteristic of the induced representation $\phi[\rho] \uparrow_H^{S_n}$ is $\mathsf{f}[\mathsf{g}]$.

## 3. Relating the top homology of the $p$-cycle complex to that of the Quillen complex

Much (but not all) of the material in this section can be found in [Ks1] and [Sh]. We include details here for the sake of self-containment.

Say $P \in \mathcal{A}_p(S_n)$ has orbits $\Omega_1, \ldots, \Omega_r$ on $[n]$. For each $i \in [r]$, we have a homomorphism

$$\omega_i : P \to S_{\Omega_i} \tag{1}$$

determined by the action of $P$ on $\Omega_i$. Set

$$\overline{P} := \prod_{i=1}^{r} \omega_i(P) \leq S_n.$$

Each $\omega_i(P)$ is a quotient of the elementary abelian $p$-group $P$ and is therefore an elementary abelian $p$-group. Since elements of $S_n$ whose supports are disjoint commute, we see that $\overline{P}$ is elementary abelian, that is, $\overline{P} \in \mathcal{A}_p(S_n)$. Certainly $P \leq \overline{P}$. Each $\omega_i(P)$ is a transitive abelian subgroup of $S_{\Omega_i}$, and is therefore regular. In other words, $|\omega_i(P)| = |\Omega_i|$. It follows that

$$\mathsf{rank}(\overline{P}) = \sum_{i=1}^{r} \mathsf{rank}(\omega_i(P)) = \sum_{i=1}^{r} \log_p(|\Omega_i|). \tag{2}$$

**Lemma 3.1.** *Let $P \in \mathcal{A}_p(S_n)$. Then $\mathsf{rank}(P) \leq \lfloor \frac{n}{p} \rfloor$.*

*Proof.* Our claim holds by observation when $n \leq p$, and we proceed by induction on $n$. It suffices to prove the claim when $P$ is such that

$P = \overline{P}$. Given the orbits $\Omega_i$ and maps $\omega_i$ as defined above, we may assume that $|\Omega_1| > 1$. Set

$$Q = \prod_{i=2}^{r} \omega_i(P) \leq S_{\bigcup_{i=2}^{r} \Omega_i}.$$

By equation (2) and our inductive hypothesis, we have

$$\mathsf{rank}(P) = \log_p(|\Omega_1|) + \mathsf{rank}(Q) \leq \log_p(|\Omega_1|) + \lfloor \frac{n - |\Omega_1|}{p} \rfloor. \tag{3}$$

The lemma now follows from the fact that for any positive integer $a$, we have

$$\lfloor \frac{n - p^a}{p} \rfloor + a \leq \lfloor \frac{n}{p} \rfloor.$$

□

**Theorem 3.2** (Ksontini [Ks1, Proposition 8.1])**.** *Let $p$ be a prime and $n$ a positive integer. Set $t = \lfloor \frac{n}{p} \rfloor - 1$. Then*

(1) $\dim(\Delta\mathcal{A}_p(S_n)) = \dim(\mathsf{C}_p(n)) = t$, *and*

(2) $\widetilde{H}_j(\Delta\mathcal{A}_p(S_n)) \cong_{S_n} \widetilde{H}_j(\mathsf{C}_p(n))$ *for all* $j \geq t - (p-3)$.

*Proof.* Set

$$\mathcal{M} = \{P \in \mathcal{A}_p(S_n) : P \text{ is generated by } \lfloor \frac{n}{p} \rfloor \text{ pairwise disjoint } p\text{-cycles}\},$$

and let

$$\mathsf{I}_p(n) = \{Q \in \mathcal{A}_p(S_n) : Q \leq P \text{ for some } P \in \mathcal{M}\}.$$

Then $\mathsf{I}_p(n)$ is an $S_n$-invariant subposet of $\mathcal{A}_p(S_n)$, and by Lemma 3.1 we have

$$\dim \Delta\mathcal{A}_p(S_n) = \dim \Delta\mathsf{I}_p(n) = t.$$

The elements of any maximal face of $\mathsf{C}_p(n)$ together generate an element of $\mathcal{M}$. It follows that

$$\dim \mathsf{C}_p(n) = t$$

and claim (1) follows.

Next we show that if $j \geq t - p + 3$ then

$$\widetilde{H}_j(\Delta\mathcal{A}_p(S_n)) \cong_{S_n} \widetilde{H}_j(\Delta\mathsf{I}_p(n)). \tag{4}$$

We prove this by first establishing the implication

$$P \in \mathcal{A}_p(S_n) - \mathsf{I}_p(n) \implies \mathsf{rank}(P) \leq t - (p-3). \tag{5}$$

It suffices to prove the implication when $P$ is a maximal element of $\mathcal{A}_p(S_n)$, in which case $P = \overline{P}$. Let $\Omega_i$ and $Q$ be defined as in the proof of Lemma 3.1, with $|\Omega_1| \geq |\Omega_i|$ for all $i$. It follows from Lemma 3.1

that the string of equalities and inequalities given in (3) holds. Let $a = \log(|\Omega_1|)$. Then $a$ is a positive integer and it follows from (3) that $\mathsf{rank}(P) \leq t + 1 + a - p^{a-1}$. It is easy to see that $a - p^{a-1} \leq 2 - p$ for all $a \geq 2$. Hence $\mathsf{rank}(P) \leq t - (p-3)$, unless $a = 1$, which we claim is impossible. Indeed, if $a = 1$ then all orbits have size $p$ or 1, which means that $P$ is generated by pairwise disjoint $p$-cycles. Since $P$ is maximal, the number of $p$-cycles is $\lfloor \frac{n}{p} \rfloor$, which means that $P \in \mathcal{M}$, contradicting $P \notin \mathsf{I}_p(n)$.

It follows from the implication (5) that every chain of $\mathcal{A}_p(S_n)$ with at least $t - p + 4$ elements is a chain of $\mathsf{I}_p(n)$ since its top element has rank at least $t - p + 4$. Hence (4) holds.

Now let

$$\mathsf{L}_p(n) = \{Q \in \mathsf{I}_p(n) : Q = \bigcap_{P \in \mathcal{N}} P \text{ for some } \mathcal{N} \subseteq \mathcal{M}\}.$$

For $Q \in \mathsf{L}_p(n)$, let $\mathcal{M}_Q$ be the set of elements of $\mathcal{M}$ which contain $Q$, so

$$Q = \bigcap_{P \in \mathcal{M}_Q} P.$$

For (nonidentity) $q \in Q$, write

$$q = \prod_{i=1}^{m} q_i,$$

where the $q_i$ are pairwise disjoint $p$-cycles. Let $P \in \mathcal{M}_Q$ be generated by pairwise disjoint $p$-cycles $p_1, \ldots, p_t$. The elements of $P$ have the form

$$\prod_{j=1}^{t} p_j^{a_j}, (0 \leq a_j < p),$$

since the $p_j$ commute. Since $q \in P$, we see that each $q_i$ is a power of some $p_j$ and it follows that $q_i \in P$ for each $i$. Since $P$ is an arbitrary element of $\mathcal{M}_Q$, we see that each $q_i$ lies in $Q$. Therefore, $Q$ is generated by $p$-cycles. We may now apply Lemma 2.2 twice to get

$$\widetilde{H}_i(\Delta \mathsf{I}_p(n)) \cong_{S_n} \widetilde{H}_i(\Delta \mathsf{L}_p(n)) \cong_{S_n} \widetilde{H}_i(\Delta \mathsf{PC}_p(n)),$$

for all $i$, and claim (2) now follows from the isomorphism (4) and Lemma 2.1. □

## 4. Relating the top homology of the $p$-cycle complex to that of the hypergraph matching complex

Let $\Delta$ be a simplicial complex on vertex set $\{x_1, \ldots, x_n\}$ and let $\mathbf{m} = (m_1, \ldots, m_n)$ be an $n$-tuple of positive integers. As defined in

[BjWaWe, Section 6], the $\mathbf{m}$-inflation $\Delta_{\mathbf{m}}$ of $\Delta$ is the complex on vertex set $\{(x_i, j_i) : i \in [n], j \in [m_i]\}$, such that $\{(x_{i_1}, j_{i_1}), \ldots, (x_{i_k}, j_{i_k})\}$ is a $(k-1)$-simplex in $\Delta_{\mathbf{m}}$ if and only if

- $\{x_{i_1}, \ldots, x_{i_k}\}$ is a $(k-1)$-simplex in $\Delta$, and
- $j_{i_l} \in [m_{i_l}]$ for all $l \in [k]$.

Roughly, $\Delta_{\mathbf{m}}$ is obtained from $\Delta$ by taking $m_i$ copies of vertex $x_i$ and then allowing every possible "version" of a face of $\Delta$ to be a face of $\Delta_{\mathbf{m}}$.

The relevance of inflations to the matter at hand is made clear by the following easy lemma (which is discussed in [Sh]).

**Lemma 4.1.** *Let $p$ be any prime and let $n$ be any positive integer. Let $\mathbf{m}$ be the $\binom{n}{p}$-tuple each of whose entries is $(p-2)!$. Then*

$$\mathsf{C}_p(n) \cong \mathsf{M}_p(n)_{\mathbf{m}}.$$

*Proof.* The vertices of $\mathsf{M}_p(n)$ are the $p$-sets from $[n]$. For each such $p$-set $X$, there are $(p-2)!$ cyclic subgroups of order $p$ in $S_n$ with support $X$. A $(k-1)$-simplex of $\mathsf{M}_p(n)$ is a collection of $k$ disjoint $p$-sets, while a $(k-1)$-simplex of $\mathsf{C}_p(n)$ is a collection of $k$ cyclic subgroups of $S_n$ with disjoint supports, each subgroup generated by a $p$-cycle. □

For a complex $\Delta$ and inflation $\Delta_{\mathbf{m}}$, the homotopy type of $\Delta_{\mathbf{m}}$ is determined by the homotopy types of the links of the faces of $\Delta$ (see [BjWaWe, Theorem 6.2]). Thus the homology of $\Delta_{\mathbf{m}}$ is determined by the homology of links in $\Delta$. There is an equivariant version of this homology result, which we will state below after making the appropriate definitions.

When a group $G$ acts (simplicially) on a complex $\Delta$, $G_F$ will denote the stabilizer in $G$ of a face $F$ of $\Delta$ and $\Delta/G$ will denote an arbitrary set of representatives for the orbits of $G$ on the set of faces of $\Delta$ (including the empty face). If $\Delta_{\mathbf{m}}$ is an inflation of $\Delta$ ($\mathbf{m} = (m_1, \ldots, m_n)$) and $F = \{x_{i_1}, \ldots, x_{i_k}\}$ is a face of $\Delta$ then $\mathbf{m}(F)$ will denote the $k$-tuple $(m_{i_1}, \ldots, m_{i_k})$. The *deflating map* $\delta : \Delta_{\mathbf{m}} \to \Delta$ is the simplicial map induced by the function on vertex sets which sends $(x_i, j_i)$ to $x_i$. For a face $F$ of $\Delta$, $\dot{F}$ will denote the set of all subsets of $F$ (a subcomplex of $\Delta$). If the actions of $G$ on the complexes $\Delta_{\mathbf{m}}$ and $\Delta$ are intertwined by the deflating map then $\dot{F}_{\mathbf{m}(F)}$ is a $G_F$-invariant subcomplex of $\Delta_{\mathbf{m}}$. In any case, $\mathsf{lk}_\Delta(F)$ is a $G_F$-invariant subcomplex of $\Delta$. Thus if $\delta$ intertwines the given actions then for any integers $i, j$, the tensor product

$$\widetilde{H}_i(\dot{F}_{\mathbf{m}(F)}) \otimes \widetilde{H}_j(\mathsf{lk}_\Delta(F))$$

is a $\mathbb{C}[G_F]$-module.

**Lemma 4.2** ([BjWaWe, Corollary 9.5])**.** *Let $\Delta$ be a simplicial complex on vertex set $\{x_1, \dots, x_n\}$ and let $\mathbf{m}$ be an $n$-tuple of positive integers. Let $G$ be a group which acts simplicially on $\Delta$ and $\Delta_{\mathbf{m}}$ so that these actions of $G$ are intertwined by the deflating map $\delta : \Delta_{\mathbf{m}} \to \Delta$. Then for each integer $r$, we have*

$$\widetilde{H}_r(\Delta_{\mathbf{m}}) \cong_G \bigoplus_{F \in \Delta/G} \left( \widetilde{H}_{|F|-1}(\dot{F}_{\mathbf{m}(F)}) \otimes \widetilde{H}_{r-|F|}(\mathsf{lk}_\Delta F) \right) \uparrow_{G_F}^{G} .$$

For a prime $p$ and an integer $n$, the deflating map $\delta : \mathsf{C}_p(n) \to \mathsf{M}_p(n)$ intertwines the actions of $S_n$ on the two complexes, so Lemma 4.2 applies. The following facts are straightforward to prove.

- For each integer $k$, the group $S_n$ acts transitively on the set of $k$-simplices of $\mathsf{M}_p(n)$. Therefore, $\mathsf{M}_p(n)/S_n$ consists of one matching with $k$ hyperedges for $0 \leq k \leq \lfloor \frac{n}{p} \rfloor$.
- Let $F \in \mathsf{M}_p(n)/S_n$ with $|F| = k$, so $F$ contains $k$ hyperedges $E_1, \dots, E_k$. Set $\Omega^+ = \bigcup_{i=1}^k E_i$ and $\Omega^- = [n] \setminus \Omega^+$.
  - The stabilizer $G_F$ of $F$ in $S_n$ is $H \times S_{\Omega^-}$, where $H \leq S_{\Omega^+}$ is isomorphic to the wreath product $S_k[S_p]$. The action of the kernel $K \cong (S_p)^k$ of $H$ is the componentwise action, that is, the $i^{th}$ component of $K$ permutes the $p$ vertices of $E_i$, while a standard complement $C \cong S_k$ in $H$ permutes the $k$ hyperedges $E_1, \dots, E_k$, as described in Section 2.
  - The link $\mathsf{lk}_{\mathsf{M}_p(n)}(F)$ is isomorphic to $\mathsf{M}_p(n - kp)$ (it is the $p$-regular hypergraph matching complex on vertex set $\Omega^-$).
  - The group $H$ acts trivially on $\Omega^-$ and therefore acts trivially on $\mathsf{lk}_{\mathsf{M}_p(n)}(F)$. The group $S_{\Omega^-}$ acts trivially on $\Omega^+$ and therefore acts trivially on $\dot{F}_{\mathbf{m}(F)}$. Thus for any integer $r$, the action of $G_F$ on the module $\widetilde{H}_{|F|-1}(\dot{F}_{\mathbf{m}(F)}) \otimes \widetilde{H}_{r-|F|}(\mathsf{lk}_{\mathsf{M}_p(n)}(F))$ is the usual tensor product action - that is, for $h \in H$, $\sigma \in S_{\Omega^-}$, $v \in \widetilde{H}_{|F|-1}(\dot{F}_{\mathbf{m}(F)})$ and $w \in \widetilde{H}_{r-|F|}(\mathsf{lk}_{\mathsf{M}_p(n)}(F))$, we have $(v \otimes w)(h, \sigma) = vh \otimes w\sigma$.

For $F \in \mathsf{M}_p(n)/S_n$ and $r$ any integer, let $V_r(F)$ be the $\mathbb{C}[G_F]$-module $\widetilde{H}_{|F|-1}(\dot{F}_{\mathbf{m}(F)}) \otimes \widetilde{H}_{r-|F|}(\mathsf{lk}_{\mathsf{M}_p(n)(F)})$, with action as described above. We are interested in the induced module $V_r(F) \uparrow_{G_F}^{S_n}$, which is a direct summand in the $\mathbb{C}[S_n]$-module $\widetilde{H}_r(\mathsf{C}_p(n))$ by Lemma 4.2. Basic facts from the theory of induced modules give

$$\begin{aligned} V_r(F)\uparrow_{G_F}^{S_n} &\cong_{S_n} (V_r(F)\uparrow_{G_F}^{S_{\Omega^+}\times S_{\Omega^-}})\uparrow_{S_{\Omega^+}\times S_{\Omega^-}}^{S_n} \\ &\cong_{S_n} (\widetilde{H}_{|F|-1}(\dot{F}_{\mathbf{m}(F)})\uparrow_H^{S_{\Omega^+}} \otimes \widetilde{H}_{r-|F|}(\mathsf{lk}_{\mathsf{M}_p(n)}(F)))\uparrow_{S_{\Omega^+}\times S_{\Omega^-}}^{S_n}. \end{aligned}$$

As noted above, if $|F| = k$ then the $\mathbb{C}[S_{\Omega^-}]$-module $\widetilde{H}_{r-|F|}(\mathsf{lk}_{\mathsf{M}_p(n)}(F))$ is equivalent to the $\mathbb{C}[S_{n-kp}]$-module $\widetilde{H}_{r-|F|}(\mathsf{M}_p(n-kp))$. In addition, if $r = \dim(\mathsf{C}_p(n))$ then $r-|F| = \dim(\mathsf{M}_p(n-kp))$. This gives the connection between the top homology modules of $p$-cycle complexes and those of $p$-regular hypergraph matching complexes, which will be made more precise once we understand the $\mathbb{C}[S_{\Omega^+}]$-module $\widetilde{H}_{|F|-1}(\dot{F}_{\mathbf{m}(F)})\uparrow_H^{S_{\Omega^+}}$.

Let us first understand the complex $\dot{F}_{\mathbf{m}(F)}$. If the matching $F$ contains hyperedges $E_1,\dots,E_k$ then (for $0 \le l \le k-1$) an $l$-simplex in $\dot{F}_{\mathbf{m}(F)}$ is obtained by selecting $l+1$ of the $k$ hyperedges of $F$, and for each of the selected hyperedges $E_i$, selecting one of the $(p-2)!$ subgroups of order $p$ from $S_{\Omega^+}$ whose support is $E_i$. It follows that

$$\dot{F}_{\mathbf{m}(F)} = \Delta_1 * \ldots * \Delta_k$$

where $*$ denotes the join of complexes and each $\Delta_i$ is a set of $(p-2)!$ points (equivalently, a wedge of $(p-2)!-1$ spheres of dimension 0). These points are the nontrivial $p$-subgroups of $S_{E_i}$.

The action of $H \cong S_k[S_p]$ on $\dot{F}_{\mathbf{m}(F)}$ is quite transparent when the complex is represented as the join of the subcomplexes $\Delta_i$. Namely, a standard complement $C \cong S_k$ acts by permuting the $k$ subcomplexes $\Delta_i$, and for any $i \in [k]$, the $i^{th}$ component of the kernel $K \cong (S_p)^k$ acts on the $(p-2)!$ points of $\Delta_i$ as it acts by conjugation on its $(p-2)!$ Sylow $p$-subgroups.

Let $P$ be the poset of $p$-subgroups of $S_p$ including the trivial subgroup; so $P$ has one minimum element $\hat{0}_P$ and $(p-2)!$ maximal elements. The symmetric group $S_p$ acts on elements of $P$ by conjugation. Clearly $P$ is isomorphic to the poset $P_i$ of faces of $\Delta_i$ for each $i$, and the action of $S_p$ on $P$ is equivalent to the action of $S_{E_i}$ on $P_i$. The action of $S_p$ on $P$ induces an obvious action of the wreath product $S_k[S_p]$ on the $k$-fold direct product $P^{\times k}$. It is straightforward to show that if $\hat{0}$ is the (unique) minimum element of $P^{\times k}$, then

$$\mathsf{P}(\Delta_1 * \ldots * \Delta_k) \cong P^{\times k} \setminus \{\hat{0}\}.$$

and that the action of $H$ on $\mathsf{P}(\Delta_1 * \ldots * \Delta_k)$ is equivalent to the action of $S_k[S_p]$ on $P^{\times k} \setminus \{\hat{0}\}$. By [Su, Proposition 2.7] the representation of $S_k[S_p]$ on $\widetilde{H}_{k-1}(\Delta P^{\times k} \setminus \{\hat{0}\})$ is $\mathrm{sgn}_k[\nu]$, where $\mathrm{sgn}_k$ is the alternating (or sign) representation of $S_k$ and $\nu$ is the representation of $S_p$ on

$\widetilde{H}_0(\Delta(P \setminus \{\hat{0}_P\}))$ induced by the action of $S_p$ on $P$ described above. Hence the representation of $H$ on $\widetilde{H}_{k-1}(\dot{F}_{\mathbf{m}(F)})$ is equivalent to the representation of $S_k[S_p]$ on $\mathrm{sgn}_k[\nu]$. Thus by viewing $\mathrm{sgn}_k[\nu]$ as a $\mathbb{C}[H]$-module via the isomorphism between $H$ and $S_k[S_p]$, we have

$$\widetilde{H}_{k-1}(\dot{F}_{\mathbf{m}(F)}) \uparrow_H^{S_{\Omega+}} \cong_{S_{\Omega+}} \mathrm{sgn}_k[\nu] \uparrow_H^{S_{\Omega+}} . \tag{6}$$

It is well known and straightforward to show that if $\Delta$ is a complex consisting of $m$ points, $K \leq S_m$ permutes these points transitively, $W$ is the permutation module for $\mathbb{C}[K]$ associated with this action, and $T$ is the (unique) trivial $\mathbb{C}[K]$-submodule of $W$ then

$$\widetilde{H}_0(\Delta) \cong_K W/T. \tag{7}$$

**Theorem 4.3.** *Let $N_p$ be the normalizer of a Sylow $p$-subgroup of $S_p$. Let $\mathsf{f}_p$ be the Frobenius characteristic of the induced character $1_{N_p}^{S_p}$. Then the $\mathbb{C}[S_{\Omega^+}]$-module $\widetilde{H}_{|F|-1}(\dot{F}_{\mathbf{m}(F)}) \uparrow_H^{S_{\Omega+}}$ is isomorphic to the $\mathbb{C}[S_{kp}]$-module whose character has Frobenius characteristic*

$$\mathsf{e}_k[\mathsf{f}_p - \mathsf{h}_p].$$

*Proof.* This follows from the isomorphisms (6) and (7), and facts about Frobenius characteristic reviewed in Section 2, after noting that the permutation character associated with the action of a group $X$ on the conjugates of a subgroup $Y$ by conjugation is $1 \uparrow_{N_X(Y)}^X$, where $N_X(Y)$ is the normalizer of $Y$ in $X$. □

Now we examine the symmetric function $\mathsf{f}_p$ of Theorem 4.3. By [St, (7.119)], we know that $\mathsf{f}_p$ is the cycle index $Z_{N_p}$, that is,

$$\mathsf{f}_p = \sum_{g \in N_p} \mathsf{p}_{\rho(g)},$$

where $\rho(g)$ is the partition of $p$ determined by the cycle shape of $g$. It is well known and straightforward to show that $N_p = CP$, where $P$ is cyclic of order $p$ (generated by a $p$-cycle) and $C$ is cyclic of order $p-1$ (generated by a $(p-1)$-cycle). Moreover, each nonidentity element of $N_p$ is contained in either $P$ or exactly one of the $p$ conjugates of $C$. For each divisor $d$ of $p-1$, the group $C$ contains exactly $\phi(d)$ elements with one fixed point and $\frac{p-1}{d}$ $d$-cycles, where $\phi$ is Euler's totient function. It now follows that

$$\mathsf{f}_p = \mathsf{p}_1^p + (p-1)\mathsf{p}_p + p\mathsf{p}_1 \sum_{d|p-1} \phi(d)\mathsf{p}_d^{(p-1)/d}. \tag{8}$$

We can also express $\mathsf{f}_p$ in terms of Schur functions, that is, we can find a formula for the multiplicity of each irreducible character of $S_p$ in $1\uparrow_{N_p}^{S_p}$. First, we introduce some notation used in [St, Exercise 7.88]. Namely, for an $n$-cycle $w$ in $S_n$ and an integer $m$, we denote by $\psi_{m,n}$ the character of $S_n$ obtained by inducing from $\langle w\rangle$ the character which maps $w$ to $e^{2\pi i m/n}$.

**Lemma 4.4.** *For any prime $p$, we have*

$$1\uparrow_{N_p}^{S_p} = \psi_{0,p-1}\uparrow_{S_{p-1}}^{S_p} - \psi_{1,p}.$$

*Proof.* Note first that for any group $X$, any subgroup $Y \le X$, any character $\chi$ of $Y$ and any $g \in X$ we have

$$\chi\uparrow_Y^X(g) = \frac{1}{|Y|}|C_X(g)|\sum_{h\in g^X\cap Y}\chi(h), \tag{9}$$

where $C_X(g)$ is the centralizer of $g$ and $g^X$ is the conjugacy class of $g$ in $X$. Writing $N_p = CP$ as above, we have (by definition)

$$\psi_{0,p-1}\uparrow_{S_{p-1}}^{S_p} = 1\uparrow_C^{S_p}$$

and

$$\psi_{1,p} = \theta\uparrow_P^{S_p},$$

where $\theta$ is a one dimensional character of $P$ which maps a generator to $e^{\frac{2\pi i}{p}}$. As noted above, for $c \in C\setminus\{1\}$ we have

$$|c^{S_p}\cap N_p| = p|c^{S_p}\cap C|,$$

and it follows from (9) that

$$1\uparrow_{N_p}^{S_p}(c) = 1\uparrow_C^{S_p}(c).$$

Since no conjugate of $c$ lies in $P$, we have

$$\psi_{1,p}(c) = 0$$

and

$$1\uparrow_{N_p}^{S_p}(c) = \psi_{0,p-1}\uparrow_{S_{p-1}}^{S_p}(c) - \psi_{1,p}(c).$$

For $g \in P\setminus\{1\}$, we have

$$\sum_{x\in g^{S_p}\cap P}\theta(x) = \sum_{j=1}^{p-1}e^{2\pi i j/p} = -1.$$

Using (9) twice, we get

$$-\psi_{1,p}(g) = \frac{1}{p}|C_{S_p}(g)|(1) = \frac{1}{p(p-1)}|C_{S_p(g)}|(p-1) = 1\uparrow_{N_p}^{S_p}(g).$$

Since no conjugate of $g$ lies in $S_{p-1}$, we have

$$\psi_{0,p-1}\uparrow_{S_{p-1}}^{S_p}(g)=0$$

and

$$1\uparrow_{N_p}^{S_p}(g)=\psi_{0,p-1}\uparrow_{S_{p-1}}^{S_p}(g)-\psi_{1,p}(g).$$

Finally, we have

$$\begin{aligned}1\uparrow_{N_p}^{S_p}(1) &= [S_p:N_p]=(p-2)!=p(p-2)!-(p-1)! \\ &= [S_p:C]-[S_p:P]=\psi_{0,p-1}\uparrow_{S_{p-1}}^{S_p}(1)-\psi_{1,p}(1).\end{aligned}$$

Since every element of $N_p$ is conjugate to an element of $P$ or an element of $C$, we are done. □

We can now apply a result discovered by W. Kraśkiewicz and J. Weyman and independently by R. Stanley (see [KrWe, Corollary 8.10] or [St, Exercise 7.88]) to find the decomposition of $\mathsf{f}_p$ into Schur functions. Namely, for a standard Young tableau $T$ with $n$ boxes, let $D(T)$ be the set of all $i\in[n]$ such that $i+1$ appears in a row of $T$ below the row which contains $i$. Define

$$maj(T):=\sum_{i\in D(T)} i.$$

Now, for integers $m,n,k$ and a partition $\lambda$ of $n$, let $M_{m,k,\lambda}$ be the number of standard Young tableaux $T$ of shape $\lambda$ which satisfy $maj(T)\equiv m \bmod k$.

**Lemma 4.5** ([KrWe, Corollary 8.10],[St, Exercise 7.88])**.** *For integers $m,n$, we have*

$$\psi_{m,n}=\sum_{\lambda\vdash n} M_{m,n,\lambda}\mathcal{S}^{\lambda}.$$

**Corollary 4.6.** *For any prime $p$, we have*

$$\mathsf{f}_p=\sum_{\lambda\vdash p}(M_{0,p-1,\lambda}-M_{1,p,\lambda})\mathsf{s}_{\lambda}.$$

*Proof.* It follows directly from Lemma 4.5 that

$$\psi_{1,p}=\sum_{\lambda\vdash p} M_{1,p,\lambda}\mathcal{S}^{\lambda}.$$

It also follows that

$$\psi_{0,p-1}=\sum_{\rho\vdash p-1} M_{0,p-1,\rho}\mathcal{S}^{\rho}.$$

Now let $\lambda$ be a partition of $p$ and let $T$ be a standard Young tableau of shape $\lambda$. Let $T'$ be the standard Young tableau obtained by removing the box containing $p$ from $T$. Then

$$maj(T') = \begin{cases} maj(T) & \text{if } p-1 \notin D(T), \\ maj(T)-(p-1) & \text{if } p-1 \in D(T). \end{cases}$$

In particular,

$$maj(T') \equiv maj(T) \bmod (p-1).$$

It now follows from the branching rule (see for example [Sa, Theorem 2.8.3]) that

$$\psi_{0,p-1} \uparrow_{S_{p-1}}^{S_p} = \sum_{\lambda \vdash p} M_{0,p-1,\lambda} \mathcal{S}^\lambda,$$

and our corollary follows from Lemma 4.4. □

Collecting our results from this section and the last, we get the following theorem.

**Theorem 4.7.** *Let $n$ be a nonnegative integer and $p$ be a prime. Let $\mathsf{q}_{n,p,i}, \mathsf{c}_{n,p,i}, \mathsf{d}_{n,p,i}$ be the Frobenius characteristics of the $\mathbb{C}[S_n]$-modules $\widetilde{H}_{\dim \Delta\mathcal{A}_p(S_n)-i}(\Delta\mathcal{A}_p(S_n))$, $\widetilde{H}_{\dim \mathsf{C}_p(n)-i}(\mathsf{C}_p(n))$ and $\widetilde{H}_{\dim \mathsf{M}_p(n)-i}(\mathsf{M}_p(n))$, respectively. Then for all $i \le p-3$,*

$$\begin{aligned} q_{n,p,i} &= \mathsf{c}_{n,p,i} \\ &= \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \mathsf{d}_{n-kp,p,i} \mathsf{e}_k[-\mathsf{h}_p + \mathsf{p}_1^p + (p-1)\mathsf{p}_p + p\mathsf{p}_1 \sum_{d|p-1} \phi(d) \mathsf{p}_d^{(p-1)/d}] \\ &= \sum_{k=0}^{\lfloor \frac{n}{p} \rfloor} \mathsf{d}_{n-kp,p,i} \mathsf{e}_k[\sum_{\substack{\lambda \vdash p \\ \lambda \neq (p)}} (M_{0,p-1,\lambda} - M_{1,p,\lambda}) \mathsf{s}_\lambda]. \end{aligned}$$

**Remark 4.8.** *Note that the restriction $i \le p-3$ is needed only for the first equation of Theorem 4.7. Note also that the theorem is vacuous for $p=2$ and that the theorem refers only to top homology in the case $p=3$. In the cases $p=2,3$, it is easy to check that the sum inside the second plethysm vanishes giving $\mathsf{c}_{n,p,i} = \mathsf{d}_{n,p,i}$, which also follows from the fact that $\mathsf{C}_p(n)$ and $\mathsf{M}_p(n)$ are isomorphic complexes when $p=2,3$.*

Say $p$ is any integer greater than 1 that divides $n$. Then $\mathsf{M}_p(n)$ has dimension $t := \frac{n}{p} - 1$, but every face of dimension $t-1$ is contained in a unique face of dimension $t$. It follows that we can reduce $\mathsf{M}_p(n)$ to a complex of dimension $t-1$ by a series of elementary collapses (see for example [Co]), giving,

$$\widetilde{H}_{\dim(\mathsf{M}_p(n))}(\mathsf{M}_p(n)) = 0, \tag{10}$$

unless $n = 0$ in which case $\mathrm{ch}\widetilde{H}_{\dim(\mathsf{M}_p(n))}(\mathsf{M}_p(n)) = 1$. The following result is thus a consequence of Theorem 4.7.

**Corollary 4.9.** *If $p$ is an odd prime that divides $n$ then*

$$\begin{aligned}
&\mathrm{ch}\widetilde{H}_{\dim(\Delta\mathcal{A}_p(S_n))}(\Delta\mathcal{A}_p(S_n)) \\
&\quad= \mathrm{ch}\widetilde{H}_{\dim(\mathsf{C}_p(n))}(\mathsf{C}_p(n)) \\
&\quad= e_{\frac{n}{p}}[-\mathsf{h}_p + \mathsf{p}_1^p + (p-1)\mathsf{p}_p + p\mathsf{p}_1 \sum_{d|p-1} \phi(d)\mathsf{p}_d^{(p-1)/d}] \\
&\quad= \mathsf{e}_{\frac{n}{p}}[\sum_{\substack{\lambda \vdash p \\ \lambda \neq (p)}} (M_{0,p-1,\lambda} - M_{1,p,\lambda})\mathsf{s}_\lambda].
\end{aligned}$$

**Remark 4.10.** *As noted above, we have $C_3(3k) \cong M_3(3k)$ for all $k$. It follows from the first equality in Theorem 4.7 and (10) that*

$$\widetilde{H}_{k-1}(\Delta\mathcal{A}_3(S_{3k})) = 0.$$

*This was observed previously in* [AsSm]*, with attribution to J. Thompson.*

## 5. The top homology of some hypergraph matching complexes

In this section, we present a conjecture on the $\mathbb{C}[S_n]$-module structure of the top homology group of $\mathsf{M}_p(n)$ in the case $n \equiv 1 \bmod p$ and prove our conjecture in the cases $p \leq 3$ and $\frac{n-1}{p} \leq 2$. For a symmetric function $\mathsf{f} = \sum_\lambda a_\lambda \mathsf{s}_\lambda$ and a positive integer $r$, define

$$\mathsf{f}|_r := \sum_{l(\lambda)=r} a_\lambda \mathsf{s}_\lambda,$$

where $l(\lambda)$ is the number of parts of the partition $\lambda$.

**Conjecture 5.1.** *For integers $k \geq 1$, $p > 1$ and $n = kp+1$, we have*

$$\mathrm{ch}\widetilde{H}_{k-1}(\mathsf{M}_p(n)) = (\mathsf{e}_k[\mathsf{h}_p]\mathsf{h}_1)|_{k+1}.$$

Note that by Proposition 5.3 below and Pieri's rule (see for example [St, Theorem 7.15.7]), Conjecture 5.1 implies that if the coefficient of $\mathsf{s}_\lambda$ in the expansion of $\mathrm{ch}\widetilde{H}_{k-1}(\mathsf{M}_p(n))$ is nonzero, then $l(\lambda) = k+1$ and $\lambda_{k+1} = 1$

Our conjecture in the case $k = 1$ is easy to verify. The complex $\mathsf{M}_p(p+1)$ is a discrete point set, and thus its $0^{th}$ chain space is a permutation module on $p$-sets and its $0^{th}$ homology group is obtained by taking the quotient of this permutation module by the trivial module. It follows that

$$\widetilde{H}_0(\mathsf{M}_p(p+1)) \cong_{S_{p+1}} \mathcal{S}^{(p,1)}. \tag{11}$$

By Pieri's rule, we have $\mathsf{s}_{(p,1)} = (\mathsf{e}_1[\mathsf{h}_p]\mathsf{h}_1)|_2$.

Now let us consider that case $k = 2$. It is known (see [Mac, I.8]) that

$$\mathsf{e}_2[\mathsf{h}_p] = \sum_{\substack{(\lambda_1,\lambda_2)\,\vdash\, 2p \\ \lambda_1,\lambda_2 \text{ odd}}} \mathsf{s}_{(\lambda_1,\lambda_2)}. \tag{12}$$

Hence by Pieri's rule, the case $k = 2$ of the conjecture is equivalent to the following result.

**Theorem 5.2** (Ksontini [Ks3, Lemma 3.5])**.** *For all integers $p \geq 2$,*

$$\mathrm{ch}\widetilde{H}_1(\mathsf{M}_p(2p+1)) = \sum_{\substack{(\lambda_1,\lambda_2)\,\vdash\, 2p \\ \lambda_1,\lambda_2 \text{ odd}}} \mathsf{s}_{(\lambda_1,\lambda_2,1)}.$$

In [Ks3] this result is stated with the unnecessary hypothesis that $p$ is an odd prime. We will need this result later and will give a proof that is slightly simpler than that of [Ks3] (but quite similar).

Using results of C. Carre, we obtain an alternative formulation of Conjecture 5.1. Since $\mathsf{h}_p^k - \mathsf{e}_k[\mathsf{h}_p] = \mathsf{h}_1^k[\mathsf{h}_p] - \mathsf{e}_k[\mathsf{h}_p] = (\mathsf{h}_1^k - \mathsf{e}_k)[\mathsf{h}_p]$ and $\mathsf{h}_1^k - \mathsf{e}_k$ is the Frobenius characteristic of a representation, we have that $\mathsf{h}_p^k - \mathsf{e}_k[\mathsf{h}_p]$ is the Frobenius characteristic of a representation. Hence $\mathsf{h}_p^k - \mathsf{e}_k[\mathsf{h}_p]$ is Schur positive, as are $\mathsf{h}_p^k$ and $\mathsf{e}_k[\mathsf{h}_p]$. It follows that the coefficient of each Schur function in the Schur function expansion of $\mathsf{e}_k[\mathsf{h}_p]$ is at most the corresponding coefficient in the expansion of $\mathsf{h}_p^k$. Using Pieri's rule repeatedly, we get the following result (which is noted in [Ca]).

**Proposition 5.3.** *For nonnegative integers $k, p, j, r$, if $r > k + \min\{j, 1\}$ then*

$$(\mathsf{e}_k[\mathsf{h}_p]h_j)|_r = 0.$$

For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$ and a positive integer $r$, define $\lambda^{(r)}$ to be the partition obtained from $\lambda$ by adding 1 to $\lambda_i$ for $1 \leq i \leq r$, that is

$$\lambda^{(r)} = (\lambda_1 + 1 \geq \lambda_2 + 1 \geq \dots \geq \lambda_r + 1 \geq \lambda_{r+1} \geq \dots).$$

(Here we are viewing the partition $\lambda$ as an infinite sequence with all but the first $l(\lambda)$ entries equal to 0.) Equivalently, the Young diagram of $\lambda^{(r)}$ is obtained from that of $\lambda$ by adding a box to each of the first $r$ rows (including empty rows). Define the linear map $\gamma_r$ on the ring of symmetric functions by

$$\gamma_r(\sum_\lambda a_\lambda \mathsf{s}_\lambda) = \sum_\lambda a_\lambda \mathsf{s}_{\lambda^{(r)}}.$$

**Theorem 5.4** (Carre, [Ca, Theorem 24(2)])**.** *For positive integers $p, r$, we have*

$$\mathsf{e}_r[\mathsf{h}_p]|_r = \gamma_r(\mathsf{h}_r[\mathsf{h}_{p-1}]).$$

By Pieri' s rule, Proposition 5.3 and Theorem 5.4, we see that the following conjecture is equivalent to Conjecture 5.1.

**Conjecture 5.5.** *For integers $k \geq 1$, $p > 1$ and $n = kp + 1$, we have*

$$\mathrm{ch}\widetilde{H}_{k-1}(\mathsf{M}_p(n)) = \gamma_{k+1}(\mathsf{h}_k[\mathsf{h}_{p-1}]). \tag{13}$$

In the case $p = 2$, equation (13) is equivalent to

$$\widetilde{H}_{k-1}(\mathsf{M}_2(2k+1)) \cong_{S_n} \mathcal{S}^{(k+1,1^k)},$$

which is a special case of the following formula of Bouc[Bo1] (see also [DoWa]), giving the homology of the matching complex in each dimension,

$$\widetilde{H}_{r-1}(\mathsf{M}_2(n)) \cong_{S_n} \bigoplus_{\substack{\lambda\,:\,\lambda \vdash n \\ \lambda = \lambda' \\ d(\lambda) = n-2r}} \mathcal{S}^\lambda, \tag{14}$$

where $\lambda'$ denotes the conjugate of $\lambda$ and $d(\lambda)$ denotes the size of the Durfee square of $\lambda$.

Before establishing the conjecture in the cases $p = 3$ and $k = 2$, we obtain some information on $\widetilde{H}_{k-1}(\mathsf{M}_p(n))$ for arbitrary $p$, $n$ and $k = \lfloor \frac{n}{p} \rfloor$. We understand the $\mathbb{C}[S_n]$-module $C_{r-1}(\mathsf{M}_p(n))$ for arbitrary $p, n$ and $r \leq \lfloor \frac{n}{p} \rfloor$. It is induced from a one-dimensional (over $\mathbb{C}$) module $X$ of the stabilizer $G$ of an $(r-1)$-dimensional face $F$. As noted above, this stabilizer is a direct product $H \times T$. Here $H$ is isomorphic to the wreath product $S_r[S_p]$, whose kernel $K \cong S_p^r$ acts trivially on $X$. A standard complement $C$ to $K$ in $H$ permutes the $r$ components $S_p$ of $K$ by conjugation in the same manner that it permutes the vertices (hyperedges) of $F$, and $C$ acts on $X$ according to the sign character of this permutation action. The group $T \cong S_{n-rp}$ acts trivially on $X$. We now have the following result.

**Proposition 5.6.** *For integers $p, n \geq 2$ and $r \leq \lfloor \frac{n}{p} \rfloor$, the Frobenius characteristic of $C_{r-1}(\mathsf{M}_p(n))$ is given by*

$$\mathrm{ch} C_{r-1}(\mathsf{M}_p(n)) = \mathsf{e}_r[\mathsf{h}_p]\mathsf{h}_{n-rp}.$$

Now let $p, n \geq 2$ and $k = \lfloor \frac{n}{p} \rfloor$. For $\lambda \vdash n$, define $C_\lambda$ to be the (direct) sum of all simple submodules of the top chain space $C_{k-1}(\mathsf{M}_p(n))$ that are isomorphic to the Specht module $\mathcal{S}^\lambda$ (and so have Frobenius characteristic $\mathsf{s}_\lambda$). We can view $\widetilde{H}_{k-1}(\mathsf{M}_p(n))$ as a submodule of $C_{k-1}(\mathsf{M}_p(n))$ since $\dim \mathsf{M}_p(n) = k - 1$.

**Corollary 5.7.** *The submodule $\bigoplus_{l(\lambda)=k+1} C_\lambda$ of $C_{k-1}(\mathsf{M}_p(n))$ is contained in $\widetilde{H}_{k-1}(\mathsf{M}_p(n))$.*

*Proof.* If $l(\lambda) = k + 1$ then $C_{k-2}(\mathsf{M}_p(n))$ contains no submodule isomorphic to $\mathcal{S}^\lambda$ by Propositions 5.3 and 5.6 and Pieri's rule; so $C_\lambda$ is contained in the kernel of the $(k-1)^{st}$ boundary map. □

**Corollary 5.8.** *For integers $k \geq 1$, $p > 1$ and $n = kp + 1$, we have*

$$\mathrm{ch}\widetilde{H}_{k-1}(\mathsf{M}_p(n)) = \gamma_{k+1}(\mathsf{h}_k[\mathsf{h}_{p-1}]) + \mathsf{f} = (\mathsf{e}_k[\mathsf{h}_p]\mathsf{h}_1)|_{k+1} + \mathsf{f},$$

*where $\mathsf{f}$ is a symmetric function such that $\mathsf{f}|_r = 0$ for all $r \geq k + 1$. Thus Conjecture 5.1 (and 5.5) holds if and only if $\mathsf{f} = 0$.*

The following long exact sequence appears in the thesis [Ks1] of Ksontini and is a generalization of a sequence used by Bouc for graph matching complexes in [Bo2]. It is straightforward to show that this sequence is the standard long exact sequence associated with the ($S_{n-1}$-equivariant) embedding of $\mathsf{M}_p(n-1)$ into $\mathsf{M}_p(n)$ determined by the identity embedding of $[n-1]$ into $[n]$.

**Lemma 5.9** ([Ks1, Proposition 4.12])**.** *For positive integers $p, n > 1$ there is a long exact sequence*

$$\begin{aligned} \ldots \to \widetilde{H}_r(\mathsf{M}_p(n-1)) \to \widetilde{H}_r(\mathsf{M}_p(n)) \downarrow_{S_{n-1}}^{S_n} \to \widetilde{H}_{r-1}(\mathsf{M}_p(n-p)) \uparrow_{S_{n-p} \times S_{p-1}}^{S_{n-1}} \\ \to \widetilde{H}_{r-1}(\mathsf{M}_p(n-1)) \to \widetilde{H}_{r-1}(\mathsf{M}_p(n)) \downarrow_{S_{n-1}}^{S_n} \to \ldots \end{aligned}$$

*of $\mathbb{C}[S_{n-1}]$-modules, where the action of the component $S_{p-1}$ on $\widetilde{H}_*(\mathsf{M}_p(n-p))$ is trivial.*

By (10) we have the following special case.

**Corollary 5.10.** *For positive integers $k, p > 1$, the long exact sequence of Lemma 5.9 with $n = kp + 1$ is*

$$0 \to \widetilde{H}_{k-1}(\mathsf{M}_p(kp+1)) \downarrow_{S_{kp}}^{S_{kp+1}} \to \widetilde{H}_{k-2}(\mathsf{M}_p((k-1)p+1)) \uparrow_{S_{(k-1)p+1}\times S_{p-1}}^{S_{kp}} \to \widetilde{H}_{k-2}(\mathsf{M}_p(kp)) \to \dots .$$

*Proof of Theorem 5.2.* By Corollary 5.10, we have a long exact sequence

$$0 \to \widetilde{H}_1(\mathsf{M}_p(2p+1)) \downarrow_{S_{2p}}^{S_{2p+1}} \to \widetilde{H}_0(\mathsf{M}_p(p+1)) \uparrow_{S_{p+1}\times S_{p-1}}^{S_{2p}} \to \dots .$$

By (11) and Pieri's rule, the $\mathbb{C}[S_{2p}]$-module $\widetilde{H}_0(\mathsf{M}_p(p+1)) \uparrow_{S_{p+1}\times S_{p-1}}^{S_{2p}}$ is isomorphic to the direct sum of Specht modules $\mathcal{S}^\mu$ over all partitions $\mu \vdash 2p$ of length 2 or 3, such that if $l(\mu) = 3$ then the smallest part of $\mu$ must be 1. Each Specht module has multiplicity 1 in this decomposition. Our exact sequence shows that $\widetilde{H}_1(\mathsf{M}_p(2p+1)) \downarrow_{S_{2p}}^{S_{2p+1}}$ embeds into $\widetilde{H}_0(\mathsf{M}_p(p+1)) \uparrow_{S_{p+1}\times S_{p-1}}^{S_{2p}}$. Hence each Specht module in $\widetilde{H}_1(\mathsf{M}_p(2p+1)) \downarrow_{S_{2p}}^{S_{2p+1}}$ must be of the form just described and must also have multiplicity 1.

Assume for contradiction that $\widetilde{H}_1(\mathsf{M}_p(2p+1))$ has a submodule isomorphic to $\mathcal{S}^\lambda$, where $l(\lambda) \leq 2$. Since the restriction of $\mathcal{S}^\lambda$ must be isomorphic to a submodule of $\widetilde{H}_0(\mathsf{M}_p(p+1)) \uparrow_{S_{p+1}\times S_{p-1}}^{S_{2p}}$, the partition $\lambda$ can't have length 1; so it must have length 2 and consist of an even part and an odd part. By reducing the even part by one, we get a partition $\mu \vdash 2p$ with 2 odd parts, $\mu_1, \mu_2$. By the branching rule, the restriction $\widetilde{H}_1(\mathsf{M}_p(2p+1)) \downarrow_{S_{2p}}^{S_{2p+1}}$ has a submodule isomorphic to $\mathcal{S}^\mu$.

Let $\tau = (\mu_1, \mu_2, 1)$. By Corollary 5.8, equation (12) and Pieri's rule, $\widetilde{H}_1(\mathsf{M}_p(2p+1))$ has a submodule isomorphic to $\mathcal{S}^\tau$. Hence by the branching rule, $\widetilde{H}_1(\mathsf{M}_p(2p+1)) \downarrow_{S_{2p}}^{S_{2p+1}}$ has an additional submodule isomorphic to $\mathcal{S}^\mu$, contradicting the multiplicity 1 requirement.

It follows that if $\mathcal{S}^\lambda$ is isomorphic to a submodule of $\widetilde{H}_1(\mathsf{M}_p(2p+1))$, then $\lambda$ has length 3. Hence by Corollary 5.8

$$\mathrm{ch}\widetilde{H}_1(\mathsf{M}_p(2p+1)) = \mathsf{e}_2[\mathsf{h}_p]h_1|_3 = \sum_{\substack{(\lambda_1,\lambda_2)\vdash 2p \\ \lambda_1,\lambda_2 \text{ odd}}} \mathsf{s}_{(\lambda_1,\lambda_2,1)}$$

$\square$

Now we turn our attention to the case $p = 3$ of the conjecture. For a partition $\lambda = (\lambda_1, \dots, \lambda_t)$, write $2\lambda$ for the partition $(2\lambda_1, \dots, 2\lambda_t)$.

It is known (see for example [St, Example A2.9]) that

$$\mathsf{h}_k[\mathsf{h}_2] = \sum_{\lambda \vdash k} \mathsf{s}_{2\lambda}.$$

Hence

$$\gamma_{k+1}(\mathsf{h}_k[\mathsf{h}_2]) = \sum_{\lambda \in \Lambda(k)} \mathsf{s}_\lambda, \tag{15}$$

where $\Lambda(k)$ is the set of all partitions of $3k+1$ into $k+1$ odd parts. It follows that the next result is the case $p = 3$ of Conjecture 5.5.

**Theorem 5.11.** *Let $k$ be any positive integer. Then*

$$\widetilde{H}_{k-1}(\mathsf{M}_3(3k+1)) \cong_{S_{3k+1}} \bigoplus_{\lambda \in \Lambda(k)} \mathcal{S}^\lambda.$$

*Proof.* We may assume $k \geq 3$ since the result for $k \leq 2$ has already been established by the isomorphism (11) and Theorem 5.2. By Corollary 5.8 and equation (15), we need only show that if $\widetilde{H}_{k-1}(\mathsf{M}_3(3k+1))$ has a submodule isomorphic to $\mathcal{S}^\lambda$ then $\lambda$ has at least $k+1$ parts. By Corollary 5.10, we have a long exact sequence

$$0 \to \widetilde{H}_{k-1}(\mathsf{M}_3(3k+1)) \downarrow^{S_{3k+1}}_{S_{3k}} \to \widetilde{H}_{k-2}(\mathsf{M}_3(3k-2)) \uparrow^{S_{3k}}_{S_{3k-2} \times S_2} \to \dots .$$

By inductive hypothesis, the $\mathbb{C}[S_{3k-2}]$-module $\widetilde{H}_{k-2}(\mathsf{M}_3(3k-2))$ is the direct sum of the Specht modules $\mathcal{S}^\lambda$ over all partitions $\lambda$ of $3k-2$ into $k$ odd parts. It follows from Pieri's rule that if $\mu$ is a partition of $3k$ and $\widetilde{H}_{k-2}(\mathsf{M}_3(3k-2)) \uparrow^{S_{3k}}_{S_{3k-2} \times S_2}$ has a submodule isomorphic to $\mathcal{S}^\mu$ then

- $\mu$ has either $k$ or $k+1$ parts, and
- $\mu$ has at most two even parts.

Our exact sequence shows that $\widetilde{H}_{k-1}(\mathsf{M}_3(3k+1)) \downarrow^{S_{3k+1}}_{S_{3k}}$ embeds into $\widetilde{H}_{k-2}(\mathsf{M}_3(3k-2)) \uparrow^{S_{3k}}_{S_{3k-2} \times S_2}$, and it follows from the branching rule (and simple arithmetic) that if $\lambda$ is a partition of $3k+1$ and $\widetilde{H}_{k-1}(\mathsf{M}_3(3k+1))$ has a submodule isomorphic to $\mathcal{S}^\lambda$ then

- $\lambda$ has at least $k$ parts, and
- if $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$ then $\lambda_k \in \{2, 3\}$.

(Note that when $\lambda$ as above has $k$ parts, we cannot have $\lambda_k = 1$ since then an irreducible constituent of the restriction of $\mathcal{S}^\lambda$ has $k-1$ parts.)

If $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition of $3k+1$ with $\lambda_k = 3$ then $\lambda = (4, 3, \dots, 3)$. Assume for contradiction that in this case $\widetilde{H}_{k-1}(\mathsf{M}_3(3k+$

1)) has a submodule isomorphic to $\mathcal{S}^\lambda$. The restriction of this submodule to $S_{3k}$ has a submodule isomorphic to $\mathcal{S}^{(3,\dots,3)}$ by the branching rule. By Corollary 5.8 and equation (15), we know that $\widetilde{H}_{k-1}(\mathsf{M}_3(3k+1))$ has a submodule isomorphic to $\mathcal{S}^{(3,\dots,3,1)}$ and the restriction of this submodule to $S_{3k}$ produces an additional submodule isomorphic to $\mathcal{S}^{(3,\dots,3)}$. On the other hand, by Pieri's rule, $\widetilde{H}_{k-2}(\mathsf{M}_3(3k-2)) \uparrow_{S_{3k-2}\times S_2}^{S_{3k}}$ has a unique submodule isomorphic to $\mathcal{S}^{(3,\dots,3)}$, namely, a constituent of the module induced from $\mathcal{S}^{(3,\dots,3,1)}$. This gives the desired contradiction.

Finally, say $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition of $3k+1$ with $\lambda_k = 2$. If $k = 3$ then the only possibilities for $\lambda$ are $(6,2,2), (5,3,2)$ and $(4,4,2)$. The Young diagram $(5,2,2)$ can be obtained from either of the first two possibilities by removing a cell, and the Young diagram $(4,4,1)$ can be obtained from the third possibility by removing a cell. However, again using Pieri's rule we see that no submodule of $\widetilde{H}_{k-2}(\mathsf{M}_3(3k-2)) \uparrow_{S_{3k-2}\times S_2}^{S_{3k}}$ is isomorphic to $\mathcal{S}^{(5,2,2)}$ or $\mathcal{S}^{(4,4,1)}$. (One cannot add two boxes in the same column of a Young diagram, so one cannot obtain two equal even parts from a partition into odd parts.)

Now suppose $k > 3$. Since no restriction of $\mathcal{S}^\lambda$ can have more than 2 even parts, it follows that $\lambda_k = 2$ is the only even part of $\lambda$, and $\lambda = (5,3,\dots,3,2)$. Now $\mathcal{S}^{(5,3,\dots,3,2,2)}$ is a submodule of the restriction of $\mathcal{S}^\lambda$. However, again using Pieri's rule, we see that no submodule of $\widetilde{H}_{k-2}(\mathsf{M}_3(3k-2)) \uparrow_{S_{3k-2}\times S_2}^{S_{3k}}$ is isomorphic to $\mathcal{S}^{(5,3,\dots,3,2,2)}$. By this final contradiction, we see that if $\widetilde{H}_{k-1}(\mathsf{M}_3(3k+1))$ has a submodule isomorphic to $\mathcal{S}^\lambda$ then $\lambda$ has at least $k+1$ parts. $\square$

The next result now follows from Theorem 3.2 and the isomorphism between the complexes $\mathsf{C}_3(n)$ and $\mathsf{M}_3(n)$.

**Corollary 5.12.** *Let $k$ be any positive integer. Then*

$$\widetilde{H}_{k-1}(\Delta\mathcal{A}_3(3k+1)) \cong_{S_{3k+1}} \widetilde{H}_{k-1}(\Delta\mathsf{C}_3(3k+1)) \cong_{S_{3k+1}} \bigoplus_{\lambda\in\Lambda(k)} \mathcal{S}^\lambda.$$

The following result of Athanasiadis on nonvanishing homology of $\mathsf{M}_n(p)$ enables one to obtain precise information on $\widetilde{H}_r(\mathsf{M}_n(p))$ when $n$ is small relative to $p$.

**Theorem 5.13** (Athanasiadis [At])**.** *For $n, p \geq 2$, the homology of $\mathsf{M}_p(n)$ vanishes below dimension $\lfloor \frac{n-p}{p+1} \rfloor$.*

**Corollary 5.14.** *If $k \leq p+3$ then the homology of $\mathsf{M}_p(kp+1)$ is nonvanishing in at most two dimensions.*

We obtain the table below giving all nonvanishing $\widetilde{H}_r(\mathsf{M}_3(n))$ for $n \leq 13$ by using Maple to compute the right hand side of the equivariant

Euler-Poincaré formula,

$$\sum_{r=0}^{\lfloor \frac{n}{3} \rfloor} (-1)^r \mathrm{ch} \widetilde{H}_{r-1}(\mathsf{M}_3(n)) = \sum_{r=0}^{\lfloor \frac{n}{3} \rfloor} (-1)^r \mathsf{e}_r[\mathsf{h}_3]\mathsf{h}_{n-3r}.$$

The computation of homology for $n \neq 10, 13$ then follows from Theorem 5.13, which guarantees that the left hand side has at most one nonzero term. When $n = 10, 13$, the computation follows from Theorem 5.11 and Corollary 5.14.

| $n$ | $r$ | $\widetilde{H}_r(\mathsf{M}_3(n))$ |
|---|---|---|
| 4 | 0 | $\mathcal{S}^{(3,1)}$ |
| 5 | 0 | $\mathcal{S}^{(4,1)} \oplus \mathcal{S}^{(3,2)}$ |
| 6 | 0 | $\mathcal{S}^{(4,2)}$ |
| 7 | 1 | $\mathcal{S}^{(5,1,1)} \oplus \mathcal{S}^{(3,3,1)}$ |
| 8 | 1 | $\mathcal{S}^{(6,1,1)} \oplus \mathcal{S}^{(5,2,1)} \oplus \mathcal{S}^{(4,3,1)} \oplus \mathcal{S}^{(3,3,2)} \oplus \mathcal{S}^{(5,3)}$ |
| 9 | 1 | $\mathcal{S}^{(6,2,1)} \oplus \mathcal{S}^{(5,3,1)} \oplus \mathcal{S}^{(4,3,2)} \oplus \mathcal{S}^{(5,4)}$ |
| 10 | 1 | $\mathcal{S}^{(5,5)}$ |
| 10 | 2 | $\mathcal{S}^{(7,1,1,1)} \oplus \mathcal{S}^{(5,3,1,1)} \oplus \mathcal{S}^{(3,3,3,1)}$ |
| 11 | 2 | $\mathcal{S}^{(8,1,1,1)} \oplus \mathcal{S}^{(7,3,1)} \oplus \mathcal{S}^{(7,2,1,1)} \oplus \mathcal{S}^{(6,4,1)} \oplus \mathcal{S}^{(6,3,2)} \oplus \mathcal{S}^{(6,3,1,1)}$ $\oplus \mathcal{S}^{(5,4,2)} \oplus \mathcal{S}^{(5,4,1,1)} \oplus \mathcal{S}^{(5,3,3)} \oplus \mathcal{S}^{(5,3,2,1)} \oplus \mathcal{S}^{(4,3,3,1)} \oplus \mathcal{S}^{(3,3,3,2)}$ |
| 12 | 2 | $\mathcal{S}^{(8,2,1,1)} \oplus \mathcal{S}^{(7,4,1)} \oplus \mathcal{S}^{(7,3,2)} \oplus \mathcal{S}^{(7,3,1,1)} \oplus \mathcal{S}^{(6,5,1)} \oplus \mathcal{S}^{(6,4,2)}$ $\oplus \mathcal{S}^{(6,4,1,1)} \oplus \mathcal{S}^{(6,3,3)} \oplus \mathcal{S}^{(6,3,2,1)} \oplus \mathcal{S}^{(5,5,2)} \oplus \mathcal{S}^{(5,4,3)} \oplus \mathcal{S}^{(5,4,2,1)}$ $\oplus \mathcal{S}^{(5,3,3,1)} \oplus \mathcal{S}^{(4,3,3,2)}$ |
| 13 | 2 | $\mathcal{S}^{(7,5,1)} \oplus \mathcal{S}^{(7,3,3)} \oplus \mathcal{S}^{(6,5,2)} \oplus \mathcal{S}^{(5,5,3)}$ |
| 13 | 3 | $\mathcal{S}^{(9,1,1,1,1)} \oplus \mathcal{S}^{(7,3,1,1,1)} \oplus \mathcal{S}^{(5,5,1,1,1)} \oplus \mathcal{S}^{(5,3,3,1,1)} \oplus \mathcal{S}^{(3,3,3,3,1)}$ |

Upon looking at this table one might be tempted to conjecture that whenever $n \equiv 1 \mod p$, the Specht modules in the decomposition of each homology (not just the top homology) are multiplicity free. Indeed, this holds when $p = 2$ by Bouc's result (14). However, this is not the case, as the multiplicity of $\mathcal{S}^{(7,5,3,1)}$ in $\widetilde{H}_3(\mathsf{M}_3(16))$ is two. Also, while it is the case that for each $k \leq 4$, every Specht module appearing as a submodule of $\widetilde{H}_{k-2}(\mathsf{M}_3(3k+1))$ is indexed by a partition with $k-1$ parts, this fails for $k = 5$, as $\widetilde{H}_3(\mathsf{M}_3(16))$ has a submodule isomorphic to $\mathcal{S}^{(7,6,3)}$. (Similarly, using (14), one sees that for $k \leq 5$, every Specht module appearing as a submodule of $\widetilde{H}_{k-2}(\mathsf{M}_2(2k+1))$ is indexed by a partition with $k-1$ parts, but this fails for $k = 6$.) The given examples involving $\widetilde{H}_3(\mathsf{M}_3(16))$ can be derived using the technique described above.

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