MATCHING COMPLEXES, BOUNDED DEGREE GRAPH COMPLEXES AND WEIGHT SPACES OF GL_n-COMPLEXES

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Abstract. We present a technique for lifting results on the homology of matching complexes of certain types of graphs and hypergraphs to general bounded degree graph complexes. This technique is based on our observation that homological results of Bouc on matching complexes, of Reiner and Roberts on bounded degree graph complexes and of Józefiak and Weyman on Koszul complexes are all equivalent.

1. Introduction

The matching complex of a graph G is the abstract simplicial complex whose vertex set is the set of edges of G and whose faces are sets of edges of G with no two edges meeting at a vertex. The matching complex of the complete graph (known simply as the matching complex) and the matching complex of the complete bipartite graph (known as the chessboard complex) have arisen in a number of contexts in the literature [7] [13] [3] [15] [4] [5] [10] [11] [1]. Closely related complexes have been considered in [9] and [12].

The matching complex admits a natural action of the symmetric group and the chessboard complex admits a natural action of a direct product of symmetric groups. These actions induce representations on the homology of the complexes which have been computed in at least four independent studies: [4] [10] [11] [5]. Bouc [4] was the first to obtain the remarkable result giving the decomposition of the representation of the symmetric group on the homology of the matching complex into irreducibles indexed by self conjugate partitions. Prior to this Józefiak and Weyman [9] produced a similar decomposition of the homology of a Koszul complex of general linear group modules. This

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result was recently rediscovered by Sigg [12] who gave a simpler proof. Friedman and Hanlon [5] obtained a result analogous to Bouc's which decomposes the homology of the chessboard complex into irreducibles.

Reiner and Roberts [11] consider a more general class of subgraphs of a graph G that also forms a simplicial complex, namely the subgraphs of G for which the degree of each graph node is bounded from above by some fixed integer. When that integer is 1 for each node, one gets the matching complex of G. When G is a complete graph or complete bipartite graph there is a natural Young subgroup action on the complex. In [11] the representation of the Young subgroup on the homology is computed. These computations specialize to the Bouc and Friedman-Hanlon decompositions, respectively.

In this paper we show that the more general results of Reiner-Roberts are actually equivalent to the results Bouc and Friedman-Hanlon, respectively. To establish the equivalence of the Bouc result and the Reiner-Roberts result for the complete graph, we observe that the homology modules of the bounded degree graph complexes are weight spaces of the general linear group homology modules considered by Józefiak and Weyman. Thus the Bouc result, the Reiner-Roberts result and the Józefiak-Weyman results are all equivalent. The equivalence of the Friedman-Hanlon result and the Reiner-Roberts result for the complete bipartite graph is established similarly.

The technique that is used to prove the equivalence is abstracted and applied in more general settings. In particular, we apply it to hypergraph versions of the bounded degree graph complexes. We express the representation of the Young subgroup on the homology of the bounded degree hypergraph complex in terms of the representation on the homology of the hypergraph matching complex. This enables us to lift acyclicity results of Björner, Lovaśz, Vrećica and Živaljević [3] from hypergraph matching complexes to bounded degree hypergraph complexes and from higher dimensional chessboard complexes to bounded degree higher dimensional chessboard complexes. More generally, we lift an acyclicity result of Björner and Eriksson [2] on packing complexes to a bounded degree version of the packing complex.

2. Three homological results

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be an n-tuple of nonnegative integers. The γ-bounded degree graph complex Δ_γ is defined to be the simplicial complex of all graphs on node set $[n] = \{1, 2 \ldots, n\}$ in which node i has degree γ_i or less. Loops are allowed and contribute 2 to the degree of the node. Multiple edges are not allowed. We may assume without loss of generality that γ is a weakly increasing sequence of positive integers (i.e. partition) since relabeling the nodes and removing nodes with degree bound 0 gives an isomorphic simplicial complex.

Let Σ_n denote the symmetric group on $[n]$. For $\gamma = 1^{a_1}2^{a_2}\cdots j^{a_j}$ we let $\Sigma(\gamma)$ denote the Young subgroup $\Sigma_{a_1}\times\cdots\times\Sigma_{a_j}\hookrightarrow\Sigma_n$. The Young subgroup $\Sigma(\gamma)$ acts as an automorphism group on Δ_γ by permuting the nodes of like degree bound. When $\gamma = 1^n$, the bounded degree graph complex Δ_γ is the matching complex M_n and $\Sigma(\gamma)$ is Σ_n. The action of $\Sigma(\gamma)$ on Δ_γ induces a representation of $\Sigma(\gamma)$ on the reduced simplicial homology $\tilde{H}_*(\Delta_\gamma;k)$, where throughout this paper k is assumed to be a field of characteristic 0.

The Specht module (or irreducible representation) of Σ_n over k indexed by the partition $\lambda\vdash n$, is denoted by S^λ. Given a partition $\lambda\vdash n$, we will say that $|\lambda| = n$. The length of λ, denoted $\ell(\lambda)$, is the number of parts of λ. Let λ' denote the conjugate of the partition λ and let $d(\lambda)$ denote the size of the main diagonal (or Durfee square) of the Ferrers diagram for λ.

Theorem 2.1 (Bouc [4]). *For all $p\in\mathbb{N}$, the following isomorphism of Σ_n-modules holds*

$$\tilde{H}_{p-1}(M_n;k)\cong_{\Sigma_n}\bigoplus_{\substack{\lambda\,:\,\lambda\vdash n\\ \lambda=\lambda'\\ d(\lambda)=|\lambda|-2p}} S^\lambda.$$

In order to state the Reiner-Roberts result we need to review the notion of weight space in a $GLV)$ representation. See [6] for further information on weight spaces. Throughout this paper let V denote an m-dimensional vector space over k. Fix an ordered basis $v_1,\ldots,v_m$ of V, so that we can view the general linear group $GL(V)$ as a group of matrices. For $x_1,\ldots,x_m\in k$, let $\mathrm{diag}(x_1,\ldots,x_m)$ denote the diagonal matrix in $GL(V)$ with these entries. Given a sequence $\gamma=(\gamma_1,\ldots,\gamma_m)$ of nonnegative integers, let x^γ denote the product $x_1^{\gamma_1}\cdots x_m^{\gamma_m}$. Given a polynomial representation U of $GL(V)$, the *weight space* of U corresponding to *weight* γ is denoted by U_γ and is defined to be

$$U_\gamma=\{u\in U:\mathrm{diag}(x_1,\ldots,x_m)\cdot u = x^\gamma u\quad\forall x_1,\ldots,x_n\in k\}.$$

For example, if $U=V^{\otimes n}$ then U_γ is generated by elements of the form $v_{i_1}\otimes\cdots\otimes v_{i_n}$ where the sequence $i_1,\ldots,i_n$ has γ_j occurrences of j for each $j\in[m]$.

Now suppose γ is a partition of length n, where $n\le m$. We shall view γ as a weight, i.e. a sequence of m nonnegative integers, by attaching $m-n$ copies of 0. Then the Young subgroup $\Sigma(\gamma)\hookrightarrow\Sigma_n\hookrightarrow$

$\Sigma_m \hookrightarrow GL(V)$ acts on U and preserves U_γ. So the weight space U_γ is a $\Sigma(\gamma)$-module. When $\gamma = 1^n$, one has that U_γ is a Σ_n-module.

Recall that the theory of highest weights gives an indexing of irreducible polynomial representations of $GL(V)$ by partitions with at most m parts. Let V^λ be the irreducible polynomial representation of $GL(V)$ with highest weight λ if $\ell(\lambda) \le \dim V$ and 0 otherwise.

Theorem 2.2 (Reiner and Roberts [11]). *Let γ be a partition and V be a k-vector space where $\dim V \ge \ell(\gamma)$. Then for all $p \in \mathbb{N}$, the following isomorphism of $\Sigma(\gamma)$-modules holds*

$$\tilde{H}_{p-1}(\Delta_\gamma; k) \cong_{\Sigma(\gamma)} \bigoplus_{\substack{\lambda:\ \lambda \vdash |\gamma| \\ \lambda = \lambda' \\ d(\lambda) = |\lambda| - 2p}} V^\lambda_\gamma.$$

The weight space V^λ_γ has a nice concrete construction as a space of polytabloids. Let $S^\lambda(\gamma)$ be the vector space generated by polytabloids of shape λ and content γ. By shape and content of a polytabloid we are referring to the shape and content of the tableau that indexes it. The content of a tableau is a sequence whose ith entry is the number of i's appearing in the tableau. The Young subgroup $\Sigma(\gamma)$ acts on $S^\lambda(\gamma)$ by replacing the entries of a polytabloid. By a basic construction of irreducible $GL(V)$-modules (cf. [6, §8.1]), the weight space V^λ_γ is isomorphic to the $\Sigma(\gamma)$-module $S^\lambda(\gamma)$. Hence the result of Reiner and Roberts can be rewritten as

Theorem 2.3. *For all $p \in \mathbb{N}$, the following isomorphism of $\Sigma(\gamma)$-modules holds*

$$\tilde{H}_{p-1}(\Delta_\gamma; k) \cong_{\Sigma(\gamma)} \bigoplus_{\substack{\lambda:\ \lambda \vdash |\gamma| \\ \lambda = \lambda' \\ d(\lambda) = |\lambda| - 2p}} S^\lambda(\gamma).$$

Since $S^\lambda(1^n) = S^\lambda$, Bouc's Theorem is a special case of the Reiner-Roberts Theorem. Since the column-strict polytabloids of shape λ and content γ form a basis for $S^\lambda(\gamma)$ we have the following corollary.

Corollary 2.4. *The reduced Betti number $\tilde{\beta}_{p-1}(\Delta_\gamma)$ is the number of column strict tableau of self-conjugate shape λ and content γ, where λ satisfies $d(\lambda) = |\gamma| - 2p$.*

Next we state the result of Józefiak and Weyman [9]. For each $p \in \mathbb{N}$ form the $GL(V)$ module

$$U_p(V) = \wedge^p(\mathrm{Sym}^2 V) \otimes \mathrm{Sym}^* V,$$

where $\wedge^p$ is the pth exterior power, Sym^p is the pth symmetric power and $\mathrm{Sym}^* := \oplus_p \mathrm{Sym}^p$ is the symmetric algebra. The map $\delta_p : U_p(V) \to U_{p-1}(V)$ defined on generators by

$$\delta_p(v_{i_1}v_{i_2} \wedge \cdots \wedge v_{i_{2p-1}}v_{i_{2p}} \otimes u)$$

$$= \sum_{j=1}^{p} (-1)^j v_{i_1}v_{i_2} \wedge \cdots \wedge \widehat{v}_{i_{2j-1}}\widehat{v}_{i_{2j}} \wedge \cdots \wedge v_{i_{2p-1}}v_{i_{2p}} \otimes uv_{i_{2j-1}}v_{i_{2j}},$$

where $\hat{}$ denotes deletion and $u \in \mathrm{Sym}^* V$, is the differential of the $GL(V)$ complex

$$U(V) := \cdots \to U_p(V) \to U_{p-1}(V) \to \ldots.$$

This complex arose in Józefiak-Weyman [9] as the Koszul complex over $\mathrm{Sym}^* V$ of the ideal generated by $\mathrm{Sym}^2 V$ in $\mathrm{Sym}^* V$. We refer to it as the JW-complex. The following theorem is stated in [9] and implicitly proved in [8].

Theorem 2.5 (Józefiak-Weyman [9]). *For all $p \in \mathbb{N}$, the following isomorphism of $GL(V)$-modules holds*

$$H_p(U(V)) \cong_{GL(V)} \bigoplus_{\substack{\lambda\,:\,\lambda=\lambda' \\ d(\lambda)=|\lambda|-2p}} V^\lambda. \tag{2.1}$$

The Reiner-Roberts Theorem (Theorem 2.2) and the Józefiak-Weyman Theorem (Theorem 2.5) together imply that the homology of the bounded degree graph complex is a weight space of the homology of the JW complex. We give a simple direct proof of this fact in the next section.

3. Equivalence of the results

In this section we will show that the Bouc Theorem, the Reiner-Roberts Theorem and the Józefiak-Weyman Theorem are all equivalent. In addition we show that the Betti number corollary to the Reiner-Roberts Theorem is also equivalent to the other results.

We begin by describing the chain complex of the bounded degree graph complex Δ_γ. Let $C_p(\Delta_\gamma)$ be the (augmented) $(p-1)$-chain space (with coefficients in k) of Δ_γ. That is, $C_p(\Delta_\gamma)$ is the k-vector space generated by oriented simplices of dimension $p-1$. An oriented simplex in $C_p(\Delta_\gamma)$ is simply a p-tuple of distinct edges. Two oriented simplices are related by

$$(e_1, e_2, \ldots, e_p) = \mathrm{sgn}\,\sigma\,(e_{\sigma(1)}, e_{\sigma(2)}, \ldots, e_{\sigma(p)}),$$

where $\sigma \in \Sigma_p$. The boundary map $\partial_p : C_p(\Delta_\gamma) \to C_{p-1}(\Delta_\gamma)$ is defined on generators by

$$\partial_p(e_1, \ldots, e_p) = \sum_{j=1}^{p} (-1)^j (e_1, \ldots, \hat{e}_j, \ldots, e_p).$$

Since ∂_* commutes with the action of the Young subgroup $\Sigma(\gamma)$ on $C_*(\Delta_\gamma)$,

$$C(\Delta_\gamma) := \cdots \to C_p(\Delta_\gamma) \to C_{p-1}(\Delta_\gamma) \to \cdots$$

is a complex of $\Sigma(\gamma)$-modules.

Now consider the γ-weight space of the JW complex $U(V)$. Since $\delta_p(U_p(V)_\gamma) \subseteq U_{p-1}(V)_\gamma$, we have that

$$U(V)_\gamma := \cdots \to U_p(V)_\gamma \to U_{p-1}(V)_\gamma \to \cdots$$

is a complex of $\Sigma(\gamma)$-modules.

Proposition 3.1. *For all $p \in \mathbb{N}$, we have that $H_p(U(V))$ is a polynomial representation of $GL(V)$.*

Proof. This follows from the fact that $H_p(U(V))$ is a subquotient of $U_p(V)$ and submodules and quotient modules of polynomial representations are again polynomial. $\square$

Theorem 3.2. *Let γ be a partition and let V be a k-vector space where $\dim V \geq \ell(\gamma)$. Then the following isomorphism of $\Sigma(\gamma)$-complexes holds*

$$C(\Delta_\gamma) \cong_{\Sigma(\gamma)} U(V)_\gamma. \tag{3.1}$$

Moreover the following isomorphism of $\Sigma(\gamma)$-modules holds for all $p \in \mathbb{N}$

$$\tilde{H}_{p-1}(\Delta_\gamma; k) \cong_{\Sigma(\gamma)} H_p(U(V))_\gamma. \tag{3.2}$$

Proof. Recall that $v_1, v_2, \ldots, v_m$ is an ordered basis for V and let $\phi : U_p(V)_\gamma \to C_p(\Delta_\gamma)$ be the map defined on generators by

$$\phi(v_{i_1} v_{i_2} \wedge \cdots \wedge v_{i_{2p-1}} v_{i_{2p}} \otimes v_{i_{2p+1}} \cdots v_{i_{|\gamma|}}) = (\{i_1, i_2\}, \ldots, \{i_{2p-1}, i_{2p}\}).$$

It is easy to see that this is a well defined vector space isomorphism that commutes with the action of $\Sigma(\gamma)$. It is also immediate that ϕ commutes with the differentials of the two complexes. Hence (3.1) holds.

It follows that

$$\tilde{H}_{p-1}(\Delta_\gamma; k) \cong_{\Sigma(\gamma)} H_p(U(V)_\gamma). \tag{3.3}$$

Since $U_p(V)$ is a polynomial representation of $GL(V)$, it decomposes into a direct sum of its weight spaces (cf. [6, §8.2]). This implies that the $GL(V)$-complex $U(V)$ decomposes into a direct sum of its weight complexes $U(V)_\gamma$. Consequently,

$$H_p(U(V)_\gamma) \cong_{\Sigma(\gamma)} H_p(U(V))_\gamma.$$

Combining this with (3.3) yields (3.2). $\square$

Corollary 3.3. *Let $n \leq \dim V$. Then*

$$\tilde{H}_{p-1}(M_n; k) \cong_{\Sigma_n} H_p(U(V))_{1^n}.$$

Theorem 3.2 and Corollary 3.3 show that the Józefiak-Weyman Theorem implies the Reiner-Roberts Theorem and the Bouc Theorem. To show that Bouc implies Reiner-Roberts we will use Corollary 3.3 and the following proposition.

Proposition 3.4. *Let W be a polynomial representation of $GL(V)$. Then for each partition λ of t, where $t \leq \dim V$, the multiplicity of the irreducible Σ_t-module S^λ in the 1^t-weight space of W equals the multiplicity of the irreducible $GL(V)$-module V^λ in W.*

Proof. This follows immediately from the semisimplicity of polynomial representations of $GL(V)$ and the fact that $V^\lambda_{1^t} = S^\lambda$ $\square$

Bouc $\implies$ Reiner-Roberts. Let γ be a partition and V a k-vector space such that $|\gamma| \leq \dim V$. Since $H_p(U(V))$ is a polynomial representation of $GL(V)$, Bouc's Theorem implies, by Corollary 3.3 and Proposition 3.4, that for any $\lambda \vdash |\gamma|$ the multiplicity of V^λ in $H_p(U(V))$ is 1 if $\lambda = \lambda'$ and $d(\lambda) = |\lambda| - 2p$, and is 0 otherwise. This and Theorem 3.1 implies that

$$\tilde{H}_{p-1}(\Delta_\gamma; k) \cong_{\Sigma(\gamma)} H_p(U(V))_\gamma \cong_{\Sigma(\gamma)} \bigoplus_{\substack{\lambda\,:\,\lambda \vdash |\gamma| \\ \lambda = \lambda' \\ d(\lambda) = |\lambda| - 2p}} V^\lambda_\gamma,$$

whenever $\dim V \geq |\gamma|$. The requirement that $\dim V \geq |\gamma|$ is easily relaxed to $\dim V \geq \ell(\gamma)$ because the $\Sigma(\gamma)$-module V^λ_γ is isomorphic to $S^\lambda(\gamma)$ for all V such that $\dim V \geq \ell(\gamma)$. Hence the Reiner-Roberts Theorem follows from Bouc.

Proposition 3.4 and Bouc's Theorem yield the coefficient of V^λ in $H_p(U(V))$ when $\dim V \geq |\lambda|$. To obtain all the coefficients, that is to derive the Józefiak-Weyman Theorem from Bouc's Theorem, one can either use the Weyl construction discussed below or the fact (which we are about to establish) that Corollary 2.4 implies the Józefiak-Weyman Theorem.

Corollary 2.4 $\implies$ *Józefiak-Weyman.* The character of any polynomial representation W of $GL(V)$ can be expressed as a generating function for the dimensions of the weight spaces. More precisely

$$\mathrm{char}(W) = \sum_{\gamma} \dim W_\gamma \, x^\gamma.$$

Hence Corollary 2.4 implies, via (3.2), that

$$\mathrm{char}(H_p(U(V))) = \sum_{\gamma} \tilde{\beta}_{p-1}(\Delta_\gamma)\, x^\gamma = \sum_{\substack{\lambda\,:\ \lambda=\lambda' \\ d(\lambda)=|\lambda|-2p}} s_\lambda(x_1,\ldots,x_m).$$

where s_λ is a Schur function and $m = \dim V$. Since $s_\lambda(x_1,\ldots,x_m) = \mathrm{char} V^\lambda$, we can use the fact that the character of a polynomial representation of $GL(V)$ determines the representation to conclude that (2.1) holds. Hence the Józefiak-Weyman Theorem follows from the Bouc Theorem.

An alternative way to view the equivalence of Józefiak-Weyman and Bouc is by the Weyl construction which we now recall (cf. [6, §8.3]). Given a Σ_n-module X and a finite dimensional k-vector space V, form the $GL(V)$-module

$$W(X,V) = V^{\otimes n} \otimes_{k[\Sigma_n]} X.$$

This construction is functorial. If $\phi : X \to Y$ is a homomorphism of Σ_n-modules then $W(\phi,V) : W(X,V) \to W(Y,V)$ defined on generators by

$$W(\phi,V)(v_{i_1}\otimes\cdots\otimes v_{i_n}\otimes x) = (v_{i_1}\otimes\cdots\otimes v_{i_n}\otimes \phi(x))$$

is a well-defined homomorphism of $GL(V)$ modules. Also a direct sum decomposition $X = \oplus X_i$ determines a direct sum decomposition $W(X,V) = \oplus W(X_i,V)$. If $\lambda \vdash n$ then $W(S^\lambda, V)$ is the irreducible $GL(V)$ module indexed by λ when $\ell(\lambda) \le \dim V$ and is 0 otherwise.

The Weyl construction applied to the chain complex of M_n yields a $GL(V)$-complex

$$W(C(M_n),V) := \cdots \to W(C_p(M_n),V) \to W(C_{p-1}(M_n),V) \to \cdots,$$

which turns out to be the degree n homogeneous piece of the JW complex. Let $U_p^n(V)$ be the degree n homogeneous component of $U_p(V)$, i.e.,

$$U_p^n(V) = \wedge^p(\mathrm{Sym}^2 V) \otimes \mathrm{Sym}^{n-2p} V.$$

Since $\delta_p(U_p^n(V)) \subseteq U_{p-1}^n(V)$, we have that $U(V)$ is a graded $GL(V)$-complex with nth component equal to

$$U^n(V) := \cdots \to U_p^n(V) \to U_{p-1}^n \to \cdots.$$

Theorem 3.5. *The following isomorphism of $GL(V)$-complexes holds*

$$W(C(M_n), V) \cong_{GL(V)} U^n(V) \tag{3.4}$$

Moreover the following isomorphism of $GL(V)$-modules holds

$$W(\tilde{H}_{p-1}(M_n; k), V) \cong_{GL(V)} H_p(U^n(V)). \tag{3.5}$$

Proof. Recall that $v_1, v_2, \ldots, v_m$ is an ordered basis for V. Let $\psi : W(C_p(M_n), V) \to U_p^n(V)$ be the map defined on generators by

$$\begin{aligned}\psi\,(v_{i_1} \otimes \cdots \otimes v_{i_n} \otimes_{\Sigma_n} (\{1,2\}, \ldots, \{2p-1, 2p\}))&\\ = v_{i_1} v_{i_2} \wedge \cdots \wedge v_{i_{2p-1}} v_{i_{2p}} \otimes v_{i_{2p+1}} \cdots v_{i_n}&.\end{aligned}$$

One can easily see that ψ is a well-defined $GL(V)$-module isomorphism. To see that

$$\psi \circ W(\partial_p, V) = \delta_p \circ \psi,$$

observe that

$$\begin{aligned}v_{i_1} \otimes \cdots \otimes v_{i_n} \quad &\otimes_{\Sigma_n} (\{1,2\}, \ldots, \widehat{\{2j-1, 2j\}}, \ldots, \{2p-1, 2p\})\\ &= v_{i_1} \otimes \cdots \otimes \hat{v}_{i_{2j-1}} \otimes \hat{v}_{i_{2j}} \otimes \cdots \otimes v_{i_n} \otimes v_{i_{2j-1}} \otimes v_{i_{2j}}\\ &\qquad \otimes_{\Sigma_n} (\{1,2\}, \ldots, \{2p-3, 2p-2\}).\end{aligned}$$

Hence (3.4) holds.

It follows from (3.4) that $H_p(U^n(V)) \cong_{GL(V)} H_p(W(C(M_n), V))$. Since $k[\Sigma_n]$ is semisimple, $V^{\otimes n}$ is projective as a right Σ_n-module. Consequently,

$$H_p(W(C(M_n), V)) \cong_{GL(V)} W(H_p(M_n), V)$$

which completes the proof of (3.5). □

4. Hypergraph complexes

Our argument in the previous section that Bouc implies Reiner-Roberts can be abstracted to provide a technique for lifting results on variations of the matching complex to corresponding variations of the bounded degree graph complexes. This technique is applied to hypergraph complexes in this section and to chessboard complexes in the next section.

Theorem 4.1 (Lifting Lemma). *Let Λ be a $\Sigma(\gamma)$-simplicial complex and let M be a Σ_t-simplicial complex, where $\gamma \vdash t$. Suppose there exists a polynomial $GL_t(k)$-complex U whose γ-weight complex U_γ is $\Sigma(\gamma)$-isomorphic to the chain complex $C(\Lambda; k)$ and whose 1^t-weight complex U_{1^t} is Σ_t-isomorphic to the chain complex $C(M; k)$. Then for all $p \in \mathbb{N}$*

$$\tilde{H}_p(\Lambda; k) \cong_{\Sigma(\gamma)} \bigoplus_{\lambda:\, \lambda \vdash t} c_\lambda S^\lambda(\gamma)$$

where c_λ is the multiplicity of S^λ in $\tilde{H}_p(M;k)$.

Proof. Since the complex U decomposes into a direct sum of its weight complexes, it follows that

$$\tilde{H}_{p-1}(\Lambda;k) \cong_{\Sigma(\gamma)} H_p(U_\gamma) \cong_{\Sigma(\gamma)} H_p(U)_\gamma.$$

That $H_p(U)$ is a polynomial representation of $GL_t(k)$ follows from the fact that it is a subquotient of polynomial representations. Hence, by Proposition 3.4, the multiplicity of the irreducible representation of $GL_t(k)$ indexed by λ in $H_p(U)$ equals c_λ, the multiplicity of S^λ in $\tilde{H}_{p-1}(M;k)$. Consequently by taking γ-weight spaces we have

$$\tilde{H}_{p-1}(\Lambda;k) \cong_{\Sigma(\gamma)} H_p(U)_\gamma \cong_{\Sigma(\gamma)} \bigoplus_{\lambda:\, \lambda\vdash t} c_\lambda S^\lambda(\gamma).$$

□

Corollary 4.2. *Let Δ and M satisfy the hypothesis of Theorem 4.1. Then for all $p \in \mathbb{N}$, the Betti number $\tilde{\beta}_p(\Lambda)$ is equal to the coefficient of x^γ in the Frobenius characteristic of $\tilde{H}_p(M;k)$.*

Let $r \geq 2$. An r-hypergraph on node set $[n]$ is a set of r element multisets on $[n]$. The multisets are called *edges*. The *degree* of a node i is the total number of occurrences of i in the edges of the hypergraph. For γ an n-tuple of nonnegative integers the γ-bounded degree r-hypergraph complex Δ^r_γ is defined to be the simplicial complex of r-hypergraphs on $[n]$ for which each node i has degree at most γ_i. Again we can assume without loss of generality that γ is a partition. The Young subgroup $\Sigma(\gamma)$ acts as an automorphism group on Δ^r_γ by permuting the hypergraph nodes of like degree bound. When $r = 2$, the bounded degree hypergraph complex Δ^r_γ is the bounded degree graph complex Δ_γ.

When $\gamma = 1^n$, we get the matching complex of the complete r-hypergraph complex on node set $[n]$ introduced in [3]. This is the Σ_n-simplicial complex whose faces are set partitions of subsets of $[n]$ into blocks of size r. We will call this the *r-matching complex* and denote it by M^r_n.

Theorem 4.3. *For any partition γ, integer $r \geq 2$ and nonnegative integer p, the following isomorphism of $\Sigma(\gamma)$-modules holds*

$$\tilde{H}_p(\Delta^r_\gamma;k) \cong_{\Sigma(\gamma)} \bigoplus_{\lambda\vdash|\gamma|} c_\lambda S^\lambda(\gamma), \tag{4.1}$$

where c_λ is the multiplicity of S^λ in $\tilde{H}_p(M^r_{|\gamma|};k)$. Consequently $\tilde{\beta}_p(\Delta^r_\gamma)$ is equal to the coefficient of x^γ in the Frobenius characteristic of $\tilde{H}_p(M^r_{|\gamma|};k)$.

Proof. Let V be a $|\gamma|$-dimensional k-vector space. Consider the $GL(V)$-complex $(\wedge^p(S^rV)\otimes S^*V,\delta_p)$ whose differential $\delta_p:\wedge^p(S^rV)\otimes S^*V\to\wedge^{p-1}(S^rV)\otimes S^*V$ is defined on generators by

$$\delta_p(u_1\wedge\cdots\wedge u_p\otimes u)=\sum_{i=1}^{p}(-1)^i\,u_1\wedge\cdots\wedge\hat{u}_i\wedge\cdots\wedge u_p\otimes u\cdot u_i,$$

where $u_j\in\operatorname{Sym}^r V$ and $u\in\operatorname{Sym}^* V$. It is easy to check that δ_p is a well-defined differential. The γ weight space of this complex is $\Sigma(\gamma)$-isomorphic to the chain complex of Δ^r_γ. Hence the result follows from the Lifting Lemma and its corollary. □

Corollary 4.4. *If $\gamma=d^n$ then the Σ_n-module $\tilde{H}_p(\Delta^r_\gamma;k)$ decomposes into irreducibles as follows*

$$\tilde{H}_p(\Delta^r_\gamma;k)\cong_{\Sigma_n}\bigoplus_{\lambda\vdash n}\langle\operatorname{ch}(\tilde{H}_p(M^r_{nd};k)),s_\lambda[h_d]\rangle\,S^\lambda,$$

where ch *denotes the Frobenius characteristic, s_λ denotes the Schur function corresponding to λ, $\langle\cdot,\cdot\rangle$ denotes the bilinear form on the ring of symmetric functions for which the Schur functions form an orthonormal basis, and $f[g]$ denotes the plethysm of the symmetric functions f and g.*

Proof. See [11, Remark 3.4]. □

In [3] Björner, Lovász, Vrećica and Živaljević show that the matching complex M^r_n is ν^r_n-connected, where

$$\nu^r_n=\left\lfloor\frac{n-2}{2r-1}\right\rfloor-1.$$

Although we suspect that this connectivity result can be lifted to general bounded degree hypergraph complexes Δ^r_γ, our technique only allows us to lift the rational homology version of this result.

Corollary 4.5. *The bounded degree hypergraph complex Δ^r_γ is $\nu^r_{|\gamma|}$-acyclic (over k).*

Proof. Let $p\le\nu^r_{|\gamma|}$. Then $\tilde{H}_p(M^r_{|\gamma|};k)=0$ since by [3, Theorem 4.1] $M^r_{|\gamma|}$ is $\nu^r_{|\gamma|}$-connected. It follows from (4.1) that $H_p(\Delta^r_\gamma;k)=0$. Hence Δ^r_γ is $\nu^r_{|\gamma|}$-acyclic. □

5. Chessboard Complexes

Next we consider the t-dimensional chessboard complexes introduced in [3]. Let $t\ge 2$ and $(n_1,\ldots,n_t)$ be a t-tuple of positive integers. An admissible rook configuration on an $n_1\times\cdots\times n_t$ chessboard is a subset

A of $[n_1] \times \cdots \times [n_t]$ such that for $i = 1, \ldots, t$, each $j \in [n_i]$ appears as the ith coordinate of at most one element of A. The collection of admissible rook configurations forms a simplicial complex $M_{n_1,\ldots,n_t}$ called the t-dimensional chessboard complex on an $n_1 \times \cdots \times n_t$ chessboard. The group $\Sigma_{n_1} \times \cdots \times \Sigma_{n_t}$ acts on the chessboard complex. When $t = 2$, an admissible rook configuration corresponds to a placement of non-taking rooks on an $n_1 \times n_2$ chessboard. The 2-dimensional chessboard complex, known simply as the chessboard complex, is also viewed as the matching complex of a complete bipartite graph. The homology of the chessboard complex M_{n_1,n_2} and the representation of $\Sigma_{n_1} \times \Sigma_{n_2}$ on it were computed by Friedman and Hanlon [5].

For each $i = 1, \ldots, t$, let $\gamma^{(i)}$ be a weakly increasing n_i-tuple of positive integers $(\gamma_1^{(i)}, \ldots, \gamma_{n_i}^{(i)})$ and let $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(t)})$. A γ-rook configuration on an $n_1 \times \cdots \times n_t$ chessboard is a subset A of $[n_1] \times \cdots \times [n_t]$ such that for $i = 1, \ldots, t$, each $j \in [n_i]$ appears as the ith coordinate of at most $\gamma_j^{(i)}$ elements of A. The collection of γ-rook configurations on an $n_1 \times \cdots \times n_t$ chessboard forms a simplicial complex Δ_γ that we call the *γ-chessboard complex.*

Note that Δ_γ can be viewed as a bounded degree multipartite hypergraph complex. That is, Δ_γ is the simplicial complex of all sub-t-hypergraphs of the complete t-partite t-hypergraph $K_{n_1,\ldots,n_t}$ on node set $[n_1] \uplus \cdots \uplus \cup[n_t]$, in which each node j of part i has degree at most $\gamma_j^{(i)}$. The group $\Sigma(\gamma) := \Sigma(\gamma^{(1)}) \times \cdots \times \Sigma(\gamma^{(t)})$ acts on Δ_γ. The representation of $\Sigma(\gamma)$ on the homology of Δ_γ for the $t = 2$ case was computed by Reiner and Roberts [11]. The $t = 2$ case of the next theorem equates the Reiner-Roberts result with the Friedman-Hanlon result.

Theorem 5.1. *For any sequence of partitions $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(t)})$ and nonnegative integer p, the following isomorphism of $\Sigma(\gamma)$-modules holds*

$$\tilde{H}_p(\Delta_\gamma; k) \cong_{\Sigma(\gamma)} \bigoplus_{\lambda_1 \vdash |\gamma^{(1)}|, \ldots, \lambda_t \vdash |\gamma^{(t)}|} c_{\lambda_1,\ldots,\lambda_t} S^{\lambda_1}(\gamma^{(1)}) \otimes \cdots \otimes S^{\lambda_t}(\gamma^{(t)}),$$

where $c_{\lambda_1,\ldots,\lambda_t}$ is the multiplicity of the irreducible $S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_t}$ in $\tilde{H}_p(M_{|\gamma^{(1)}|,\ldots,|\gamma^{(t)}|}; k)$.

Proof. The proof is similar to that of Theorem 4.3. For $i = 1, \ldots, t$, let V_i be a $|\gamma^{(i)}|$-dimensional vector space. Consider the $GL(V_1) \times \cdots \times GL(V_t)$-complex $(\wedge^p(\otimes_{i=1}^t V_i) \otimes (\otimes_{i=1}^t S^* V_i), \delta_p)$ whose differential

$$\delta_p : \wedge^p(\otimes_{i=1}^t V_i) \otimes (\otimes_{i=1}^t S^* V_i) \to \wedge^{p-1}(\otimes_{i=1}^t V_i) \otimes (\otimes_{i=1}^t S^* V_i)$$

is defined on generators by

$$\delta_p\left(\left(\otimes_{i=1}^t v_{i,1}\wedge\cdots\wedge\otimes_{i=1}^t v_{i,p}\right)\ \otimes\ \left(\otimes_{i=1}^t u_i\right)\right)\ =$$
$$\sum_{j=1}^{p}(-1)^j\left(\otimes_{i=1}^t v_{i,1}\wedge\cdots\wedge\widehat{\otimes_{i=1}^t v_{i,j}}\wedge\cdots\wedge\otimes_{i=1}^t v_{i,p}\right)\ \otimes\ \left(\otimes_{i=1}^t u_i v_{i,j}\right),$$

where $v_{i,h}\in V_i$ and $u_i\in\operatorname{Sym}^* V_i$. Clearly the γ weight space of this complex is $\Sigma(\gamma)$-isomorphic to the chain complex of Δ_γ. The result now follows from an obvious generalization of the Lifting Lemma. □

Given a t-tuple $(j_1,\ldots,j_t)$ of positive integers, define

$$\nu_{j_1,\ldots,j_t}=\min\left\{j_1'-1,\left\lfloor\frac{j_1'+j_2'-2}{3}\right\rfloor,\ldots,\left\lfloor\frac{j_1'+j_2'+\cdots+j_t'-t}{2t-1}\right\rfloor\right\}-1,$$

where $(j_1',\ldots,j_t')$ is a weakly increasing rearrangement of $(j_1,\ldots,j_t)$.

Corollary 5.2. *The γ-chessboard complex Δ_γ is $\nu_{|\gamma^{(1)}|,\ldots,|\gamma^{(t)}|}$-acyclic.*

Proof. It follows from [3, Theorem 3.1] that $M_{|\gamma^{(1)}|,\ldots,|\gamma^{(t)}|}$ is $\nu_{|\gamma^{(1)}|,\ldots,|\gamma^{(t)}|}$-acyclic. □

Björner and Eriksson generalize and unify connectivity results for the matching complex and chessboard complex by introducing a simplicial complex called a packing complex [2]. Here we consider a bounded degree version of the packing complex which yields a generalization and unification of Theorems 4.3 and 5.1. Let $(n_1,\ldots,n_t)$ and $(r_1,\ldots,r_t)$ be t-tuples of positive integers where $t\geq 1$. A $(r_1,\ldots,r_t)$-*hypergraph* on node set $[n_1]\uplus\cdots\uplus[n_t]$ is a collection of multisets on $[n_1]\uplus\cdots\uplus[n_t]$ such that exactly r_i elements (counting multiplicity) of each multiset are chosen from the ith set $[n_i]$ of the disjoint union for all $i=1,\ldots,t$. Since an $(r_1,\ldots,r_t)$-hypergraph is just a special type of $(r_1+\cdots+r_t)$-hypergraph, the definition of *node degree* from Section 4 applies.

For a t-tuple $\gamma=(\gamma^{(1)},\ldots,\gamma^{(t)})$, where each $\gamma^{(i)}$ is a weakly increasing n_i-tuple of positive integers, the γ-*bounded degree* $(r_1,\ldots,r_t)$-*hypergraph complex* $\Delta_\gamma^{r_1,\ldots,r_t}$ is defined to be the simplicial complex of $(r_1,\ldots,r_t)$-hypergraphs for which each node $j\in[n_i]$ has degree $\gamma_j^{(i)}$ or less. The group $\Sigma(\gamma):=\Sigma(\gamma^{(1)})\times\cdots\times\Sigma(\gamma^{(t)})$ acts on $\Delta_\gamma^{r_1,\ldots,r_t}$. When $\gamma^{(i)}=1^{n_i}$ for all i, we have the packing complex of Björner and Eriksson [2] which we denote by $M^{r_1,\ldots,r_t}_{n_1,\ldots,n_t}$. When $t=1$ and $r_1\geq 2$, we have the γ-bounded degree r_1-hypergraph complex $\Delta^{r_1}_{\gamma^{(1)}}$ of Section 4. When $t\geq 2$ and $r_i=1$ for all i, we have the γ-chessboard complex Δ_γ.

Theorem 5.3. *For any $p \in \mathbb{N}$, the following isomorphism of $\Sigma(\gamma)$-modules holds*

$$\tilde{H}_p(\Delta_\gamma^{r_1,\dots,r_t};k) \cong_{\Sigma(\gamma)} \bigoplus_{\lambda_1 \vdash |\gamma^{(1)}|,\dots,\lambda_t \vdash |\gamma^{(t)}|} c_{\lambda_1,\dots,\lambda_t}\, S^{\lambda_1}(\gamma^{(1)}) \otimes \cdots \otimes S^{\lambda_t}(\gamma^{(t)}),$$

where $c_{\lambda_1,\dots,\lambda_t}$ is the multiplicity of the irreducible $S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_t}$ in $\tilde{H}_p(M^{r_1,\dots,r_t}_{|\gamma^{(1)}|,\dots,|\gamma^{(t)}|};k)$.

Proof. For $i = 1,\dots,t$, let V_i be a $|\gamma^{(i)}|$-dimensional vector space. We apply an obvious generalization of the Lifting Lemma to the $GL(V_1) \times \cdots \times GL(V_t)$-complex $(\wedge^p(\otimes_{i=1}^t S^{r_i}V_i) \otimes (\otimes_{i=1}^t S^* V_i), \delta_p)$ whose differential

$$\delta_p : \wedge^p(\otimes_{i=1}^t S^{r_i}V_i) \otimes (\otimes_{i=1}^t S^* V_i) \to \wedge^{p-1}(\otimes_{i=1}^t S^{r_i}V_i) \otimes (\otimes_{i=1}^t S^* V_i)$$

is defined on generators by

$$\delta_p\left((\otimes_{i=1}^t u_{i,1} \wedge \cdots \wedge \otimes_{i=1}^t u_{i,p}) \;\otimes\; (\otimes_{i=1}^t u_i)\right) \;=$$
$$\sum_{j=1}^p (-1)^j \left(\otimes_{i=1}^t u_{i,1} \wedge \cdots \wedge \widehat{\otimes_{i=1}^t u_{i,j}} \wedge \cdots \wedge \otimes_{i=1}^t u_{i,p}\right) \;\otimes\; (\otimes_{i=1}^t u_i u_{i,j}),$$

where $u_{i,h} \in \mathrm{Sym}^{r_i} V_i$ and $u_i \in \mathrm{Sym}^* V_i$. $\square$

We assume without loss of generality that the parameters are ordered so that

$$\frac{|\gamma^{(1)}|}{r_1} \le \cdots \le \frac{|\gamma^{(t)}|}{r_t} \quad \text{and} \quad |\gamma^{(i)}| \le |\gamma^{(i+1)}| \text{ if } \frac{|\gamma^{(i)}|}{r_i} = \frac{|\gamma^{(i+1)}|}{r_{i+1}}.$$

Corollary 5.4. *The γ-bounded degree $(r_1,\dots,r_t)$-hypergraph complex $\Delta_\gamma^{(r_1,\dots,r_t)}$ is ν-acyclic, where*

$$\nu = \min_{i \ge 1} \left\lfloor \frac{|\gamma^{(1)}| + \cdots + |\gamma^{(i)}| - i - \epsilon}{2(r_1 + \cdots + r_i) - 1} \right\rfloor - 1,$$

and $\epsilon = \begin{cases} 1 & \textit{if } r_i \ge 2 \;\; \forall i \\ 0 & \textit{otherwise} \end{cases}$.

Proof. We apply the packing complex connectivity result of Björner and Eriksson [2, Theorem 2.1]. $\square$

Remark. Another application of the lifting technique is given in [14] where the homology representations of bounded degree multigraph and digraph complexes are computed.

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