Supplemental Examples:
Limits and Tangent Lines

1. Fill in the tables below to estimate the limit of $f(x) = \dfrac{x^4 - 16}{x - 2}$ as x approaches 2.

x	1.9	1.99	1.999	1.9999
$f(x)$				

x	2.1	2.01	2.001	2.0001
$f(x)$				

Solution: Filling in the tables is straightforward, but you might need some practice with your particular calculator, the order in which to punch things in, etc.

You don't want to spend valuable test time fumbling with your calculator!

So make sure you get the hang of these calculations. **Double check that you get the entries I get.** When it comes to the limit, we make an educated guess.

x	1.9	1.99	1.999	1.9999
$f(x)$	29.679	31.761	31.976	31.9976

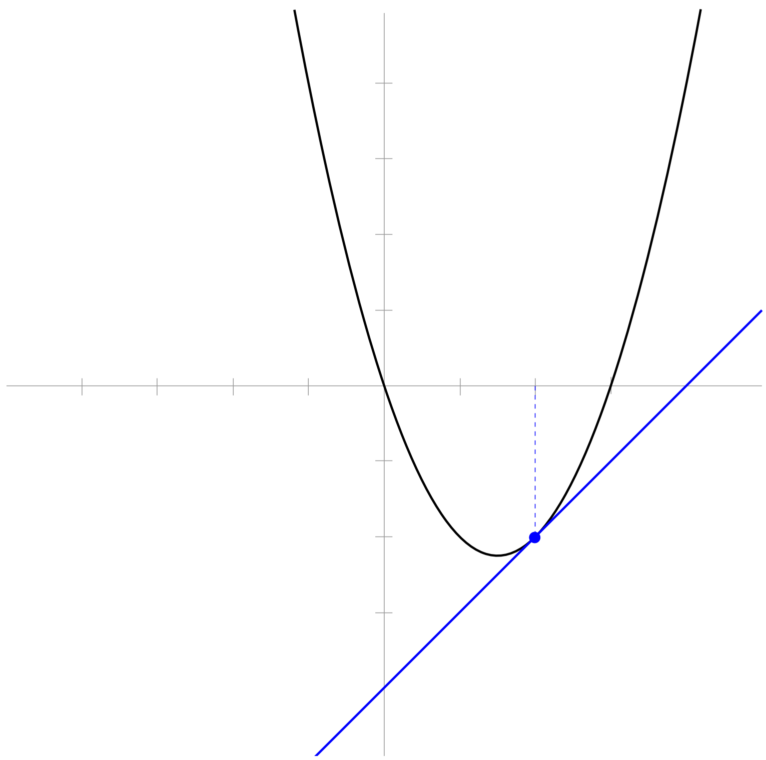
x	2.1	2.01	2.001	2.0001
$f(x)$	34.481	32.241	32.024	32.002

From the looks of it, as x gets closer and closer to 2, from either side, $f(x)$ gets closer and closer to 32. Hence, we guess:

$$\lim_{x \to 2} \left(\frac{x^4 - 16}{x - 2} \right) = 32$$

2. Consider the function $f(x) = x^2 - 3x$. Use the limit definition to find the slope of the tangent line at $x = 2$.

Solution: We don't need the picture to answer the question, but it is helpful to remember what we are doing. The graph of $f(x) = x^2 - 3x$ is shown in black. The tangent line at $x = 2$ is shown in blue.



First, recall the (limit) definition of the slope of the tangent line at any point:

$$\text{slope of tangent line at } x \quad = \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Method 1: Focus on $x = 2$ from the start.

$$\text{slope of tangent line at } x = 2 \quad = \quad \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

Hence, we will need the following pieces: (You fill in the 'dot dot dot'!)

- $f(2) = (2)^2 - 3(2) = 4 - 6 = -2$
- $f(2+h) = (2+h)^2 - 3(2+h) = \;\; ... \;\; = h^2 + h - 2$

So we have:

$$\begin{aligned}
\text{slope of tangent line at } x = 2 \quad &= \quad \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\
&= \quad \lim_{h \to 0} \frac{[h^2 + h - 2] - [-2]}{h} \\
&= \quad \lim_{h \to 0} \frac{h^2 + h - \not{2} + \not{2}}{h} \\
&= \quad \lim_{h \to 0} \frac{h^2 + h}{h} \\
&= \quad \lim_{h \to 0} \frac{h(h+1)}{h} \\
&= \quad \lim_{h \to 0} \frac{\not{h}(h+1)}{\not{h}} \\
&= \quad \lim_{h \to 0} (h+1) \\
&= \quad 1
\end{aligned}$$

We can think of the last step as just letting $h = 0$.

Method 2: First find the slope at any point x. Then plug in $x = 2$ (at the end).

$$\text{slope of tangent line at } x \quad = \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Hence, we will need the following pieces: (You fill in the 'dot dot dot'!)

- $f(x) = x^2 - 3x$
- $f(x+h) = (x+h)^2 - 3(x+h) = \quad ... \quad = x^2 + 2xh + h^2 - 3x - 3h$

So we have:

$$\begin{aligned}
\text{slope of tangent line at } x \quad &= \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \quad \lim_{h \to 0} \frac{[x^2 + 2xh + h^2 - 3x - 3h] - [x^2 - 3x]}{h} \\
&= \quad \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 3x - 3h - x^2 + 3x}{h} \\
&= \quad \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{x^2} + \cancel{3x}}{h} \\
&= \quad \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h} \\
&= \quad \lim_{h \to 0} \frac{h(2x + h - 3)}{h} \\
&= \quad \lim_{h \to 0} \frac{\cancel{h}(2x + h - 3)}{\cancel{h}} \\
&= \quad \lim_{h \to 0} (2x + h - 3) \\
&= \quad 2x - 3
\end{aligned}$$

Again, we can think of the last step as just letting $h = 0$. (continued on next page)

Hence, we have found:

$$\text{slope of tangent line at } x \quad = \quad 2x - 3$$

This gives us a lot more information; We know the slope of *any* tangent line. For example, the tangent line at $x = 1$ has slope $2(1) - 3 = -1$. The tangent line at $x = 0$ has slope $2(0) - 3 = -3$. In any case, the question is about the tangent line at $x = 2$. Plugging this in:

$$\text{slope of tangent line at } x = 2 \quad = \quad 2(2) - 3 \quad = \quad 1$$