Supplemental Example:
Increasing/Decreasing, Concavity, etc.

Example: Consider the function

$$f(x) \;=\; 4 \;-\; \frac{2}{x} \;+\; \frac{3}{x^2}$$

a) Determine the intervals on which f is increasing, decreasing, and give the coordinates of any and all relative max and mins.

b) Determine the intervals on which f is concave up, down, and give the coordinates of any and all inflection points.

Solution: a) Make a sign chart for f'.

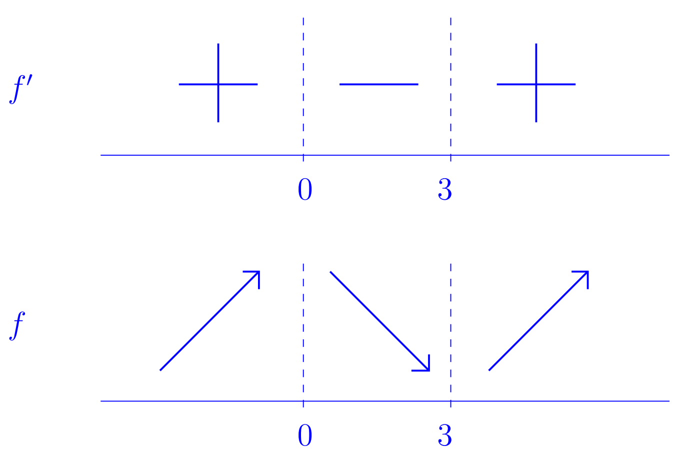
$$\begin{aligned} f &= 4 - \frac{2}{x} + \frac{3}{x^2} \\ &= 4 - 2x^{-1} + 3x^{-2} \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad f' &= 0 - 2(-1)x^{-2} + 3(-2)x^{-3} \\ &= 2x^{-2} - 6x^{-3} \\ &= \frac{2}{x^2} - \frac{6}{x^3} \\ &= \frac{2x}{x^3} - \frac{6}{x^3} \\ &= \frac{2x-6}{x^3} \end{aligned}$$

$$
\begin{aligned}
f' = 0 \quad &\iff \quad \frac{2x-6}{x^3} = 0 \\
&\iff \quad 2x - 6 = 0 \\
&\iff \quad 2x = 6 \\
&\iff \quad x = 3
\end{aligned}
$$

$$
\begin{aligned}
f' \ \text{undefined} \quad &\iff \quad \frac{2x-6}{x^3} \ \ \text{undefined} \\
&\iff \quad x^3 = 0 \\
&\iff \quad x = 0
\end{aligned}
$$

Hence, there are two critical points, and testing for the sign of f' on the resulting pieces, gives:



Hence, f is increasing on $(-\infty, 0)$ and $(3, \infty)$ and decreasing on $(0, 3)$.

0 ‘looks like’ a relative max, and 3 ‘looks like’ a min ...

... however, if we try plugging in $x = 0$,

$$f(0) = 4 - \frac{2}{(0)} + \frac{3}{(0)^2}$$

... there's a problem. f is not defined at $x = 0$. Consequently, the critical point $x = 0$ does not correspond to a point on the graph, and hence there is no 'max' at $x = 0$.

On the other hand, plugging in $x = 3$, we get:

$$f(3) \;=\; 4 - \frac{2}{(3)} + \frac{3}{(3)^2} \;=\; 4 - \frac{2}{3} + \frac{3}{9} \;=\; \frac{36}{9} - \frac{6}{9} + \frac{3}{9} \;=\; \frac{33}{9} \;\approx\; 3.67$$

Hence, f has no relative maxima and one relative minimum at $(3, 33/9) \approx (3, 3.67)$.

b) Make a sign chart for f''.

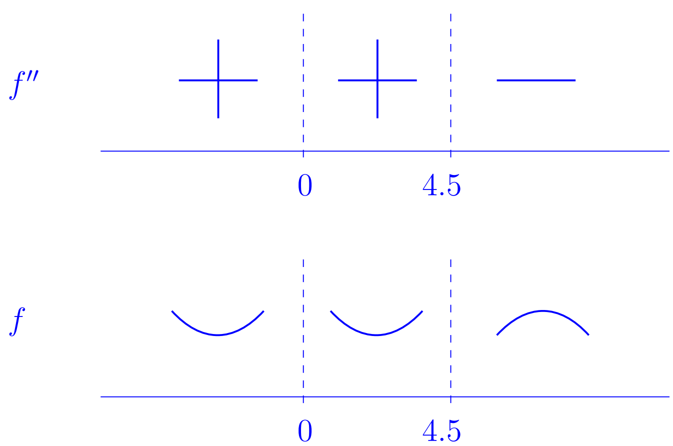
$$f' \;=\; 2x^{-2} - 6x^{-3}$$

$$\begin{aligned}
\Longrightarrow \quad f'' &= 2(-2)x^{-3} - 6(-3)x^{-4} \\
&= -4x^{-3} + 18x^{-4} \\
&= -\frac{4}{x^3} + \frac{18}{x^4} \\
&= -\frac{4x}{x^4} + \frac{18}{x^4} \\
&= \frac{-4x + 18}{x^4}
\end{aligned}$$

$$
\begin{aligned}
f'' = 0 \quad &\iff \quad \frac{-4x+18}{x^4} = 0 \\
&\iff \quad -4x + 18 = 0 \\
&\iff \quad 4x = 18 \\
&\iff \quad x = \frac{18}{4} = \frac{9}{2} = 4.5
\end{aligned}
$$

$$
\begin{aligned}
f'' \text{ undefined} \quad &\iff \quad \frac{-4x+18}{x^4} \text{ undefined} \\
&\iff \quad x^4 = 0 \\
&\iff \quad x = 0
\end{aligned}
$$

Hence, there are two ‘critical points’, and testing for the sign of f'' on the resulting pieces, gives:



Hence, f is concave up on $(-\infty, 0)$ and $(0, 4.5)$, and concave down on $(4.5, \infty)$.

Since f changes concavity at 4.5, plugging in we see that f has one inflection point, at $(4.5, f(4.5)) \approx (4.5, 3.7)$.