PRACTICE PROBLEMS FOR EXAM 1

1. Let \( \mathbf{v} = 2\mathbf{i} + 2\mathbf{k} \) and \( \mathbf{w} = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \). (a) Find \( |\mathbf{v}| \). (b) Find the unit vector with the same direction as \( \mathbf{v} \). (c) Find \( \mathbf{v} \cdot \mathbf{w} \). (d) State a formula for the angle between two vectors. (e) Find the angle between \( \mathbf{v} \) and \( \mathbf{w} \). (f) Find \( \mathbf{v} + \mathbf{w} \). (g) Find the component of \( \mathbf{w} \) in the direction of \( \mathbf{v} \). (h) Write \( \mathbf{w} \) as the sum of two vectors, one of them parallel to \( \mathbf{v} \), the other perpendicular to \( \mathbf{v} \). (i) Find \( \mathbf{v} \times \mathbf{w} \).

2. True or false? (a) \( \mathbf{u} \times \mathbf{v} = 0 \) if and only if \( \mathbf{u} \) and \( \mathbf{v} \) are perpendicular. (b) \( \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \times (\mathbf{v} \cdot \mathbf{w}) \) for all vectors \( \mathbf{u}, \mathbf{v}, \mathbf{w} \) in space. (c) If \( \mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w} \) and \( \mathbf{u} \neq 0 \) then \( \mathbf{v} = \mathbf{w} \).

3. (a) Check that the two lines with the parametric equations \( \mathbf{r}_1(t) = \langle 0, 2, -3 \rangle + t\langle 1, 0, -1 \rangle \) and \( \mathbf{r}_2(s) = \langle -2, 3, -6 \rangle + s\langle 3, -1, 2 \rangle \) intersect at the point \( \mathbf{P} = (1, 2, -4) \). (b) What is the acute angle between the two lines? You can leave your answer in terms of \( \arccos \).

4. (a) Find a nonzero vector perpendicular to the vectors \( \mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k} \) and \( \mathbf{v} = 2\mathbf{j} - 3\mathbf{k} \). (b) Find the area of the parallelogram spanned by \( \mathbf{u} \) and \( \mathbf{v} \).

5. Find the equation of the plane passing through the point \( (0, 1, -2) \) and containing the line \( \mathbf{r}(t) = \langle 0, 2, -3 \rangle + t\langle 1, 0, -1 \rangle \).

6. Consider the parametric line \( \mathbf{r}(t) = \langle -2, 3, -6 \rangle + t\langle 3, -1, -2 \rangle \) and the plane given by the equation \( 2x - y + 3z = 5 \). (a) Show that the line intersects the plane at precisely one point and find its coordinates. (b) Find the acute angle that the line makes with a vector normal to the plane.

7. Determine if there exists a plane that contains all four points \( P(3, 1, 2) \), \( Q(6, -1, 6) \), \( R(1, 4, 7) \), and \( S(2, 1, 5) \).

8. Find symmetric equations for the line of intersection of the planes \( x + y - z = 2 \) and \( 3x - 4y + 5z = 6 \).

9. The velocity of a particle moving in space is given by
   \[
   \frac{d\mathbf{r}}{dt} = -\sin t \cdot \mathbf{i} + \cos t \cdot \mathbf{j} + 3\mathbf{k}.
   \]
   (a) Find the position vector \( \mathbf{r}(t) \) of the particle if \( \mathbf{r}(0) = \mathbf{i} + 3\mathbf{k} \). (b) Find the unit tangent vector to the trajectory of the particle when \( t = \pi \). (c) Find particle’s acceleration when \( t = \pi \). (d) Find the distance traveled by the particle along the curve \( \mathbf{r}(t) \) from \( t = 0 \) to \( t = \pi \).
10. The surface with equation \( z + \cos(xy) = x^2 + y^2 \) can be described both as a level surface of a function \( f \) and as the graph of a function \( g \). Give explicit formulas for \( f \) and \( g \).
11. (a) Let \( f(x, y, z) = \sin(xy + z) \). Find all first and second order partial derivatives of \( f \).
(b) Evaluate \( \partial^{100}_{x^{95} y^2 z^3} (ye^x \cos(x)) \).
12. Find the equation of the tangent plane to the graph \( z = -x^2 + 4y^2 + 1 \) at the point \((2, 1, 1)\).
13. Let \( f(x, y) \) be a differentiable function such that \( f(1, 1) = 3 \), \( f_x(1, 1) = 2 \), and \( f_y(1, 1) = -1 \). From the available information, what is the best estimate you can give of \( f(1.1, 0.9) \)?
14. The radius of a right circular cone is measured at 120 in with a possible error of 1.8 in, while its height measured at 140 in with a possible error of 2.5 in. Estimate the maximal possible error if these measurements are used to compute the volume of this cone.

**SOLUTIONS**

1. (a) \( 2\sqrt{2} \); (b) \( \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle \); (c) 6; (e) \( \pi/6 \); (f) \( \langle 4, 1, 3 \rangle \);
(g) \( \frac{3}{\sqrt{2}} \); (h) \( \langle 1.5, 0, 1.5 \rangle + \langle 0.5, 1, -0.5 \rangle \); (i) \( \langle -2, 2, 2 \rangle \).
2. All three are false. 3. (b) \( \arccos \left( \frac{1}{2\sqrt{7}} \right) \). 4. (a) \( \langle 1, 3, 2 \rangle \); (b) \( \sqrt{14} \).
5. \( x + y + z + 1 = 0 \). 6. (a) \( P(88, -27, -66) \); (b) \( \arccos \left( \frac{1}{14} \right) \).
7. There is no such plane. 8. \( x - 2 = y/(-8) = z/(-7) \).
9. (a) \( \langle \cos t, \sin t, 3 + 3t \rangle \); (b) \( \mathbf{T} = (-\mathbf{j} + 3\mathbf{k})/\sqrt{10} \); (c) \( \mathbf{r''} = \mathbf{i} \); (d) \( \sqrt{10} \pi \).
10. \( f(x, y, z) = x^2 + y^2 - \cos(xy) - z, \quad g(x, y) = x^2 + y^2 - \cos(xy) \).
11. (a) \( f_x = y \cos(xy + z), \quad f_y = x \cos(xy + z), \quad f_z = \cos(xy + z) \); (b) 0.
12. \( z = 1 - 4x + 8y \). 13. \( f(1.1, 0.9) \approx f(1, 1) + 2 \cdot 0.1 + (-1) \cdot (-0.1) = 3.3 \).
14. \( V(r, h) = \frac{1}{3} \pi r^2 h, \quad dV = V_r(120, 140) \, dr + V_h(120, 140) \, dh = 32160 \pi \).