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ON THE ZEROS OF TRANSCENDENTAL FUNCTIONS WITH APPLICATIONS TO STABILITY OF DELAY DIFFERENTIAL EQUATIONS WITH TWO DELAYS

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Abstract. In this paper, we first establish a basic theorem on the zeros of general transcendental functions. Based on the basic theorem, we develop a decomposition technique to investigate the stability of some exponential polynomials, that is, to find conditions under which all zeros of the exponential polynomials have negative real parts. The technique combines the D-decomposition and τ-decomposition methods so that it can be used to study differential equations with multiple delays. As an application, we study the stability and bifurcation of a scalar equation with two delays modeling compound optical resonators.

Keywords. Transcendental polynomials, delay differential equations, stability, bifurcation, compound optical resonators.

AMS (MOS) subject classification: 30C15, 34K20.

1 Introduction

For an ordinary differential equation, the trivial solution is asymptotically stable if and only if all roots of the corresponding characteristic equation of the linearized equation have negative real parts. Since the characteristic function is a polynomial, the well-known Routh-Hurwitz criterion can be used to determine the negativity of the real parts of the characteristic roots. Similar equivalence holds for delay differential equations, especially the discrete delay differential equations. However, the characteristic functions corresponding to

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the linearized delay differential equations are no longer ordinary polynomials, rather, they are *exponential polynomials* or *quasi-polynomials* as named in Bellman and Cooke [5].

Consider the delay differential equation

$$\sum_{j=0}^{m}\sum_{k=0}^{n} a_{jk}x^{(j)}(t-\tau_k)=0, \tag{1.1}$$

where $0=\tau_0<\tau_1<\tau_2<\cdots<\tau_n$, the associated characteristic exponential polynomial is a transcendental function of the form

$$p(\lambda, e^{-\lambda\tau_1},\cdots,e^{-\lambda\tau_n})=\sum_{j=0}^{m}\sum_{k=0}^{n} a_{jk}\lambda^j e^{-\lambda\tau_k}. \tag{1.2}$$

We introduce some notations (see Bellman and Cooke [5] and Hale and Verduyn Lunel [21]). Equation (1.1) is called a *retarded* or *delay* differential equation if the highest derivative term does not have a delay, i.e., if $a_{m0}\neq 0$, and $a_{mk}=0$ for $k=1,2,...,n$, and a *neutral* differential equation if at least one of of the highest derivative terms has delay, i.e., if $a_{mk}\neq 0$ for some $k=1,2,...,n$. For a neutral differential equation, if $a_{mn}\neq 0$, then $a_{mn}\lambda^m e^{-\lambda\tau_n}$ is called a *principal term.* It is known that if (1.1) is a retarded differential equation and all roots of the exponential polynomial (1.2) have negative real parts, then the zero solution of the delay equation (1.1) is asymptotically stable. Occasionally, we also say that the exponential polynomial is *stable* if all its roots have negative real parts.

Special cases of the exponential polynomial (1.2) have been studied by many researchers. For example, Hayes [23] and Wright [47] investigated the first degree exponential polynomials; Bellman and Cooke [5], Baptistini and Tátoas [2], Boese [8], Chuma and van den Driessche [11], etc. studied the second degree exponential polynomials; Ruan [41] studied the distribution of zeros of some second order exponential polynomials and used the results to investigate the stability and bifurcations in Kolmogorov types of predator-prey systems, see also Martin and Ruan [33]; Ruan and Wei [42] analyzed a third degree exponential polynomial and applied the obtained results to a delayed model for the control of testosterone secretion; Cooke and van den Driessche [13], Brauer [10], Beretta and Kuang [6], etc. considered the higher degree exponential polynomials without the principal term; Freedman and Kuang [17] studied the case with the principal term; see also Avellar and Hale [1], Cooke [12], Hale and Verduyn Lunel [21], Kolmanovskii and Nosov [25], Walther [44], Wei and Ruan [45, 46], and the references cited therein.

For the general exponential polynomial (1.2), the first and most fundamental criterion was due to Pontryagin [38] who studied the special case of (1.2) when $-\tau_k=k, k=0,1,2,\cdots,n$, namely,

$$p(\lambda, e^{\lambda})=\sum_{j=0}^{m}\sum_{k=0}^{n} a_{jk}\lambda^j e^{k\lambda}. \tag{1.3}$$

Pontryagin gave necessary and sufficient conditions for the stability of (1.3). Pontryagin criterion is widely used in engineering and control theory. However, it has strong limitations and becomes very complicated for equations with multiple delays. Other methods which have been developed to study the stability of the exponential polynomials include the *Nyquist criterion* (see Krall [26]), the *Michailov criterion* (see Krall [27]), the *direct stability investigation* method (see Stépán [43]), to name a few. We refer to Stépán [43] for a brief introduction of these methods.

For a fixed delay τ, the zeros of the exponential polynomial associated with a delay differential equation are continuous functions of its coefficients. Divide the coefficient space into different regions by means of hypersurfaces, the points of the hypersurfaces correspond to exponential polynomials with at least one zero on the imaginary axis. Such a partition is called a *D-decomposition*, proposed by Neĭmark [35]. The points of each region of a D-decomposition correspond to exponential polynomials with the same number of zeros with positive real parts (by number of zeros it means the sum of their multiplicities). To every region D_k of the D-decomposition, it is possible to assign a number k which is the number of zeros with positive real parts of the exponential polynomial. Suppose that the region D_0 (if exists) is also found, corresponding to the exponential polynomial which does not have any root with positive real part. Then this region is the stability region for the solution of the delay differential equation. For a more detailed description and applications of the D-decomposition method, we refer to the monographs of El'sgol'ts [16] and Pinney [37].

A variant of the D-decomposition method is the τ*-decomposition* method (see Popov [39], Lee and Hsu [29], and Huang [24]) which involves first decomposing the delay τ-axis into intervals such that within each interval the same stability character prevails, and then studying the change of stability character of the system as the boundary points of the intervals are crossed. Both the D-decomposition and the τ-decomposition methods are very useful in applications. However, they cannot be used to study equations with multiple delays.

In this paper, we first prove a basic theorem on the zeros of some general transcendental functions and then use the basic theorem (and its corollaries) to investigate the stability of some exponential polynomials by choosing either one of the coefficients or one of the delays as a parameter. The technique can be regarded as a combination of the D-decomposition and the τ-decomposition methods so that it can be used to study equations with multiple delays. As an application, we consider a first order differential equation with two delays. By choosing one of the coefficients as a parameter, we discuss the stability of the steady state solution and the Hopf bifurcation of the equation. The obtained results can be applied to analyze the two-delay equations of Mizuno and Ikeda [34] and Marriot, Vallée and Delisle [32] modeling compound optical resonators.

2 Basic Theorems

In this section, we state and prove some basic results on zeros of some transcendental functions.

Theorem 2.1 *Suppose that $B \subset \mathbf{R}^n$ is an open connected set, $h(\lambda,\mu)$ is continuous in $(\lambda,\mu) \in \mathbf{C} \times B$ and analytic in $\lambda \in \mathbf{C}$, and the zeros of $h(\lambda,\mu)$ in the right half plane*

$$\{\lambda \in \mathbf{C} : \mathrm{Re}\lambda \geq 0\}$$

are uniformly bounded. If for any $\mu \in B_1 \subset B$, where B_1 is a bounded, closed, and connected set, $h(\lambda,\mu)$ has no zeros on the imaginary axis, then the sum of the orders of the zeros of $h(\lambda,\mu)$ in the open right half plane (Re $\lambda > 0$) is a fixed number for B_1, that is, it is independent of the parameter $\mu \in B_1$.

Proof. Since the zeros of the function $h(\lambda,\mu)$ in the right half plane are uniformly bounded, there exists a constant $r > 0$ such that for any zero λ of $h(\lambda,\mu)$ with $\mathrm{Re}\lambda > 0$, we have $|\lambda| < r$. Let

$$A = \{\lambda \in \mathbf{C} : \mathrm{Re}\lambda \geq 0,\ |\lambda| < \mathrm{r}\}.$$

If λ is a zero of $h(\lambda,\mu)$ satisfying $\mathrm{Re}\lambda > 0$, then $\lambda \in \mathring{A}$ (the interior of A). The assumptions imply that for any $\mu \in B,\ h(\lambda,\mu)$ has no zeros on

$$\partial A = \{\lambda \in A : \mathrm{Re}\lambda = 0 \ \text{ or } \ |\lambda| = \mathrm{r}\},$$

the boundary of A. By Rouché's Theorem (Dieudonné [15, Theorem 9.17.4]), for any $\mu_0 \in B_1$, there exists an open neighborhood of $\mu_0,\ W(\mu_0) \subset B$, such that for any $\mu \in W(\mu_0)$, the sum of the orders of the zeros of $h(\lambda,\mu)$ belonging to InA is independent of μ. Clearly,

$$\cup_{\mu_0 \in B_1} W(\mu_0)$$

is an open covering of B_1. Since B_1 is compact, there is a finite integer $N \geq 1$ such that

$$\cup_{i=1}^N W(\mu_0^{(i)}) \supset B_1.$$

Since for any $\mu \in W(\mu_0)$, the sum of the orders of the zeros of $h(\lambda,\mu)$ belonging to InA is independent of μ, and for any $W(\mu_0^{(i)}) \in \{W(\mu_0^{(i)})\}_{1\leq i \leq N}$, there exists at least another $W(\mu_0^{(j)}) \in \{W(\mu_0^{(i)})\}_{1\leq i\leq N}$ which intersects with $W(\mu_0^{(i)})$ because of the connectness of $\cup_{i=1}^N W(\mu_0^{(i)})$, we know that the sum of the orders of the zeros of $h(\lambda,\mu)$ belonging to InA is independent of $\mu \in B_1$. This completes the proof.

Remark 2.2 Theorem 2.1 can be applied to study the distribution of zeros of a transcendental function either by using a delay as the bifurcation parameter or by dividing the coefficient space into different regions with purely imaginary zeros on the boundaries. In other words, both the $\tau-$decomposition and

$D-$decomposition methods can be derived from Theorem 2.1. Thus, Theorem 2.1 can be regarded as a combination of the $\tau-$ and $D-$decomposition methods.

Corollary 2.3 *Under the assumptions of Theorem 2.1, as μ varies, the sum of the orders of the zeros of $h(\lambda,\mu)$ in the open right half plane can change only if a zero appears on or crosses the imaginary axis.*

Since we focus on retarded differential equations, we now consider the following exponential polynomial:

$$\begin{aligned} &P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}) \\ &\quad = \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &\quad\quad +[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\tau_1} + \cdots \\ &\quad\quad +[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\tau_m}, \end{aligned} \tag{2.1}$$

where $\tau_i \geq 0 (i = 1, 2, ..., m)$ and $p_j^{(i)} (i = 0, 1, \cdots, m; j = 1, 2, \cdots, n)$ are constants.

Corollary 2.4 *As $(\tau_1, \tau_2, \cdots, \tau_m)$ vary, the sum of the orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ in the open right half plane can change only if a zero appears on or crosses the imaginary axis.*

Proof. It suffices to prove that the zeros of $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ in the open right half plane are uniformly bounded. Without loss of generality, let λ be a zero of $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ satisfying $\mathrm{Re}\lambda \geq 0$ and $|\lambda| > 1$. Then we have

$$|\lambda| \leq \sum_{j=1}^{n}(\sum_{i=0}^{m}|p_j^{(i)}|).$$

Hence, $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ satisfies the assumptions in Theorem 2.1 and the conclusion follows.

Remark 2.5 Corollary 2.4 is a generalization of the Lemma in Cooke and Grossman [14] in which a second order degree exponential polynomial was studied. See also Kuang [28].

Remark 2.6 Theorem 2.1 can be used to discuss the zeros of $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ by regarding either the delays $(\tau_1, \tau_2, \cdots, \tau_m)$ or some coefficients as parameters.

3 A Scalar Equation with Two Delays

In the last two decades, great attention has been paid to equations with two delays which not only have significant biological and physical background but also exhibit very rich dynamics. We refer to Bélair and Campbell [4] and Beuter, Larocque and Glass [7] for a simple motor control equation with two delays; to Braddock and van den Driessche [9] and Gopalsamy [18] for a logistic model with two delays; and to Bélair [3], Hale and Huang [19], Hale and Tanaka [20], Hassard [22], Li, Ruan and Wei [30], Mahaffy, Zak and Joiner [31], Marriot, Vallée and Delisle [32], Nussbaum [36], Ragazzo and Malta [40] and the references therein for related studies on scalar equations with two delays.

In modeling a compound optical resonator with competing boundary conditions, Mizuno and Ikeda [34] proposed the following equation with two delays:

$$\gamma^{-1}\dot{\phi}(t) = -\phi(t) + \eta[\cos(\phi(t-\tau_1)-\phi_1) + \cos(\phi(t-\tau_2)-\phi_2)], \tag{3.1}$$

where ϕ denotes the phase shift of the electric field across the medium; γ^{-1} is the relaxation time of the nonlinear medium; η is the input power; ϕ_1, ϕ_2 are constants, τ_1 and $\tau_2 (\tau_1 > \tau_2)$ are positive parameters related to the time it takes the electromagnetic wave (the laser beam) to make a round trip between the two mirrors and between the mirror and the semitransparent mirror, respectively.

In this section, we shall apply the basic theorems obtained in section 2 to study the stability and Hopf bifurcation of the following differential equation with two delays:

$$\dot{x}(t) = -ax(t) + [f(x(t-\tau_1)) + f(x(t-\tau_2))], \tag{3.2}$$

where a, τ_1 and $\tau_2 (\tau_1 > \tau_2)$ are positive constants, and $f \in C^1(R)$. Suppose that there exists $x^* \geq 0$ such that $ax^* = 2f(x^*)$, that is, equation (3.2) has an equilibrium x^*. Clearly, equation (3.2) is the generalization of (3.1).

The linearized equation of (3.2) at the equilibrium $x = x^*$ is

$$\dot{x}(t) = -ax(t) - b[x(t-\tau_1) + x(t-\tau_2)], \tag{3.3}$$

where $b = -f'(x^*)$. The associated characteristic equation of (3.3) has the following form:

$$\lambda = -b[e^{-\lambda\tau_1} + e^{-\lambda\tau_2}] - a. \tag{3.4}$$

Notice that $\lambda = 0$ is a real root of equation (3.4) when $b = -a/2$. Meanwhile, we know that $i\omega$ $(\omega > 0)$ is a root of equation (3.4) if and only if ω satisfies

$$i\omega = -b[(\cos\omega\tau_1 + \cos\omega\tau_2) - i(\sin\omega\tau_1 + \sin\omega\tau_2)] - a.$$

Separating the real and imaginary parts, we have

$$\begin{aligned} b(\cos\omega\tau_1 + \cos\omega\tau_2) &= -a, \\ b(\sin\omega\tau_1 + \sin\omega\tau_2) &= \omega, \end{aligned} \tag{3.5}$$

which is equivalent to

$$\begin{aligned} 2b\cos\frac{\tau_1+\tau_2}{2}\omega\ \cos\frac{\tau_1-\tau_2}{2}\omega &= -a, \\ \omega - 2b\sin\frac{\tau_1+\tau_2}{2}\omega\ \cos\frac{\tau_1-\tau_2}{2}\omega &= 0. \end{aligned} \tag{3.6}$$

By (3.6), we have

$$\tan \frac{\tau_1 + \tau_2}{2}\omega = -\frac{\omega}{a}. \tag{3.7}$$

We know that equation (3.7) has a sequence of roots $\{\omega_j\}_{j\geq 1}$, where (see Fig. 3.1)

$$\omega_j \in \Big(\frac{(2j-1)\pi}{\tau_1+\tau_2}, \frac{(2j+1)\pi}{\tau_1+\tau_2}\Big).$$

Define

$$b_j = -\frac{a}{2\cos\frac{\tau_1+\tau_2}{2}\omega_j \cos\frac{\tau_1-\tau_2}{2}\omega_j}. \tag{3.8}$$

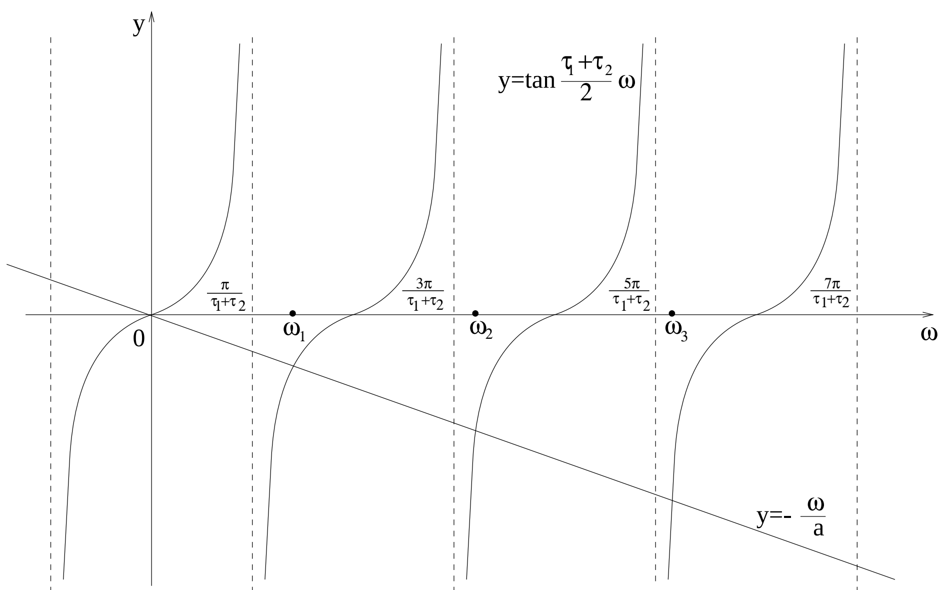

Figure 3.1: The points of intersection of $y_1 = -\frac{\omega}{a}$ and $y_2 = \tan\frac{\tau_1+\tau_2}{2}\omega$.

We have the following claim.

Claim. Equation (3.4) has purely imaginary roots if and only if $b = b_j$, and the purely imaginary roots are $\pm i\omega_j$, where b_j is defined by (3.8) and ω_j is a root of (3.7).

Lemma 3.1 *Denote* $\lambda(b) = \alpha(b) + i\omega(b)$ *the root of equation (3.4) satisfying* $\alpha(b_j) = 0,\ \omega(b_j) = \omega_j$. *Then*

$$\text{Sign } \alpha'(b_j) = \text{Sign } b_j.$$

Proof. From equation (3.4) we have

$$\frac{d\lambda}{db} + [e^{-\lambda\tau_1} + e^{-\lambda\tau_2}] + b[-\tau_1 e^{-\lambda\tau_1} - \tau_2 e^{-\lambda\tau_2}]\frac{d\lambda}{db} = 0.$$

Hence,

$$\begin{aligned}\frac{d\lambda}{db} &= -\frac{e^{-\lambda\tau_1} + e^{-\lambda\tau_2}}{1 - b(\tau_1 e^{-\lambda\tau_1} + \tau_2 e^{-\lambda\tau_2})} \\ &= \frac{\lambda + a}{b[1 - b(\tau_1 e^{-\lambda\tau_1} + \tau_2 e^{-\lambda\tau_2})]}.\end{aligned}$$

Substituting b_j into the above equation, we obtain

$$\begin{aligned}\frac{d\lambda(b_j)}{db} &= \frac{i\omega_j + a}{b_j[1 - b_j(\tau_1 e^{-i\omega_j\tau_1} + \tau_2 e^{-i\omega_j\tau_2})]} \\ &= \frac{i\omega_j + a}{b_j[1 - b_j(\tau_1\cos\omega_j\tau_1 + \tau_2\cos\omega_j\tau_2)] + ib_j^2[\tau_1\sin\omega_j\tau_1 + \tau_2\sin\omega_j\tau_2]}.\end{aligned}$$

It thus follows that

$$\begin{aligned}\alpha'(b_j) = \frac{1}{\Delta}\{ab_j[1 - b_j(\tau_1\cos\omega_j\tau_1 + \tau_2\cos\omega_j\tau_2)] \\ + b_j^2\omega_j[\tau_1\sin\omega_j\tau_1 + \tau_2\sin\omega_j\tau_2]\},\end{aligned} \tag{3.9}$$

where

$$\Delta = b_j^2[1 - b_j(\tau_1\cos\omega_j\tau_1 + \tau_2\cos\omega_j\tau_2)]^2 + b_j^4[\tau_1\sin\omega_j\tau_1 + \tau_2\sin\omega_j\tau_2]^2.$$

For convenience, set

$$h_1(\omega) = \sin\omega\tau_1 + \sin\omega\tau_2, \quad h_2(\omega) = \cos\omega\tau_1 + \cos\omega\tau_2.$$

Then we have

$$\frac{h_1(\omega)}{h_2(\omega)} = \tan\frac{\tau_1 + \tau_2}{2}\omega,$$

and hence

$$\frac{d}{d\omega}\Big(\frac{h_1(\omega)}{h_2(\omega)}\Big) > 0.$$

Meanwhile, we have

$$\frac{d}{d\omega}\Big(\frac{h_1(\omega)}{h_2(\omega)}\Big) = \frac{h_1'(\omega)h_2(\omega) - h_2'(\omega)h_1(\omega)}{h_2^2(\omega)}.$$

Thus, we must have

$$h_1'(\omega)h_2(\omega) - h_2'(\omega)h_1(\omega) > 0. \tag{3.10}$$

Since

$$h_1'(\omega) = \tau_1 \cos \omega\tau_1 + \tau_2 \cos \omega\tau_2,$$
$$h_2'(\omega) = -(\tau_1 \sin \omega\tau_1 + \tau_2 \sin \omega\tau_2),$$

by (3.5) and (3.9), we have

$$\alpha'(b_j) = \frac{b_j}{\Delta}[a - ab_j h_1'(\omega_j) - b_j\omega_j h_2'(\omega_j)]$$
$$= \frac{b_j}{\Delta}[a + b_j^2(h_1'(\omega_j)h_2(\omega_j) - h_2'(\omega_j)h_1(\omega_j))].$$

The lemma then follows from (3.10) and the fact that $a > 0$.

Now we can state and prove the following result on the distribution of roots of the characteristic equation (3.4).

Theorem 3.2 *Define*

$$b_0^+ = \min_{j\geq 1}\{b_j : b_j > 0\}, \quad b_0^- = \max_{j\geq 1}\{-\frac{a}{2}, b_j : b_j < 0\}. \tag{3.11}$$

Then all roots of equation (3.4) have negative real parts if and only if $b \in (b_0^-, b_0^+)$. *If* $\{b_j : b_j > 0\}_{j\geq 1} = \emptyset$, *then the conclusion holds for* $b \in (b_0^-, \infty)$.

Proof. Obviously, either $\{b_j : b_j > 0\}_{j\geq 1} = \emptyset$ or $\{b_j : b_j > 0\}_{j\geq 1} \neq \emptyset$, meanwhile $\{-\frac{a}{2}, b_j : b_j < 0\}_{j\geq 1} \neq \emptyset$. So b_0^- and b_0^+ are well-defined.

When $b = 0$, we know that equation (3.4) has only one root $\lambda = -a < 0$. When $b_0^+ < \infty$, b_0^+ is the first value of $b > 0$ so that equation (3.4) has roots on the imaginary axis. By Corollary 2.3, all roots of equation (3.4) have negative real parts for $b \in [0, b_0^+)$. When $b_0^+ = \infty$, this means that $b_j < 0$ for all $j \geq 1$. Once again, Corollary 2.3 implies that all roots of equation (3.4) have negative real parts for $b \in [0, \infty)$. A similar argument applies for $b \in (b_0^-, 0]$.

Denote by $\lambda(b)$ the root of equation (3.4) satisfying $\lambda(-\frac{a}{2}) = 0$. Then we have

$$\lambda'(-\frac{a}{2}) = -\frac{4}{2 + a(\tau_1 + \tau_2)} < 0.$$

By Lemma 3.1, we know that for $b < b_0^-$ and $b > b_0^+$, equation (3.4) has at least one root with positive real part. This completes the proof of the theorem.

By Theorem 3.2 and the Hopf Bifurcation Theorem (see Hale and Verduyn Lunel [21]), we have the following result on the stability and bifurcation of the equilibrium $x = x^*$ of equation (3.2).

Theorem 3.3 *Let* b_j *be defined by (3.8) and* b_0^- *and* b_0^+ *be defined by (3.11).*

(i) The equilibrium $x = x^*$ *of equation (3.2) is asymptotically stable if and only if* $b \in (b_0^-, b_0^+)$. *If* $\{b_j : b_j > 0\}_{j\geq 1} = \emptyset$, *then the conclusion holds for* $b \in (b_0^-, \infty)$.

(iii) Equation (3.2) undergoes Hopf bifurcations at the equilibrium $x = x^*$ *when* $b = b_j (j = 1, 2, \cdots)$.

Applying Theorem 3.3 to the compound optical resonator equation (3.1), one can easily derive stability and bifurcation conditions for the steady state solution.

Finally, we would like to mention that the basic theorem in section 2 can be used to study higher order delay differential equations and neutral delay differential equations, we leave this for future consideration.

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