

# ALGEBRAIC DENSITY PROPERTY OF HOMOGENEOUS SPACES

F. DONZELLI

A. DVORSKY

Institute of Mathematical Sciences  
Stony Brook University  
Stony Brook, NY 11794, USA  
fabrizio@math.sunysb.edu

Department of Mathematics  
University of Miami  
Coral Gables, FL 33124, USA  
a.dvorsky@math.miami.edu

S. KALIMAN

Department of Mathematics  
University of Miami  
Coral Gables, FL 33124, USA  
kaliman@math.miami.edu

**Abstract.** Let  $X$  be an affine algebraic variety with a transitive action of the algebraic automorphism group. Suppose that  $X$  is equipped with several fixed point free nondegenerate  $SL_2$ -actions satisfying some mild additional assumption. Then we prove that the Lie algebra generated by completely integrable algebraic vector fields on  $X$  coincides with the space of all algebraic vector fields. In particular, we show that apart from a few exceptions this fact is true for any homogeneous space of form  $G/R$  where  $G$  is a linear algebraic group and  $R$  is a closed proper reductive subgroup of  $G$ .

## 1. Introduction

In this paper we develop further methods introduced by Kutzschebauch and the third author in [KK2] which they used to obtain new results in the Andersén–Lempert theory [A], [AL]. The following notion crucial for this theory was introduced first by Varolin [V1] though its importance for Euclidean spaces was emphasized already in the earlier paper of Rosay [Ro].

**Definition 1.** A complex manifold  $X$  has the density property if in the compact-open topology the Lie algebra  $\text{Lie}_{\text{hol}}(X)$  generated by completely integrable holomorphic vector fields on  $X$  is dense in the Lie algebra  $\text{VF}_{\text{hol}}(X)$  of all holomorphic vector fields on  $X$ . An affine algebraic manifold  $X$  has the algebraic density property if the Lie algebra  $\text{Lie}_{\text{alg}}(X)$  generated by completely integrable algebraic vector fields on it coincides with the Lie algebra  $\text{VF}_{\text{alg}}(X)$  of all algebraic vector fields on it (clearly, the algebraic density property for such an  $X$  implies the density property for it).

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For any complex manifold with the density property the Andersén–Lempert theory is applicable and its effectiveness in complex analysis was demonstrated in several papers (e.g., see [FR], [Ro], [V1], [V2]). However, until recently, the class of manifolds for which this property was established was quite narrow (mostly Euclidean spaces and semisimple Lie groups, and homogeneous spaces of semisimple groups with trivial centers [TV1], [TV2]). In [KK1] and [KK2] this class was shown to contain hypersurfaces of the form  $uv = p(\bar{x})$  and connected complex algebraic groups except for  $\mathbb{C}_+$  and tori  $(\mathbb{C}^*)^n$ . Furthermore, it was proven in [KK1], [KK2] that these varieties have the algebraic density property. For  $\mathbb{C}_+$  and  $(\mathbb{C}^*)^n$  the algebraic density property is not true (for tori of dimension at least 2 this fact is nontrivial and it follows from the result of Andersén [A2] who established that for each completely integrable algebraic vector field on any torus its flow preserves the invariant volume form).

In this paper we study a smooth complex affine algebraic variety  $X$  with a transitive action of the algebraic automorphism group  $\text{Aut } X$ . This is a mild restriction because for an affine algebraic (resp. Stein) manifold with the algebraic density property (resp. density property) the group of holomorphic automorphism generated by elements of flows induced by completely integrable algebraic (resp. holomorphic) vector fields is transitive and even  $m$ -transitive for any natural  $m$  (this is a consequence of, say, Theorem 0.2 from [V2]). Though the facts we prove about such objects are rather logical extension of [KK2], in combination with Lie group theory they lead to a much wider class of homogeneous spaces with the algebraic density property. Our new technique yields, in particular, to the following.

**Theorem A.** *Let  $G$  be a linear algebraic group and let  $R$  be a closed proper reductive subgroup of  $G$  such that the homogeneous space  $G/R$  has connected components different from  $\mathbb{C}_+$  or a torus  $(\mathbb{C}^*)^n$ . Then  $G/R$  has the algebraic density property.*

Besides the criteria developed in [KK2] the main new ingredient of the proof is the Luna slice theorem. For the convenience of the readers we recall it in Section 2 together with basic facts about algebraic quotients and some crucial results from [KK2]. In Section 3 we prove our main theorem. As an application we prove Theorem A in Section 4 using some technical facts from the Lie group theory presented in the Appendix.

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## 2. Preliminaries

Let us fix some notation first. In this paper  $X$  will always be a complex affine algebraic variety and  $G$  will be an algebraic group acting on  $X$ , i.e.,  $X$  is a  $G$ -variety. The ring of regular functions on  $X$  will be denoted by  $\mathbb{C}[X]$  and its subring of  $G$ -invariant functions by  $\mathbb{C}[X]^G$ .

### 2.1. Algebraic (categorical) quotients

Recall that the algebraic quotient  $X//G$  of  $X$  with respect to the  $G$ -action is  $\text{Spec}(\mathbb{C}[X]^G)$ . By  $\pi : X \rightarrow X//G$  we denote the natural quotient morphism gener-

ated by the embedding  $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$ . The main (universal) property of algebraic quotients is that any morphism from  $X$  constant on orbits of  $G$  factors through  $\pi$ . In the case of a reductive  $G$  several important facts (e.g., see [Sch], [PV], [D], [G]) are collected in the following.

**Proposition 1.** *Let  $G$  be a reductive group.*

- (1) *The quotient  $X//G$  is an affine algebraic variety which is normal in the case of a normal  $X$  and the quotient morphism  $\pi : X \rightarrow X//G$  is surjective.*
- (2) *The closure of every  $G$ -orbit contains a unique closed orbit and each fiber  $\pi^{-1}(y)$  (where  $y \in X//G$ ) also contains a unique closed orbit  $O$ . Furthermore,  $\pi^{-1}(y)$  is the union of all those orbits whose closures contain  $O$ .*
- (3) *In particular, if every orbit of the  $G$ -action on  $X$  is closed then  $X//G$  is isomorphic to the orbit space  $X/G$ .*
- (4) *The image of a closed  $G$ -invariant subset under  $\pi$  is closed.*

If  $X$  is a complex algebraic group, and  $G$  is a closed subgroup acting on  $X$  by multiplication, then all orbits of the action are obviously closed. If  $G$  is reductive, the previous proposition implies that the quotient  $X/G$  is affine. The next proposition (Matsushima's criterion) shows that the converse is also true for quotients of reductive groups.

**Proposition 2.** *Let  $G$  be a complex reductive group, and let  $H$  be a closed subgroup of  $G$ . Then the quotient space  $G/H$  is affine if and only if  $H$  is reductive.*

Besides reductive group actions in this paper, a crucial role will be played by  $\mathbb{C}_+$ -actions. In general, algebraic quotients in this case are not affine but only quasi-affine [W]. However, we shall later use the fact that for the natural action of any  $\mathbb{C}_+$ -subgroup of  $SL_2$  generated by multiplication one has  $SL_2//\mathbb{C}_+ \cong \mathbb{C}^2$ .

## 2.2. Luna's slice theorem (e.g., see [D], [PV])

Let us recall some terminology first. Suppose that  $f : X \rightarrow Y$  is a  $G$ -equivariant morphism of affine algebraic  $G$ -varieties  $X$  and  $Y$ . Then the induced morphism  $f_G : X//G \rightarrow Y//G$  is well defined and the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X//G & \xrightarrow{f_G} & Y//G. \end{array} \quad (1)$$

**Definition 2.** A  $G$ -equivariant morphism  $f$  is called strongly étale if

- (1) the induced morphism  $f_G : X//G \rightarrow Y//G$  is étale; and
- (2) the quotient morphism  $\pi_G : X \rightarrow X//G$  induces a  $G$ -isomorphism between  $X$  and the fibered product  $Y \times_{Y//G} (X//G)$ .

From the properties of étale maps [D] it follows that  $f$  is étale (in particular, quasi-finite).

Let  $H$  be an algebraic subgroup of  $G$ , and let  $Z$  be an affine  $H$ -variety. We denote  $G \times_H Z$  the quotient of  $G \times Z$  with respect to the action of  $H$  given

by  $h(g, z) = (gh^{-1}, hz)$ . The left multiplication on  $G$  generates a left action on  $G \times_H Z$ . The next lemma is an obvious consequence of Proposition 1.

**Lemma 3.** *Let  $X$  be an affine  $G$ -variety and  $G$  be reductive. Then the  $H$ -orbits of  $G \times X$  are all isomorphic to  $H$ . Therefore, the fibers of the quotient morphism  $G \times X \rightarrow G \times_H X$  coincide with the  $H$ -orbits.*

The isotropy group of a point  $x \in X$  will be denoted by  $G_x$ . Recall also that an open set  $U$  of  $X$  is called saturated if  $\pi_G^{-1}(\pi_G(U)) = U$ . We are ready to state the Luna slice theorem.

**Theorem 4.** *Let  $G$  be a reductive group acting on an affine algebraic variety  $X$ , and let  $x \in X$  be a point in a closed  $G$ -orbit. Then there exists a locally closed affine algebraic subvariety  $V$  (called a slice) of  $X$  containing  $x$  such that:*

- (1)  $V$  is  $G_x$ -invariant;
- (2) the image of the  $G$ -morphism  $\varphi : G \times_{G_x} V \rightarrow X$  induced by the action is a saturated open set  $U$  of  $X$ ; and
- (3) the restriction  $\varphi : G \times_{G_x} V \rightarrow U$  is strongly étale.

Given a saturated open set  $U$ , we will denote  $\pi_G(U)$  by  $U//G$ . It follows from Proposition 1 that  $U//G$  is open. Theorem 4 implies that the following diagram is commutative:

$$\begin{array}{ccc} G \times_{G_x} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V//G_x & \longrightarrow & U//G \end{array} \tag{2}$$

and  $G \times_{G_x} V \simeq U \times_{U//G} V//G_x$ .

### 2.3. The compatibility criterion

This section presents the criteria for the algebraic density property, introduced in [KK2], that will be used to prove the main results of this paper.

**Definition 3.** Let  $X$  be an affine algebraic manifold. An algebraic vector field  $\sigma$  on  $X$  is semisimple if its flow is an algebraic  $\mathbb{C}^*$ -action on  $X$ . A vector field  $\delta$  is locally nilpotent if its flow is an algebraic  $\mathbb{C}_+$ -action on  $X$ . In the last case,  $\delta$  can be viewed as a locally nilpotent derivation on  $\mathbb{C}[X]$ . That is, for every nonzero  $f \in \mathbb{C}[X]$ , there is the smallest  $n = n(f)$  for which  $\delta^n(f) = 0$ . We set  $\deg_\delta(f) = n - 1$ . In particular, elements from the kernel  $\text{Ker } \delta$  have degree 0 with respect to  $\delta$ .

**Definition 4.** Let  $\delta_1$  and  $\delta_2$  be nontrivial algebraic vector fields on an affine algebraic manifold  $X$  such that  $\delta_1$  is a locally nilpotent derivation on  $\mathbb{C}[X]$ , and  $\delta_2$  is either also locally nilpotent or semisimple. That is,  $\delta_i$  generates an algebraic action of  $H_i$  on  $X$  where  $H_1 \simeq \mathbb{C}_+$  and  $H_2$  is either  $\mathbb{C}_+$  or  $\mathbb{C}^*$ . We say that  $\delta_1$  and  $\delta_2$  are semicompatible if the vector space  $\text{Span}(\text{Ker } \delta_1 \cdot \text{Ker } \delta_2)$ , generated by elements from  $\text{Ker } \delta_1 \cdot \text{Ker } \delta_2$ , contains a nonzero ideal in  $\mathbb{C}[X]$ .

A semicompatible pair is called compatible if in addition one of the following condition holds:

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- (1) when  $H_2 \simeq \mathbb{C}^*$  there is an element  $a \in \text{Ker } \delta_2$  such that  $\text{deg}_{\delta_1}(a) = 1$ , i.e.,  $\delta_1(a) \in \text{Ker } \delta_1 \setminus \{0\}$ ; and
- (2) when  $H_2 \simeq \mathbb{C}_+$  (i.e. both  $\delta_1$  and  $\delta_2$  are locally nilpotent) there is an element  $a$  such that  $\text{deg}_{\delta_1}(a) = 1$  and  $\text{deg}_{\delta_2}(a) \leq 1$ .

*Remark 1.* If  $[\delta_1, \delta_2] = 0$ , then conditions (1) and (2) with  $a \in \text{Ker } \delta_2$  hold automatically, i.e., any semicompatible pair of commutative algebraic vector fields is always compatible.

**Example 1.** Consider  $\text{SL}_2$  (or even  $\text{PSL}_2$ ) with two natural  $\mathbb{C}_+$ -subgroups: namely, the subgroup  $H_1$  (resp.  $H_2$ ) of the lower (resp. upper) triangular unipotent matrices. Denote by

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

an element of  $\text{SL}_2$ . Then the left multiplication generates actions of  $H_1$  and  $H_2$  on  $\text{SL}_2$  with the following associated locally nilpotent derivations on  $\mathbb{C}[\text{SL}_2]$ ,

$$\begin{aligned} \delta_1 &= a_1 \frac{\partial}{\partial b_1} + a_2 \frac{\partial}{\partial b_2}, \\ \delta_2 &= b_1 \frac{\partial}{\partial a_1} + b_2 \frac{\partial}{\partial a_2}. \end{aligned}$$

Clearly,  $\text{Ker } \delta_1$  is generated by  $a_1$  and  $a_2$  while  $\text{Ker } \delta_2$  is generated by  $b_1$  and  $b_2$ . Hence  $\delta_1$  and  $\delta_2$  are semicompatible. Furthermore, taking  $a = a_1 b_2$ , we see that condition (2) of Definition 4 holds, i.e., they are compatible.

It is worth mentioning the following geometrical reformulation of semicompatibility which will be needed further.

**Proposition 5.** *Suppose that  $H_1$  and  $H_2$  are as in Definition 4,  $X$  is a normal affine algebraic variety equipped with nontrivial algebraic  $H_i$ -actions where  $i = 1, 2$  (in particular, each  $H_i$  generates an algebraic vector field  $\delta_i$  on  $X$ ). Let  $X_i = X//H_i$  and let  $\rho_i : X \rightarrow X_i$  be the quotient morphisms. Set  $\rho = (\rho_1, \rho_2) : X \rightarrow Y := X_1 \times X_2$  and set  $Z$  equal to the closure of  $\rho(X)$  in  $Y$ . Then  $\delta_1$  and  $\delta_2$  are semicompatible iff  $\rho : X \rightarrow Z$  is a finite birational morphism.*

**Definition 5.** A finite subset  $M$  of the tangent space  $T_x X$  at a point  $x$  of a complex algebraic manifold  $X$  is called a generating set if the image of  $M$  under the action of the isotropy group (of algebraic automorphisms) of  $x$  generates  $T_x X$ .

It was shown in [KK2] that the existence of a pair of compatible derivations  $\delta_1$  and  $\delta_2$  from Definition 4 implies that  $\text{Lie}_{\text{alg}}(X)$  contains a  $\mathbb{C}[X]$ -submodule  $I\delta_2$  where  $I$  is a nontrivial ideal in  $\mathbb{C}[X]$ <sup>1</sup>. This yields the central criterion for the algebraic density property [KK2].

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<sup>1</sup>In the case of condition (2) in Definition 4 this fact was proven in [KK2] only for  $\text{deg}_{\delta_2}(a) = 0$  but the proof works for  $\text{deg}_{\delta_2}(a) = 1$  as well without any change.

**Theorem 6.** *Let  $X$  be a smooth homogeneous (with respect to  $\text{Aut} X$ ) affine algebraic manifold with finitely many pairs of compatible vector fields  $\{\delta_1^k, \delta_2^k\}_{k=1}^m$  such that for some point  $x_0 \in X$  vectors  $\{\delta_2^k(x_0)\}_{k=1}^m$  form a generating set. Then  $\text{Lie}_{\text{alg}}(X)$  contains a nontrivial  $\mathbb{C}[X]$ -module and  $X$  has the algebraic density property.*

As an application of this theorem we have the following.

**Proposition 7.** *Let  $X_1$  and  $X_2$  be smooth homogeneous (with respect to algebraic automorphism groups) affine algebraic varieties such that each  $X_i$  admits a finite number of completely integrable algebraic vector fields  $\{\delta_i^k\}_{k=1}^{m_i}$  whose values at some point  $x_i \in X_i$  form a generating set and, furthermore, in the case of  $X_1$  these vector fields are locally nilpotent. Then  $X_1 \times X_2$  has the algebraic density property.*

We shall also need two technical results (Lemmas 3.6 and 3.7 in [KK2]) that describe conditions under which quasi-finite morphisms preserve semicompatibility.

**Lemma 8.** *Let  $G = \text{SL}_2$  and let  $X, X'$  be normal affine algebraic varieties equipped with nondegenerate  $G$ -actions (i.e., general  $G$ -orbits are of dimension 3). Suppose that subgroups  $H_1$  and  $H_2$  of  $G$  are as in Example 1, i.e., they act naturally on  $X$  and  $X'$ . Let  $\rho_i : X \rightarrow X_i := X // H_i$  and  $\rho'_i : X' \rightarrow X'_i := X' // H_i$  be the quotient morphisms and let  $p : X \rightarrow X'$  be a finite  $G$ -equivariant morphism, i.e., we have commutative diagrams:*

$$\begin{array}{ccc} X & \xrightarrow{\rho_i} & X_i \\ p \downarrow & & \downarrow q_i \\ X' & \xrightarrow{\rho'_i} & X'_i \end{array}$$

for  $i = 1, 2$ . Treat  $\mathbb{C}[X_i]$  (resp.  $\mathbb{C}[X'_i]$ ) as a subalgebra of  $\mathbb{C}[X]$  (resp.  $\mathbb{C}[X']$ ). Let  $\text{Span}(\mathbb{C}[X_1] \cdot \mathbb{C}[X_2])$  contain a nonzero ideal of  $\mathbb{C}[X]$ . Then  $\text{Span}(\mathbb{C}[X'_1] \cdot \mathbb{C}[X'_2])$  contains a nonzero ideal of  $\mathbb{C}[X']$ .

The second result is presented here in a slightly different form but with a much simpler proof.

**Lemma 9.** *Let the assumption of Lemma 8 hold with two exceptions: we do not assume that  $G$ -actions are nondegenerate and instead of the finiteness of  $p$  we suppose that there are a surjective étale morphism  $r : M \rightarrow M'$  of normal affine algebraic varieties equipped with trivial  $G$ -actions and a surjective  $G$ -equivariant morphism  $\tau' : X' \rightarrow M'$  such that  $X$  is isomorphic to the fibered product  $X' \times_{M'} M$  with  $p : X \rightarrow X'$  being the natural projection (i.e.,  $p$  is surjective étale). Then the conclusion of Lemma 8 remains valid.*

*Proof.* By construction,  $X_i = X'_i \times_{M'} M$ . Thus we have the following commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\rho} & (X'_1 \times X'_2) \times_{(M' \times M')} (M \times M) & \longrightarrow & M \times M \\ p \downarrow & & \downarrow q & & \downarrow (r, r) \\ X' & \xrightarrow{\rho'} & X'_1 \times X'_2 & \xrightarrow{(\tau', \tau')} & M' \times M' \end{array}$$

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Set  $Z$  (resp.  $Z'$ ) equal to the closure of  $\rho(X)$  in  $X_1 \times X_2$  (resp.  $\rho'(X')$  in  $X'_1 \times X'_2$ ) and let  $D \simeq M$  (resp.  $D' \simeq M'$ ) be the diagonal subset in  $M \times M$  (resp.  $M' \times M'$ ). Since  $X = X' \times_{M'} M$  we see that  $Z = Z' \times_{D'} D$ . For any affine algebraic variety  $Y$  denote by  $Y_{\text{norm}}$  its normalization, i.e.,  $Z_{\text{norm}} = Z'_{\text{norm}} \times_{D'} D$ . By Lemma 5,  $\rho : X \rightarrow Z_{\text{norm}}$  is an isomorphism. Since  $r$  is surjective it can happen only when  $\rho' : X' \rightarrow Z'$  is an isomorphism. Hence the desired conclusion follows from Proposition 5.

The last result from [KK2] that we need allows us to switch from local to global compatibility.

**Proposition 10.** *Let  $X$  be an  $\text{SL}_2$ -variety with associated locally nilpotent derivations  $\delta_1$  and  $\delta_2$ , let  $Y$  be a normal affine algebraic variety equipped with a trivial  $\text{SL}_2$ -action, and let  $r : X \rightarrow Y$  be a surjective  $\text{SL}_2$ -equivariant morphism. Suppose that for any  $y \in Y$  there exists an étale neighborhood  $g : W \rightarrow Y$  such that the vector fields induced by  $\delta_1$  and  $\delta_2$  on the fibered product  $X \times_Y W$  are semicompatible. Then  $\delta_1$  and  $\delta_2$  are semicompatible.*

### 3. Algebraic density property and $\text{SL}_2$ -actions

**Notation 1.** We suppose that  $H_1, H_2, \delta_1$ , and  $\delta_2$  are as in Example 1. Note that if  $\text{SL}_2$  acts algebraically on an affine algebraic variety  $X$  then we have automatically the  $\mathbb{C}_+$ -actions of  $H_1$  and  $H_2$  on  $X$  that generate locally nilpotent vector fields on  $X$  which, by abuse of notation, will be denoted by the same symbols  $\delta_1$  and  $\delta_2$ . If  $X$  admits several (say,  $N$ )  $\text{SL}_2$ -actions, we denote by  $\{\delta_1^k, \delta_2^k\}_{k=1}^N$  the corresponding collection of pairs of locally nilpotent derivations on  $\mathbb{C}[X]$ .

Recall also that an action of a Lie group of a manifold is nondegenerate if the dimension of general orbits is the same as the dimension of the group. Here is the first main result of this paper.

**Theorem 11.** *Let  $X$  be a smooth complex affine algebraic variety whose group of algebraic automorphisms is transitive. Suppose that  $X$  is equipped with  $N$  fixed point free nondegenerate actions of  $\text{SL}_2$ -groups  $\Gamma_1, \dots, \Gamma_N$ . Let  $\{\delta_1^k, \delta_2^k\}_{k=1}^N$  be the corresponding pairs of locally nilpotent vector fields. If  $\{\delta_2^k(x_0)\}_{k=1}^N \subset T_{x_0}X$  is a generating set at some point  $x_0 \in X$  then  $X$  has the algebraic density property.*

*Remark 2.* Note that we can choose any nilpotent element of the Lie algebra of  $\text{SL}_2$  as  $\delta_2$ . Since the space of nilpotent elements generates the whole Lie algebra we can reformulate Theorem 11 as follows: a smooth complex affine algebraic variety  $X$  with a transitive group of algebraic automorphisms has the algebraic density property provided it admits “sufficiently many” fixed point free nondegenerate  $\text{SL}_2$ -actions, where “sufficiently many” means that at some point  $x_0 \in X$  the tangent spaces of the corresponding  $\text{SL}_2$ -orbits through  $x_0$  generate the whole space  $T_{x_0}X$ .

By virtue of Theorem 6 the main result will be a consequence of the following.

**Theorem 12.** *Let  $X$  be a smooth complex affine algebraic variety equipped with a fixed point free nondegenerate  $\text{SL}_2$ -action that induces a pair of locally nilpotent vector fields  $\{\delta_1, \delta_2\}$ . Then these vector fields are compatible.*

In preparation for the proof of Theorem 12 we must establish a number of results. From now on, we assume that all  $\mathrm{SL}_2$ -actions under consideration are nondegenerate.

**Lemma 13.** *Let the assumption of Theorem 12 hold and let  $x \in X$  be a point contained in a closed  $\mathrm{SL}_2$ -orbit. Then the isotropy group of  $x$  is either finite, or isomorphic to the diagonal  $\mathbb{C}^*$ -subgroup of  $\mathrm{SL}_2$ , or to the normalizer of this  $\mathbb{C}^*$ -subgroup (which is the extension of  $\mathbb{C}^*$  by  $\mathbb{Z}_2$ ).*

*Proof.* By Matsushima's criterion (Proposition 2) the isotropy group must be reductive and it cannot be  $\mathrm{SL}_2$  itself since the action has no fixed points. The only two-dimensional reductive group is  $\mathbb{C}^* \times \mathbb{C}^*$  [FuHa] which is not contained in  $\mathrm{SL}_2$ . Thus besides finite subgroups we are left to consider the one-dimensional reductive subgroups that include  $\mathbb{C}^*$  (which can be considered to be the diagonal subgroup since all tori are conjugated) and its finite extensions. The normalizer of  $\mathbb{C}^*$  which is its extension by  $\mathbb{Z}_2$  generated by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is reductive. If we try to find an extension of  $\mathbb{C}^*$  by another finite subgroup that contains an element  $B$  not from the normalizer then  $\mathbb{C}^*$  and  $BC^*B^{-1}$  meet at the identity matrix. In particular, the reductive subgroup must be at least two-dimensional, and we have to disregard this case.  $\square$

**Proposition 14.** *Let  $X, \delta_1, \delta_2$  be as in Theorem 12. Then there exists a regular function  $g \in \mathbb{C}[X]$  such that  $\deg_{\delta_1}(g) = \deg_{\delta_2}(g) = 1$ .*

*Proof.* Let  $x \in X$  be a point of a closed  $\mathrm{SL}_2$ -orbit. Luna's slice theorem yields diagram (2) with  $G = \mathrm{SL}_2$  and  $G_x$  being one of the subgroups described in Lemma 13. That is, we have the natural morphism  $\varphi : \mathrm{SL}_2 \times V \rightarrow U$  that factors through the étale morphism  $\mathrm{SL}_2 \times_{G_x} V \rightarrow U$  where  $V$  is the slice at  $x$ . First, consider the case when  $G_x$  is finite. Then  $\varphi$  itself is étale. Furthermore, replacing  $V$  by its Zariski open subset and  $U$  by the corresponding Zariski open  $\mathrm{SL}_2$ -invariant subset one can suppose that  $\varphi$  is also finite. Set  $f = a_1 b_2$  where  $a_i, b_i$  are as in Example 1. Note that each  $\delta_i$  generates a natural locally nilpotent vector field  $\tilde{\delta}_i$  on  $\mathrm{SL}_2 \times V$  such that  $\mathbb{C}[V] \subset \mathrm{Ker} \tilde{\delta}_i$  and  $\varphi_*(\tilde{\delta}_i)$  coincides with the vector field induced by  $\delta_i$  on  $X$ . Treating  $f$  as an element of  $\mathbb{C}[\mathrm{SL}_2 \times V]$  we have  $\deg_{\tilde{\delta}_i}(f) = 1, i = 1, 2$ . For every  $h \in \mathbb{C}[\mathrm{SL}_2 \times V]$  we define a function  $\hat{h} \in \mathbb{C}[U]$  by  $\hat{h}(u) = \sum_{y \in \varphi^{-1}(u)} h(y)$ . One can check that if  $h \in \mathrm{Ker} \tilde{\delta}_i$  then  $\delta_i(\hat{h}) = 0$ . Hence  $\delta_i^2(\hat{f}) = 0$  but we also need  $\delta_i(\hat{f}) \neq 0$  which is not necessarily true. Thus, multiply  $f$  by  $\beta \in \mathbb{C}[V]$ . Since  $\beta \in \mathrm{Ker} \tilde{\delta}_i$  we have  $\delta_i(\widehat{\beta f})(u) = \sum_{y \in \varphi^{-1}(u)} \beta(\pi_V(y)) \tilde{\delta}_i(f)(y)$ . Note that  $\tilde{\delta}_i(f)(y_0)$  is not zero at a general  $y_0 \in \mathrm{SL}_2 \times V$  since  $\tilde{\delta}_i(f) \neq 0$ . By a standard application of the Nullstellensatz we can choose  $\beta$  with prescribed values at the finite set  $\varphi^{-1}(u_0)$  where  $u_0 = \varphi(y_0)$ . Hence we can assume that  $\delta_i(\widehat{\beta f})(u_0) \neq 0$ , i.e.,  $\deg_{\delta_i}(\widehat{\beta f}) = 1$ . There is still one problem:  $\widehat{\beta f}$  is regular on  $U$  but necessarily not on  $X$ . In order to fix it we set  $g = \alpha \widehat{\beta f}$  where  $\alpha$  is a lift of a nonzero function on  $X//G$  that



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vanishes with high multiplicity on  $(X//G) \setminus (U//G)$ . Since  $\alpha \in \text{Ker } \delta_i$  we still have  $\deg_{\delta_i}(g) = 1$  which concludes the proof in the case of a finite isotropy group.

For a one-dimensional isotropy group note that  $f$  is  $\mathbb{C}^*$ -invariant with respect to the action of the diagonal subgroup of  $\text{SL}_2$ . That is,  $f$  can be viewed as a function on  $\text{SL}_2 \times_{\mathbb{C}^*} V$ . Then we can replace morphism  $\varphi$  with morphism  $\psi : \text{SL}_2 \times_{\mathbb{C}^*} V \rightarrow U$  that factors through the étale morphism  $\text{SL}_2 \times_{G_x} V \rightarrow U$ . Now  $\psi$  is also étale and the rest of the argument remains the same.  $\square$

In order to finish the proof of Theorem 12 we need to show semicompatibility of the vector fields  $\delta_1$  and  $\delta_2$  on  $X$ . Let  $U$  be a saturated set as in diagram (2) with  $G = \text{SL}_2$ . Since  $U$  is  $\text{SL}_2$ -invariant it is  $H_i$ -invariant (where  $H_i$  is from Notation 1) and the restriction of  $\delta_i$  to  $U$  is a locally nilpotent vector field which we denote again by the same letter. Furthermore, the closure of any  $\text{SL}_2$ -orbit  $O$  contains a closed orbit, i.e.,  $O$  is contained in an open set like  $U$  and, therefore,  $X$  can be covered by a finite collections of such open sets. Thus Proposition 10 implies the following.

**Lemma 15.** *If for every  $U$  as before the locally nilpotent vector fields  $\delta_1$  and  $\delta_2$  are semicompatible on  $U$  then they are semicompatible on  $X$ .*

**Notation 2.** Suppose further that  $H_1$  and  $H_2$  act on  $\text{SL}_2 \times V$  by left multiplication on the first factor. The locally nilpotent vector fields associated with these actions of  $H_1$  and  $H_2$  are, obviously, semicompatible since they are compatible on  $\text{SL}_2$  (see Example 1). Consider the  $\text{SL}_2$ -equivariant morphism  $G \times V \rightarrow G \times_{G_x} V$  where  $V, G = \text{SL}_2$ , and  $G_x$  are as in diagram (2). By definition  $G \times_{G_x} V$  is the quotient of  $G \times V$  with respect to the  $G_x$ -action whose restriction to the first factor is the multiplication from the right. Hence  $H_i$ -action commutes with  $G_x$ -action and, therefore, one has the induced  $H_i$ -action on  $G \times_{G_x} V$ . Following the convention of Notation 1 we denote the associated locally nilpotent derivations on  $G \times_{G_x} V$  again by  $\delta_1$  and  $\delta_2$ . That is, the  $\text{SL}_2$ -equivariant étale morphism  $\varphi : G \times_{G_x} V \rightarrow U$  transforms the vector field  $\delta_i$  on  $G \times_{G_x} V$  into the vector field  $\delta_i$  on  $U$ .

From Lemma 9 and Luna's slice theorem we immediately have the following.

**Lemma 16.**

- (1) *If the locally nilpotent vector fields  $\delta_1$  and  $\delta_2$  are semicompatible on  $G \times_{G_x} V$  then they are semicompatible on  $U$ .*
- (2) *If the isotropy group  $G_x$  is finite,  $\delta_1$  and  $\delta_2$  are, indeed, semicompatible on  $G \times_{G_x} V$ .*

Now we have to tackle semicompatibility in the case of a one-dimensional isotropy subgroup  $G_x$  using Proposition 5 as a main tool. We start with the case of  $G_x = \mathbb{C}^*$ .

**Notation 3.** Consider the diagonal  $\mathbb{C}^*$ -subgroup of  $\text{SL}_2$ , i.e., elements of form

$$s_\lambda = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

The action of  $s_\lambda$  on  $v \in V$  will be denoted by  $\lambda.v$ . When we speak later about the  $\mathbb{C}^*$ -action on  $V$  we shall mean exactly this action. Set  $Y = \text{SL}_2 \times V, Y' = \text{SL}_2 \times_{\mathbb{C}^*} V$

$V$ ,  $Y_i = Y//H_i$ ,  $Y'_i = Y'//H_i$ . Denote by  $\rho_i : Y \rightarrow Y_i$  the quotient morphism of the  $H_i$ -action and use the similar notation for  $Y'$ ,  $Y'_i$ . Set  $\rho = (\rho_1, \rho_2) : Y \rightarrow Y_1 \times Y_2$  and  $\rho' = (\rho'_1, \rho'_2) : Y' \rightarrow Y'_1 \times Y'_2$ .

Note that  $Y_i \simeq \mathbb{C}^2 \times V$  since  $\mathrm{SL}_2//\mathbb{C}_+ \simeq \mathbb{C}^2$ . Furthermore, looking at the kernels of  $\delta_1$  and  $\delta_2$  from Example 1, we see for

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in \mathrm{SL}_2$$

the quotient maps  $\mathrm{SL}_2 \rightarrow \mathrm{SL}_2//H_1 \simeq \mathbb{C}^2$  and  $\mathrm{SL}_2 \rightarrow \mathrm{SL}_2//H_2 \simeq \mathbb{C}^2$  are given by  $A \mapsto (a_1, a_2)$  and  $A \mapsto (b_1, b_2)$ , respectively. Hence, the morphism  $\rho : \mathrm{SL}_2 \times V = Y \rightarrow Y_1 \times Y_2 \simeq \mathbb{C}^4 \times V \times V$  is given by

$$\rho(a_1, a_2, b_1, b_2, v) = (a_1, a_2, b_1, b_2, v, v). \quad (3)$$

As we mentioned before, to define  $Y' = \mathrm{SL}_2 \times_{\mathbb{C}^*} V$  we let  $\mathbb{C}^*$  act on  $\mathrm{SL}_2$  via right multiplication. Since  $H_1$  and  $H_2$  act on  $\mathrm{SL}_2$  from the left, there are well-defined  $\mathbb{C}^*$ -actions on  $Y_1$  and  $Y_2$  and a torus  $\mathbb{T}$ -action on  $Y_1 \times Y_2$ , where  $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$ . Namely,

$$(\lambda, \mu) \cdot (a_1, a_2, b_1, b_2, v, w) = (\lambda a_1, \lambda^{-1} a_2, \mu b_1, \mu^{-1} b_2, \lambda v, \mu w) \quad (4)$$

for  $(a_1, a_2, b_1, b_2, v, w) \in Y_1 \times Y_2$  and  $(\lambda, \mu) \in \mathbb{T}$ .

Since the  $\mathbb{C}^*$ -action on  $Y$  and the action of  $H_i$ ,  $i = 1, 2$ , are commutative, the following diagram is also commutative:

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & Y_1 \times Y_2 \\ p \downarrow & & q \downarrow \\ Y' & \xrightarrow{\rho'} & Y'_1 \times Y'_2, \end{array} \quad (5)$$

where  $q$  (resp.  $p$ ) is the quotient map with respect to the  $\mathbb{T}$ -action (resp.  $\mathbb{C}^*$ -action). It is also worth mentioning that the  $\mathbb{C}^*$ -action on  $Y$  induces the action of the diagonal of  $\mathbb{T}$  on  $\rho(Y)$ , i.e., for every  $y \in Y$  we have  $\rho(\lambda y) = (\lambda, \lambda) \cdot \rho(y)$ .

**Lemma 17.** *Let  $Z = \rho(Y)$  in diagram (5) and let  $Z'$  be the closure of  $\rho'(Y')$ .*

- (i) *The map  $\rho : Y \rightarrow Z$  is an isomorphism and  $Z$  is the closed subvariety of  $Y_1 \times Y_2 = \mathbb{C}^4 \times V \times V$  that consists of points  $(a_1, a_2, b_1, b_2, v, w) \in Y_1 \times Y_2$  satisfying the equations  $a_1 b_2 - a_2 b_1 = 1$  and  $v = w$ .*
- (ii) *Let  $T$  be the  $\mathbb{T}$ -orbit of  $Z$  in  $Y_1 \times Y_2$  and let  $\bar{T}$  be its closure. Then  $T$  coincides with the  $(\mathbb{C}^* \times 1)$ -orbit (resp.  $(1 \times \mathbb{C}^*)$ -orbit) of  $Z$ . Furthermore, for each  $(a_1, a_2, b_1, b_2, v, w) \in \bar{T}$ , one has  $\pi(v) = \pi(w)$  where  $\pi : V \rightarrow V//\mathbb{C}^*$  is the quotient morphism.*
- (iii) *The restriction of diagram (5) yields the following:*

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & Z \subset \bar{T} \\ p \downarrow & & q \downarrow \\ Y' & \xrightarrow{\rho'} & \rho'(Y') \subset Z', \end{array} \quad (6)$$

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where  $Y' = Y//\mathbb{C}^* = Y/\mathbb{C}^*$ ,  $q$  is the quotient morphism of the  $\mathbb{T}$ -action (i.e.  $Z' = \bar{T}/\mathbb{T}$ ), and  $q(Z) = \rho'(Y')$ .

*Proof.* The first statement is an immediate consequence of formula (3). The beginning of the second statement follows from the fact that the action of the diagonal  $\mathbb{C}^*$ -subgroup of  $\mathbb{T}$  preserves  $Z$ . This implies that for every  $t = (a_1, a_2, b_1, b_2, v, w) \in T$  the points  $v, w \in V$  belong to the same  $\mathbb{C}^*$ -orbit and, in particular,  $\pi(v) = \pi(w)$ . This equality holds for each point in  $\bar{T}$  by continuity.

In diagram (5),  $Y' = Y//\mathbb{C}^* = Y/\mathbb{C}^*$  because of Proposition 1(3) and Lemma 3, and the equality  $q(Z) = \rho'(Y')$  is the consequence of the commutativity of that diagram. Note that  $\bar{T}$  is  $\mathbb{T}$ -invariant. Hence  $q(\bar{T})$  coincides with  $Z'$  by Proposition 1(4). Being the restriction of the quotient morphism,  $q|_{\bar{T}} : \bar{T} \rightarrow Z'$  is a quotient morphism itself (e.g., see [D]) which concludes the proof.  $\square$

**Lemma 18.** *There is a rational  $\mathbb{T}$ -quasi-invariant function  $f$  on  $\bar{T}$  such that for  $t = (a_1, a_2, b_1, b_2, w, v) \in T$  one has:*

- (1)  $(1/f(t))a_1b_2 - f(t)a_2b_1 = 1$  and  $w = f(t).v$ ;
- (2) the set  $\bar{T} \setminus T$  is contained in  $(f)_0 \cup (f)_\infty$ ; and
- (3)  $f$  generates a regular function on a normalization  $T_N$  of  $T$

*Proof.* By Lemma 17(ii) any point  $t = (a_1, a_2, b_1, b_2, w, v) \in T$  is of the form  $t = (\lambda, 1).z_0$  where  $z_0 \in Z$  and  $\lambda \in \mathbb{C}^*$ . Hence formula (4) implies that  $w = \lambda.v$  and  $\lambda^{-1}a_1b_2 - \lambda a_2b_1 = 1$ . The last equality yields two possible values (one of which can be  $\infty$  or 0 if any of the numbers  $a_1, a_2, b_1$ , or  $b_2$  vanishes)

$$\lambda_{\pm} = \frac{-1 \pm \sqrt{1 + 4a_1a_2b_1b_2}}{2a_2b_1}$$

and we assume that

$$\lambda = \lambda_- = \frac{-1 - \sqrt{1 + 4a_1a_2b_1b_2}}{2a_2b_1},$$

i.e.,  $w = \lambda_-.v$ . Note that  $\lambda_+.v = w$  as well only when

$$\tau = \frac{\lambda_+}{\lambda_-} = \frac{-1 + \sqrt{1 + 4a_1a_2b_1b_2}}{-1 - \sqrt{1 + 4a_1a_2b_1b_2}}$$

is in the isotropy group of  $v$ .

Consider the set of points  $t \in T$  such that  $v$  is not a fixed point of the  $\mathbb{C}^*$ -action on  $V$  and  $\tau.v = v$ . Denote its closure by  $S$ . Since  $S$  is a proper subvariety of  $T$ , one has a well-defined branch  $\lambda_-$  of the two-valued function  $\lambda_{\pm}$  on the complement to  $S$ . Its extension to  $\bar{T}$ , which is denoted by  $f$ , satisfies (1).

Let  $t_n \in T$  and  $t_n \rightarrow t \in \bar{T}$  as  $n \rightarrow \infty$ . By Lemma 17(ii),  $t_n$  is of the form  $t_n = (f(t_n)a_1^n, (1/f(t_n))a_2^n, b_1^n, b_2^n, f(t_n).v_n, v_n)$  where

$$\begin{pmatrix} a_1^n & a_2^n \\ b_1^n & b_2^n \end{pmatrix} \in \mathrm{SL}_2 \quad \text{and} \quad v = \lim_{n \rightarrow \infty} v_n.$$

If sequences  $\{f(t_n)\}$  and  $\{1/f(t_n)\}$  are bounded, then switching to a subsequence one can suppose that  $f(t_n) \rightarrow f(t) \in \mathbb{C}^*$ ,  $w = f(t)v$ , and  $t = (f(t)a'_1, (1/f(t))a'_2, b'_1, b'_2, f(t).v, v)$  where

$$\begin{pmatrix} a'_1 & a'_2 \\ b'_1 & b'_2 \end{pmatrix} \in \mathrm{SL}_2,$$

i.e.,  $t \in T$ . Hence  $\bar{T} \setminus T$  is contained in  $((f)_0 \cup (f)_\infty)$  which is (2).

The function  $f$  is regular on  $T \setminus S$  by construction. Consider  $t \in S$  with  $w$  and  $v$  in the same nonconstant  $\mathbb{C}^*$ -orbit, i.e.,  $w = \lambda.v$  for some  $\lambda \in \mathbb{C}^*$ . Then  $w = \lambda'.v$  if and only if  $\lambda'$  belongs to the coset  $\Gamma$  of the isotropy subgroup of  $v$  in  $\mathbb{C}^*$ . For any sequence of points  $t_n$  convergent to  $t$  one can check that  $f(t_n) \rightarrow \lambda \in \Gamma$  by continuity, i.e.,  $f$  is bounded in a neighborhood of  $t$ . Let  $\nu : T_N \rightarrow T$  be a normalization morphism. Then the function  $f \circ \nu$  extends regularly to  $\nu^{-1}(t)$  by the Riemann extension theorem. The set of points of  $S$  for which  $v$  is a fixed point of the  $\mathbb{C}^*$ -action is of codimension at least 2 in  $T$ . By the Hartogs theorem,  $f \circ \nu$  extends regularly to  $T_N$  which concludes (3).

*Remark 3.* Consider the rational map  $\kappa : T \rightarrow Z$  given by  $t \mapsto (1/f(t), 1).t$ . It is regular on  $T \setminus S$  and if  $t \in T \setminus S$  and  $z \in Z$  are such that  $t = (\lambda, 1).z$  then  $\kappa(t) = z$ . In particular,  $\kappa$  sends  $\mathbb{T}$ -orbits from  $T$  into  $\mathbb{C}^*$ -orbits of  $Z$ . Furthermore, the morphism  $\kappa_N = \kappa \circ \nu : T_N \rightarrow Z$  is regular because this is the case for the function  $f \circ \nu$ .

**Lemma 19.** *Let  $E_i = \{t = (a_1, a_2, b_1, b_2, w, v) \in \bar{T} \mid b_i = 0\}$  and let  $\bar{T}^b$  coincide with  $T \cup ((f)_0 \setminus E_2) \cup ((f)_\infty \setminus E_1)$ . Suppose that  $\bar{T}_N^b$  is a normalization of  $\bar{T}^b$ . Then there is a regular extension of  $\kappa_N : T_N \rightarrow Z$  to a morphism  $\bar{\kappa}_N^b : \bar{T}_N^b \rightarrow Z$ .*

*Proof.* Since the set  $(f)_0 \cap (f)_\infty$  is of codimension 2 in  $\bar{T}$ , the Hartogs theorem implies that it suffices to prove the regularity of  $\bar{\kappa}_N^b$  on the normalization of  $\bar{T}^b \setminus ((f)_0 \cap (f)_\infty)$ . Furthermore, by the Riemann extension theorem it is enough to construct a continuous extension of  $\kappa$  from  $T \setminus S$  to  $\bar{T}^b \setminus (S \cup ((f)_0 \cap (f)_\infty))$ .

By Lemma 18(2) we need to consider this extension, say, at  $t = (a_1, a_2, b_1, b_2, w, v) \in (f)_0 \setminus (f)_\infty$ . Let  $t_n \rightarrow t$  as  $n \rightarrow \infty$  where

$$t_n = \left( f(t_n)a_1^n, \frac{1}{f(t_n)}a_2^n, b_1^n, b_2^n, f(t_n).v_n, v_n \right) \in T$$

with  $a_1^n b_2^n - a_2^n b_1^n = 1$  and  $f(t_n) \rightarrow 0$ . Perturbing, if necessary, this sequence  $\{t_n\}$  we can suppose every  $t_n \notin S$ , i.e.,  $\kappa(t_n) = (a_1^n, a_2^n, b_1^n, b_2^n, v_n, v_n)$ . Note that  $\lim v_n = v$ ,  $b_k = \lim b_k^n$ ,  $k = 1, 2$ , and  $a_2^n \rightarrow 0$  since  $a_2$  is finite. Hence,  $1 = a_1^n b_2^n - a_2^n b_1^n \approx a_1^n b_2$  and  $a_1^n \rightarrow 1/b_2$  as  $n \rightarrow \infty$ . Now we get a continuous extension of  $\kappa$  by putting  $\kappa(t) = (1/b_2, 0, b_1, b_2, v, v)$ . This yields the desired conclusion.  $\square$

*Remark 4.* If we use the group  $(1 \times \mathbb{C}^*)$  instead of the group  $(\mathbb{C}^* \times 1)$  from Lemma 17(ii) in our construction this would lead to the replacement of  $f$  by  $f^{-1}$ . Furthermore, for the variety  $\bar{T}^a = T \cup ((f)_0 \setminus \{a_1 = 0\}) \cup ((f)_\infty \setminus \{a_2 = 0\})$ , we obtain a morphism  $\bar{\kappa}_N^a : \bar{T}_N^a \rightarrow Z$  similar to  $\bar{\kappa}_N^b$ .

The next fact is intuitively obvious but the proof requires some work.

**Lemma 20.** *The complement  $\bar{T}^0$  of  $\bar{T}^a \cup \bar{T}^b$  in  $\bar{T}$  (which is  $\bar{T}^0 = (\bar{T} \setminus T) \cap \bigcup_{i \neq j} \{a_i = b_j = 0\}$ ) has codimension at least 2.*

*Proof.* Let  $t_n \rightarrow t = (a_1, a_2, b_1, b_2, w, v)$  be as in the proof of Lemma 19. Since for a general point of the slice  $V$  the isotropy group is finite after perturbation we can suppose that each  $v_n$  is contained in a nonconstant  $\mathbb{C}^*$ -orbit  $O_n \subset V$ . Treat  $v_n$  and  $f(t_n).v_n$  as numbers in  $\mathbb{C}^* \simeq O_n$  such that  $f(t_n).v_n = f(t_n)v_n$ . Let  $|v_n|$  and  $|f(t_n).v_n|$  be their absolute values. Then one has the annulus  $A_n = \{|f(t_n).v_n| < \zeta < |v_n|\} \subset O_n$ , i.e.,  $\zeta = \eta v_n$  where  $|f(t_n)| < |\eta| < 1$  for each  $\zeta \in A_n$ . By Lemma 17(iii)  $\pi(v) = \pi(w)$  but by Lemma 18(3) the  $\mathbb{C}^*$ -orbit  $O(v)$  and  $O(w)$  are different unless  $w = v$  is a fixed point of the  $\mathbb{C}^*$ -action. In any case, by Proposition 1(2) the closures of these orbits meet at a fixed point  $\bar{v}$  of the  $\mathbb{C}^*$ -action.

Consider a compact neighborhood  $W = \{u \in V \mid \varphi(u) \leq 1\}$  of  $\bar{v}$  in  $V$  where  $\varphi$  is a plurisubharmonic function on  $V$  that vanishes at  $\bar{v}$  only. Note that the sequence  $\{(\lambda, \mu).t_n\}$  is convergent to  $(\lambda a_1, a_2/\lambda, \mu b_1, b_2/\mu, \lambda.w, \mu.v)$ . In particular, replacing  $\{t_n\}$  by  $\{(\lambda, \mu).t_n\}$  with appropriate  $\lambda$  and  $\mu$  we can suppose that the boundary  $\partial A_n$  of any annulus  $A_n$  is contained in  $W$  for sufficiently large  $n$ . By the maximum principle  $\bar{A}_n \subset W$ . The limit  $A = \lim_{n \rightarrow \infty} \bar{A}_n$  is a compact subset of  $W$  that contains both  $v$  and  $w$ , and also all points  $\eta.v$  with  $0 < |\eta| < 1$  (since  $|f(t_n)| \rightarrow 0$ ). Unless  $O(v) = \bar{v}$  only one of the closures of the subsets  $\{\eta.v \mid 0 < |\eta| < 1\}$  or  $\{\eta.v \mid |\eta| > 1\}$  in  $V$  is compact and contains the fixed point  $\bar{v}$  (indeed, otherwise the closure of  $O(v)$  is a complete curve in the affine variety  $V$ ). The argument above shows that it is the first one.

That is,  $\mu.v \rightarrow \bar{v}$  when  $\mu \rightarrow 0$ . Similarly,  $\lambda.w \rightarrow \bar{v}$  when  $\lambda \rightarrow \infty$ . It is not difficult to check now that the dimension of the set of such pairs  $(w, v)$  is at most  $\dim V$ .

Consider the set  $(\bar{T} \setminus T) \cap \{a_1 = b_2 = 0\}$ . It consists of points  $t = (0, a_2, b_1, 0, w, v)$  and, therefore, its dimension is at most  $\dim V + 2$ . Thus it has codimension at least 2 in  $\bar{T}$  whose dimension is  $\dim V + 4$ . This yields the desired conclusion.  $\square$

The next technical fact may be somewhere in the literature, but, unfortunately, we did not find a reference.

**Proposition 21.** *Let a reductive group  $G$  act on an affine algebraic variety  $X$  and let  $\pi : X \rightarrow Q := X//G$  be the quotient morphism such that one of closed  $G$ -orbits  $O$  is contained in the smooth part of  $X$ <sup>2</sup>. Suppose that  $\nu : X_N \rightarrow X$  and  $\mu : Q_N \rightarrow Q$  are normalization morphisms, i.e.,  $\pi \circ \nu = \pi_N \circ \mu$  for some morphism  $\pi_N : X_N \rightarrow Q_N$ . Then  $Q_N \simeq X_N//G$  for the induced  $G$ -action on  $X_N$  and  $\pi_N$  is the quotient morphism.*

*Proof.* Let  $\psi : X_N \rightarrow R$  be the quotient morphism. By the universal property of quotient morphisms,  $\pi_N = \varphi \circ \psi$  where  $\varphi : R \rightarrow Q_N$  is a morphism. It suffices to show that  $\varphi$  is an isomorphism. The points of  $Q$  (resp.  $R$ ) are nothing but the closed  $G$ -orbits in  $X$  (resp.  $X_N$ ) by Proposition 1, and above each closed orbit in  $X$  we have only a finite number of closed orbits in  $X_N$  because  $\nu$  is finite. Hence,  $\mu \circ \varphi : R \rightarrow Q$  and, therefore,  $\varphi : R \rightarrow Q_N$  are at least quasi-finite. There is only one closed orbit  $O_N$  in  $X_N$  above orbit  $O \subset \text{reg } X$ . Thus  $\varphi$  is injective in a

<sup>2</sup>Without this assumption on smoothness the proposition does not hold.

neighborhood of  $\psi(O_N)$ . That is,  $\varphi$  is birational and by Zariski's Main Theorem it is an embedding.

It remains to show that  $\varphi$  is proper. Recall that  $G$  is a complexification of a compact subgroup  $G^{\mathbb{R}}$  and there is a so-called Kempf–Ness real algebraic subvariety  $X^{\mathbb{R}}$  of  $X$  such that the restriction  $\pi|_{X^{\mathbb{R}}}$  is nothing but the standard quotient map  $X^{\mathbb{R}} \rightarrow X^{\mathbb{R}}/G^{\mathbb{R}} = Q$  which is automatically proper (e.g., see [Sch]). Set  $X_N^{\mathbb{R}} = \nu^{-1}(X^{\mathbb{R}})$ . Then the restriction of  $\pi \circ \nu$  to  $X_N^{\mathbb{R}}$  is proper being the composition of two proper maps. On the other hand, the restriction of  $\mu \circ \pi_N = \pi \circ \nu$  to  $X_N^{\mathbb{R}}$  is proper only when morphism  $\varphi$ , through which it factors, is proper which concludes the proof.

**Proposition 22.** *The morphism  $\rho' : Y' \rightarrow Z'$  from diagram (6) is finite birational.*

*Proof.* The morphism  $\rho'$  factors through  $\rho'_N : Y' \rightarrow Z'_N$  where  $\mu : Z'_N \rightarrow Z'$  is a normalization of  $Z'$  and the statement of the proposition is equivalent to the fact that  $\rho'_N$  is an isomorphism. Set  $Z'(b) = q(\bar{T}^b)$  and  $Z'(a) = q(\bar{T}^a)$ . Note that  $Z' \setminus (Z'(a) \cup Z'(b))$  is in the  $q$ -image of the  $\mathbb{T}$ -invariant set  $\bar{T}^0$  from Lemma 20. Hence  $Z' \setminus (Z'(b) \cup Z'(a))$  is of codimension 2 in  $Z'$  and by the Hartogs theorem it suffices to prove that  $\rho'_N$  is invertible over  $Z(b)'$  (resp.  $Z'(a)$ ).

By Remark 3,  $\bar{\kappa}_N^b$  sends each orbit of the induced  $\mathbb{T}$ -action on  $\bar{T}_N^b$  onto a  $\mathbb{C}^*$ -orbit in  $Z$ . Thus the composition of  $\bar{\kappa}_N^b$  with  $p : Z \simeq Y \rightarrow Y'$  is constant on  $\mathbb{T}$ -orbits and by the universal property of quotient spaces it must factor through the quotient morphism  $q_N^b : \bar{T}_N^b \rightarrow Q$ . By Proposition 21,  $Q = Z'_N(b)$  where  $Z'_N(b) = \mu^{-1}(Z'(b))$ . That is,  $p \circ \bar{\kappa}_N^b = \tau^b \circ q_N^b$  where  $\tau^b : Z'_N(b) \rightarrow Y'$ . Our construction implies that  $\tau^b$  is the inverse of  $\rho'_N$  over  $Z'_N(b)$ . Hence,  $\rho'_N$  is invertible over  $Z'_N(b)$  which concludes the proof.

### 3.1. Proof of Theorems 12 and 11

Let  $G = \mathrm{SL}_2$  act algebraically on  $X$  as in Theorem 12 and let  $V$  be the slice of this action at a point  $x \in X$  so that there is an étale morphism  $G \times_{G_x} V \rightarrow U$  as in Theorem 4. By Lemmas 15 and 16 for the validity of Theorem 12 it suffices to prove semicompatibility of the vector fields  $\delta_1$  and  $\delta_2$  on  $\mathcal{Y} = G \times_{G_x} V$ , which was already done in the case of a finite isotropy group  $G_x$  (see Lemma 16(2)). Consider the quotient morphisms  $\varrho_i : \mathcal{Y} \rightarrow \mathcal{Y}_i := \mathcal{Y} // H_i$  where  $H_i, i = 1, 2$ , are as in Example 1. Set  $\varrho = (\varrho_1, \varrho_2) : \mathcal{Y} \rightarrow \overline{\varrho(\mathcal{Y})} \subset \mathcal{Y}_1 \times \mathcal{Y}_2$ . By Proposition 5, Theorem 12 is true if  $\varrho$  is finite birational. If  $G_x = \mathbb{C}^*$  then  $\varrho : \mathcal{Y} \rightarrow \mathcal{Y}_1 \times \mathcal{Y}_2$  is nothing but morphism  $\rho' : Y' \rightarrow Y'_1 \times Y'_2$  from Proposition 22, i.e., we are done in this case as well. By Lemma 13 the only remaining case is when  $G_x$  is an extension of  $\mathbb{C}^*$  by  $\mathbb{Z}_2$ . Then one has a  $\mathbb{Z}_2$ -action on  $Y'$  such that it is commutative with  $H_i$ -actions on  $Y'$  and  $\mathcal{Y} = Y' // \mathbb{Z}_2$ . Since the vector fields  $\delta_1$  and  $\delta_2$  are semicompatible on  $Y'$  by Propositions 22 and 5, they also generate semicompatible vector fields on  $\mathcal{Y}$  by Lemma 8. This concludes Theorem 12 and, therefore, Theorem 11.  $\square$

*Remark 5.* (1) Consider  $\mathcal{Y} = G \times_{G_x} V$  in the case when  $G = G_x = \mathrm{SL}_2$ , i.e., the  $\mathrm{SL}_2$ -action has a fixed point. It is not difficult to show that morphism  $\varrho = (\varrho_1, \varrho_2) : \mathcal{Y} \rightarrow \overline{\varrho(\mathcal{Y})} \subset \mathcal{Y}_1 \times \mathcal{Y}_2$  as in the proof above is not quasi-finite. In particular,  $\delta_1$  and

$\delta_2$  are not compatible. However, we do not know if the condition about the absence of fixed points is essential for Theorem 11. In examples we know the presence of fixed points is not an obstacle for the algebraic density property. Say, for  $\mathbb{C}^n$  with  $2 \leq n \leq 4$  any algebraic  $\mathrm{SL}_2$ -action is a representation in a suitable polynomial coordinate system (e.g., see [Po]) and, therefore, it has a fixed point; but the algebraic density property is a consequence of the Andersén–Lempert work.

(2) Similarly, we do not know whether the nondegeneration assumption is required for Theorem 11. The simplest case of a degenerate  $\mathrm{SL}_2$ -action is presented by the homogeneous space  $\mathrm{SL}_2/\mathbb{C}^*$  where  $\mathbb{C}^*$  is the diagonal subgroup. Let

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

be a general element of  $\mathrm{SL}_2$ . Then the ring of invariants of the  $\mathbb{C}^*$ -action is generated by  $u = a_1a_2$ ,  $v = b_1b_2$ , and  $z = a_2b_1 + \frac{1}{2}$  (since  $a_1b_2 = 1 + a_2b_1 = \frac{1}{2} + z$ ). Hence  $\mathrm{SL}_2/\mathbb{C}^*$  is isomorphic to a hypersurface  $\tilde{S}$  in  $\mathbb{C}_{u,v,z}^3$  given by the equation  $uv = z^2 - \frac{1}{4}$ . This hypersurface has the algebraic density property by [KK1].

(3) The situation is a bit more complicated if we consider the normalizer  $T$  of the diagonal  $\mathbb{C}^*$ -subgroup of  $\mathrm{SL}_2$  (i.e.,  $T$  is an extension of  $\mathbb{C}^*$  by  $\mathbb{Z}_2$ ). Then  $\mathcal{P} = \mathrm{SL}_2/T$  is isomorphic to  $S/\mathbb{Z}_2$  where the  $\mathbb{Z}_2$ -action is given by  $(u, v, z) \rightarrow (-u, -v, -z)$ . This interesting surface<sup>3</sup> has also the algebraic density property. To see this, note that the diagonal  $\mathbb{C}^*$ -subgroup of  $\mathrm{SL}_2$  contains the center of  $\mathrm{SL}_2$  which implies that both  $S$  and  $\mathcal{P}$  are quotients of  $\mathrm{PSL}_2$ . Since  $\mathrm{PSL}_2$  has a trivial center it has the density property by the Toth–Varolin theorem in [TV2]. Furthermore, the analysis of their paper shows that they used only completely integrable algebraic vector fields in the proof, i.e., they established, in fact, the algebraic density property. In brief, we have the algebraic density property for any homogeneous space that is a quotient of  $\mathrm{SL}_2$ .

#### 4. Applications

Theorem 11 is applicable to a wide class of homogeneous spaces. Let us start with the following observation: given a reductive subgroup  $R$  of a linear algebraic group  $G$ , any  $\mathrm{SL}_2$ -subgroup  $\Gamma < G$  yields a natural  $\Gamma$ -action on  $G/R$ . Furthermore, for each point  $aR \in G/R$ , its isotropy subgroup under this action is isomorphic to  $\Gamma \cap aRa^{-1}$ . In particular, the action has no fixed points if  $a^{-1}\Gamma a$  is not contained in  $R$  for any  $a \in G$  and it is nondegenerate if  $\Gamma^a := a^{-1}\Gamma a \cap R \simeq \Gamma \cap aRa^{-1}$  is finite for some  $a \in R$ . Thus Theorem 11 implies the following.

**Proposition 23.** *Let  $G$  be an algebraic group and let  $\Gamma_1, \dots, \Gamma_k$  be  $\mathrm{SL}_2$ -subgroups of  $G$  such that at some  $x \in G$  the set  $\{\delta_2^i(x)\}$  is a generating one (where  $(\delta_1^i, \delta_2^i)$  is the corresponding pair of locally nilpotent vector fields on  $G$  generated by the natural  $\Gamma_i$ -action). Suppose that for each  $i = 1, \dots, k$  and any  $a \in G$ , the group*

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<sup>3</sup>It can be shown that  $\mathcal{P}$  is the only  $\mathbb{Q}$ -homology plane which is simultaneously a Danilov–Gizatullin surface (i.e., it has a trivial Makar-Limanov invariant (see [FKZ])), and its fundamental group is  $\mathbb{Z}_2$ .

$\Gamma_i^a := a^{-1}\Gamma_i a \cap R$  is not isomorphic to  $\Gamma_i$  and, furthermore,  $\Gamma_i^a$  is finite for some  $a$ . Then  $G/R$  has the algebraic density property.

Note that for a simple Lie group  $G$ , a generating set at any  $x \in G$  consists of one nonzero vector since the adjoint representation is irreducible. Therefore, in this case the algebraic density property is a consequence of the following.

**Theorem 24.** *Let  $G$  be a simple Lie group with Lie algebra different from  $\mathfrak{sl}_2$ , and let  $R$  be a proper closed reductive subgroup of  $G$ . Then there exists an  $\mathrm{SL}_2$ -subgroup  $\Gamma$  in  $G$  such that  $\Gamma^a$  is not isomorphic to  $\Gamma$  for any  $a \in G$  and, furthermore,  $\Gamma^a$  is finite for some  $a \in G$ .*

Surprisingly enough, the proof of Theorem 24 (at least in our presentation) requires some serious facts from Lie group theory and we shall postpone it until the Appendix.

**Corollary 25.** *Let  $X = G/R$  be an affine homogeneous space of a semisimple Lie group  $G$  different from  $\mathrm{SL}_2$ . Then  $X$  is equipped with  $N$  pairs  $\{\delta_1^k, \delta_2^k\}_{k=1}^N$  of compatible derivations such that the collection  $\{\delta_2^k(x_0)\}_{k=1}^N \subset T_{x_0}X$  is a generating set at some point  $x_0 \in X$ . In particular,  $X$  has the algebraic density property by Theorem 6.*

*Proof.* Note that  $R$  is reductive by Proposition 2 (Matsushima's theorem). Then  $X$  is isomorphic to a quotient of form  $G/R$  where  $G = G_1 \oplus \cdots \oplus G_N$ , each  $G_i$  is a simple Lie group, and  $R$  is not necessarily connected. However, we can suppose that  $R$  is connected by virtue of Proposition 8. Consider the projection homomorphism  $\pi_k : G \rightarrow G_k$  and  $R_k = \pi_k(R)$  which is reductive being the image of a reductive group. If  $N$  is the minimal possible then  $R_k \neq G_k$  for every  $k$ . Indeed, if say  $R_N = G_N$ , then  $X = (G_1 \oplus \cdots \oplus G_{N-1})/\tilde{R}$  where  $\tilde{R} = \mathrm{Ker} \pi_N$  which contradicts minimality.

Assume first that none of the  $G_i$ s is isomorphic to  $\mathrm{SL}_2$ . By Theorem 24 one can choose an  $\mathrm{SL}_2$ -subgroup  $\Gamma_k < G_k$  such that the natural  $\Gamma_k$ -action on  $G_k/\pi_k(R)$  and, therefore, on  $G/R$  is fixed point free. We can also assume that each  $\Gamma_k$ -action is nondegenerate. Denote by  $\delta_1^k$  and  $\delta_2^k$  the corresponding pair of locally nilpotent derivations for the  $\Gamma_k$ -action. Since the adjoint representation is irreducible for a simple Lie group,  $\{\delta_2^k(e)\}_{k=1}^N$  is a generating set of the tangent space  $T_e G$  at  $e = e_1 \oplus \cdots \oplus e_N \in G$ , where  $e_k$  is the unity of  $G_k$ . Consider  $X = G/R$  as the set of left cosets, i.e.,  $X$  is the quotient of  $G$  with respect to the action generated by multiplication by elements of  $R$  from the right. Hence, this action commutes with multiplication by elements of  $\Gamma_k$  from the left and, therefore, it commutes with any field  $\delta_k^i$ . Pushing the actions of  $\Gamma_k$ s to  $X$  we get fixed point free nondegenerate  $\mathrm{SL}_2$ -actions on  $X$  and the desired conclusion in this case follows from by Theorem 11.

In the case when some of the  $G_k$ s are isomorphic to  $\mathrm{SL}_2$  we cannot assume that each  $\Gamma_k$ -action is nondegenerate, but now  $N \geq 2$  and the  $\Gamma_k$ -actions are still fixed point free. Consider an isomorphism  $\varphi_k : \Gamma_k \rightarrow \Gamma_1$ . Then we have an  $\mathrm{SL}_2$ -group  $\Gamma^{\varphi_k} = \{(\varphi_k(\gamma), \gamma) \mid \gamma \in \Gamma_k\} < \Gamma_1 \times \Gamma_k$  acting naturally on  $G_1 \times G_k$  and, therefore, on  $G$ . This isomorphism  $\varphi_k$  can be chosen so that the  $\Gamma^{\varphi_k}$ -action is nondegenerate. Indeed, if, say a  $\mathbb{C}^*$ -subgroup  $L < \Gamma_k$  acts on  $G_k$  trivially, choose  $\varphi_k$  so that  $\varphi_k(L)$  acts nontrivially on  $G_1$  which makes the action nondegenerate. In



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particular, by Theorem 12 we get pairs of compatible locally nilpotent derivations  $\tilde{\delta}_1^{\varphi_k}$  and  $\tilde{\delta}_2^{\varphi_k}$  corresponding to such actions. Set  $G' = G_2 \oplus \dots \oplus G_N$  and  $e' = e_2 \oplus \dots \oplus e_N \in G'$ . Since the adjoint representation is irreducible for a simple Lie group the orbit of the set  $\{\tilde{\delta}_2^{\varphi_k}(e)\}_{k=2}^N$  under conjugations generates a subspace of  $S$  of  $T_e G$  such that the restriction of the natural projection  $T_e G \rightarrow T_{e'} G'$  to  $S$  is surjective. In order to enlarge  $\{\tilde{\delta}_2^{\varphi_k}(e)\}_{k=2}^N$  to a generating subset of  $T_e G$  consider an isomorphism  $\psi_2 : \Gamma_2 \rightarrow \Gamma_1$  different from  $\varphi_2$  and such that the  $\Gamma^{\psi_2}$ -action is nondegenerate. Denote the corresponding compatible locally nilpotent derivations by  $\tilde{\delta}_1^{\psi_2}$  and  $\tilde{\delta}_2^{\psi_2}$  on  $G_1 \oplus G_2$  (and also by abusing notation on  $G$ ). Note that the vectors  $\tilde{\delta}_2^{\varphi_2}(e_1 \oplus e_2)$  and  $\tilde{\delta}_2^{\psi_2}(e_1 \oplus e_2)$  can be assumed different with an appropriate choice of  $\psi_2$ . Hence these two vectors form a generating subset of  $T_{e_1 \oplus e_2} G_1 \oplus G_2$ . Taking into consideration the remark about  $S$  we see that  $\{\tilde{\delta}_2^{\varphi_k}(e)\}_{k=2}^N \cup \{\tilde{\delta}_2^{\psi_2}(e)\}$  is a generating subset of  $T_e G$ . Now pushing these  $SL_2$ -actions to  $X$  we get the desired conclusion.  $\square$

**Theorem 26.** *Let  $G$  be a linear algebraic group and let  $R$  be a closed proper reductive subgroup of  $G$  such that the homogeneous space  $G/R$  has connected components different from  $\mathbb{C}_+$  or a torus  $(\mathbb{C}^*)^n$ . Then  $G/R$  has the algebraic density property.*

*Proof.* Since all components of  $G/R$  are isomorphic as varieties we can suppose that  $G$  is connected. Furthermore, by Corollary 25 and Remark 5(3), we are done with a semisimple  $G$ .

Let us consider first the case of a reductive but not semisimple  $G$ . Then the center  $Z \simeq (\mathbb{C}^*)^n$  of  $G$  is nontrivial. Let  $S$  be the semisimple part of  $G$ . Assume for the time being that  $G$  is isomorphic as a group to the direct product  $S \times Z$  and consider the natural projection  $\tau : G \rightarrow Z$ . Set  $Z' = \tau(R) = R/R'$  where  $R' = R \cap S$ . Since we are going to work with compatible vector fields we can suppose that  $R$  is connected by virtue of Lemma 8. Then  $Z'$  is a subtorus of  $Z$  and also  $R'$  is reductive by Proposition 2. Hence  $G/R = (G/R')/Z'$  and  $G/R' = S/R' \times Z'$ . Note that there is a subtorus  $Z''$  of  $Z$  such that  $Z'' \simeq Z/Z'$  and  $Z' \cdot Z'' = Z$ . (Indeed,  $Z' \simeq (\mathbb{C}^*)^k$  generates a sublattice  $L \simeq \mathbb{Z}^k$  of homomorphisms from  $\mathbb{C}^*$  into  $Z'$  of the similar lattice  $\mathbb{Z}^n$  of  $Z \simeq (\mathbb{C}^*)^n$  such that the quotient  $\mathbb{Z}^n/L$  has no torsion, i.e., it is isomorphic to  $\mathbb{Z}^{n-k}$ . Since any short exact sequence of free  $\mathbb{Z}$ -modules splits we have a  $\mathbb{Z}$ -submodule  $K \simeq \mathbb{Z}^{n-k}$  in  $\mathbb{Z}^n$  such that  $K + L = \mathbb{Z}^n$ . This lattice  $K$  yields a desired subtorus  $Z'' \simeq (\mathbb{C}^*)^{n-k}$ .) Hence  $G/R$  is isomorphic to  $\varrho^{-1}(Z'') \simeq S/R' \times Z''$  where  $\varrho : G/R' \rightarrow Z$  is the natural projection. Note that both factors are nontrivial since otherwise  $G/R$  is either a torus or we are in the semisimple case again. Thus  $X$  has the algebraic density property by Proposition 7 with  $S/R'$  playing the role of  $X_1$  and  $Z''$  of  $X_2$ . In particular, we have a finite set of pairs of compatible vector fields  $\{\delta_1^k, \delta_2^k\}$  as in Theorem 6. Furthermore, one can suppose that the fields  $\delta_1^k$  correspond to one-parameter subgroups of  $S$  isomorphic to  $\mathbb{C}_+$  and  $\delta_2^k$  to one-parameter subgroups of  $Z$  isomorphic to  $\mathbb{C}^*$ . In the general case  $G/R$  is the factor of  $X$  with respect to the natural action of a finite (central) normal subgroup  $F < G$ . Since  $F$  is central, the fields  $\delta_1^k, \delta_2^k$  induce completely integrable vector fields  $\tilde{\delta}_1^k, \tilde{\delta}_2^k$  on  $G/R$  while  $\tilde{\delta}_2^k(x_0)$  is a generating set for some  $x_0 \in G/R$ . By Lemma 8 the pairs  $\{\tilde{\delta}_1^k, \tilde{\delta}_2^k\}$  are compatible and the algebraic density

property for  $G/R$  follows again from Theorem 6.

In the case of a general linear algebraic group  $G$  different from a reductive group,  $\mathbb{C}^n$ , or a torus  $(\mathbb{C}^*)^n$  consider the nontrivial unipotent radical  $\mathcal{R}_u$  of  $G$ . It is automatically an algebraic subgroup of  $G$ , see [Ch, p. 183]. By Mostow's theorem [Mo] (see also [Ch, p. 181])  $G$  contains a (Levi) maximal closed reductive algebraic subgroup  $G_0$  such that  $G$  is the semi-direct product of  $G_0$  and  $\mathcal{R}_u$ , i.e.,  $G$  is isomorphic as an affine variety to the product  $\mathcal{R}_u \times G_0$ . Furthermore, any other maximal reductive subgroup is conjugated to  $G_0$ . Hence, replacing  $G_0$  by its conjugate, we can suppose that  $R$  is contained in  $G_0$ . Therefore,  $G/R$  is isomorphic as an affine algebraic variety to the  $G_0/R \times \mathcal{R}_u$  and we are done now by Proposition 7 with  $\mathcal{R}_u$  playing the role of  $X_1$  and  $G_0/R$  of  $X_2$ .

*Remark 6.* (1) The algebraic density property implies, in particular, that the Lie algebra generated by completely integrable algebraic (and, therefore, holomorphic) vector fields is infinite dimensional, i.e., this is true for homogeneous spaces from Theorem 26. For Stein manifolds of dimension at least 2 that are homogeneous spaces of holomorphic actions of a connected complex Lie group the infinite dimensionality of such algebras was also established by Huckleberry and Isaev [HI].

(2) Note that as in [KK2] we actually proved a stronger fact for a homogeneous space  $X = G/R$  from Theorem 26. Namely, it follows from the construction that the Lie algebra generated by vector fields of form  $f\sigma$ , where  $\sigma$  is either locally nilpotent or semisimple and  $f \in \text{Ker } \sigma$  for semisimple  $\sigma$  and  $\deg_\sigma f \leq 1$  in the locally nilpotent case, coincides with  $\text{AVF}(X)$ .

## 5. Appendix: The proof of Theorem 24

Let us start with the following technical fact.

**Proposition 27.** *Let  $G$  be a semisimple group and let  $R$  be a semisimple subgroup of  $G$ . Suppose that the number of orbits of nilpotent elements in the Lie algebra  $\mathfrak{r}$  of  $R$  under the adjoint action is less than the number of orbits of nilpotent elements in the Lie algebra of  $G$  under the adjoint action. Then  $G$  contains an  $\text{SL}_2$ -subgroup  $\Gamma$  such that  $\Gamma^g := g^{-1}\Gamma g \cap R$  is different from  $g^{-1}\Gamma g$  for any  $g \in G$ .*

*Proof.* By the Jacobson–Morozov theorem (e.g., see Proposition 2 and Corollary in [Bo, Sect. 8.11.2]) for any semisimple group  $G$  there is a bijection between the set of  $G$ -conjugacy classes of  $\mathfrak{sl}_2$ -triples and the set of  $G$ -conjugacy classes of nonzero nilpotent elements from  $G$  which implies the desired conclusion.

In order to exploit Proposition 27 we need to recall some terminology and results from [Bo].

**Definition 6.** (1) Recall that a semisimple element  $h$  of a Lie algebra is regular, if the kernel of its adjoint action is a Cartan subalgebra. An  $\mathfrak{sl}_2$ -subalgebra of the Lie algebra  $\mathfrak{g}$  of a semisimple group  $G$  is called principal if in its triple of standard generators the semisimple element  $h$  is regular and the adjoint action of  $h$  has even eigenvalues (see Definition 3 in [Bo, Sect. 8.11.4]). The subgroup generated by this subalgebra is called a principal  $\text{SL}_2$ -subgroup of  $G$ . As an example of such a principal subgroup one can consider an  $\text{SL}_2$ -subgroup of  $\text{SL}_n$  that acts irreducibly

on the natural representation space  $\mathbb{C}^n$ . In general, principal  $\mathfrak{sl}_2$ -subalgebras exist in any semisimple Lie algebra  $\mathfrak{g}$  (see Proposition 8 in [Bo, Sect. 8.11.4]). Any two principal  $\mathrm{SL}_2$ -subgroups are conjugated (see Proposition 6 in [Bo, Sect. 8.11.3] and Proposition 9 in [Bo, Sect. 8.11.4]).

(2) A connected closed subgroup  $P$  of  $G$  is called principal if it contains a principal  $\mathrm{SL}_2$ -subgroup<sup>4</sup>. The rank of  $P$  is the rank of the maximal torus it contains. If this rank is 1 then  $P$  coincides with its principal  $\mathrm{SL}_2$ -subgroup (see Exercise 21 in [Bo, Sect. 9.4]).

**Proposition 28.** *Let  $R$  be a proper reductive subgroup of a simple group  $G$  different from  $\mathrm{SL}_2$  or  $\mathrm{PSL}_2$ . Then there exists an  $\mathrm{SL}_2$ -subgroup  $\Gamma$  of  $G$  such that  $\Gamma^g := g^{-1}\Gamma g \cap R$  is different from  $g^{-1}\Gamma g$  for any  $g \in G$ .*

*Proof.* If  $R$  is not principal it cannot contain a principal  $\mathrm{SL}_2$ -subgroup and we are done. Thus it suffices to consider the case of the principal subgroup  $R$  only.

Suppose first that  $R$  is of rank 1. If  $R$  contains  $g^{-1}\Gamma g$  it must coincide with this subgroup by the dimension argument. Hence it suffices to choose nonprincipal  $\Gamma$  to see the validity of the proposition in this case.

Suppose now that  $R$  is of rank at least 2. Then there are the following possibilities (Exercises 20c–e in [Bo, Sect. 9.4]):

- (1)  $R$  is of type  $\mathrm{B}_2$  and  $G$  is of type  $\mathrm{A}_3$  or  $\mathrm{A}_4$ ;
- (2)  $R$  is of type  $\mathrm{G}_2$  and  $G$  is of type  $\mathrm{B}_3, \mathrm{D}_4,$  or  $\mathrm{A}_6$ ;
- (3)  $G$  is of type  $\mathrm{A}_{2l}$  with  $l \geq 3$  and  $R$  is of type  $\mathrm{B}_l$ ;
- (4)  $G$  is of type  $\mathrm{A}_{2l-1}$  with  $l \geq 3$  and  $R$  is of type  $\mathrm{C}_l$ ;
- (5)  $G$  is of type  $\mathrm{D}_l$  with  $l \geq 4$  and  $R$  is of type  $\mathrm{B}_{l-1}$ ;
- (6)  $G$  is of type  $\mathrm{E}_6$  and  $R$  is of type  $\mathrm{F}_4$ .

In order to apply Proposition 27 to these cases we need the Dynkin classification of nilpotent orbits (with Elkington’s corrections) as described in the Bala–Carter paper [BaCa2, pp. 6–7].

By this classification the number  $a_n$  of such orbits in a simple Lie algebra of type  $\mathrm{A}_n$  coincides with the number of partitions  $\lambda$  of  $n + 1$ , i.e.,  $\lambda = (\lambda_1, \dots, \lambda_k)$  with natural  $\lambda_i$  such that  $|\lambda| = \lambda_1 + \dots + \lambda_k = n + 1$ .

For a simple Lie algebra of type  $\mathrm{B}_m$  the number  $b_m$  of nilpotent orbits coincides with the number of partitions  $\lambda$  and  $\mu$  such that  $2|\lambda| + |\mu| = 2m + 1$  where  $\mu$  is a partition with distinct odd parts.

For a simple Lie algebra of type  $\mathrm{C}_m$  the number  $c_m$  of nilpotent orbits coincides with the number of partitions  $\lambda$  and  $\mu$  such that  $|\lambda| + |\mu| = m$  where  $\mu$  is a partition with distinct parts.

For a simple Lie algebra of type  $\mathrm{D}_m$  the number  $d_m$  of nilpotent orbits coincides with the number of partitions  $\lambda$  and  $\mu$  such that  $2|\lambda| + |\mu| = 2m$  where  $\mu$  is a partition with distinct odd parts.

The numbers of nilpotent orbits of algebras of type  $\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$  are 5, 16, 21, 45, and 70, respectively.

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<sup>4</sup>The definition of a principal subgroup in [Bo] is different (see Exercise 18 in Section 9.4) but it coincides with this one in the case of a complex Lie group (see Exercise 21c in Section 9.4).

Now one has  $a_4 > a_3 = 5 > b_2 = 4$  which settles case (1) by Proposition 27. Then  $b_3, d_4, a_6 > 5$  which settles case (2). Similarly,  $a_{2l} > b_l$  for  $l \geq 3$ ,  $a_{2l-1} > c_l$  for  $l \geq 3$ ,  $d_l > b_{l-1}$  for  $l \geq 4$ , and  $21 > 16$  which settles cases (3)–(6) and concludes the proof.  $\square$

*Remark 7.* In fact, the statement of Proposition 28 is true for any proper maximal subgroup  $R$  of  $G$ . This can be deduced from Dynkin’s classification of maximal subalgebras in semisimple Lie algebras. We outline the argument below.

Let us consider a maximal subalgebra  $\mathfrak{r}$  in  $\mathfrak{g}$ , where  $\mathfrak{g}$  is a simple Lie algebra. If  $\mathfrak{r}$  is regular (i.e., if its normalizer contains some Cartan subalgebra in  $\mathfrak{g}$ ), then  $\mathfrak{r}$  does not contain any principal  $\mathfrak{sl}_2$ -triple [OVG, Sect. 6.2.4]. Thus we may assume that  $\mathfrak{r}$  is nonregular.

If  $\mathfrak{g}$  is exceptional, the list of such  $\mathfrak{r}$  is given in Theorems 6.3.4 and 6.3.5 from [OVG]. All of them are semisimple, and we will only consider simple subalgebras (otherwise,  $\mathfrak{r}$  once again does not contain any principal  $\mathfrak{sl}_2$ s). The list of simple maximal nonregular subalgebras of rank  $\geq 2$  in exceptional Lie algebras is short:  $B_2$  in  $E_8$ ,  $A_2$  in  $E_7$  and  $A_2, G_2, C_4, F_4$  in  $E_6$ . In all these cases, Proposition 27 applies.

It remains to consider nonregular maximal subalgebras  $\mathfrak{r}$  of classical Lie algebras. Any such  $\mathfrak{r}$  is simple, and an embedding of  $\mathfrak{r}$  in  $\mathfrak{g}$  is defined by a nontrivial linear irreducible representation  $\varphi : \mathfrak{r} \rightarrow \mathfrak{sl}(V)$ . Let  $n = \dim V$  and  $m = \lfloor n/2 \rfloor$ . If the module  $V$  is not self-dual,  $\mathfrak{r}$  is a maximal subalgebra in  $\mathfrak{g} = A_{n-1}$ . If  $V$  is self-dual and endowed with a skew-symmetric invariant form,  $\mathfrak{r}$  is a maximal subalgebra in  $\mathfrak{g} = C_m$ ; and if  $V$  is self-dual with a symmetric invariant form,  $\mathfrak{r}$  is a maximal subalgebra in  $\mathfrak{g} = B_m$  or  $D_m$ . Denote by  $o(V)$  the number of nilpotent orbits in  $\mathfrak{g}$ , then

$$o(V) \geq o_n = \begin{cases} \min(a_{n-1}, b_m) & \text{if } n \text{ odd,} \\ \min(a_{n-1}, d_m, c_m) & \text{if } n \text{ is even.} \end{cases}$$

We want to check that for any irreducible  $\mathfrak{r}$ -module  $V$  (except those corresponding to the trivial embedding  $\mathfrak{r} = \mathfrak{g}$ ), the number  $o(\mathfrak{r})$  of the nilpotent orbits in  $\mathfrak{r}$  is less than  $o(V)$ . In what follows, representations  $\varphi$  generating trivial embeddings of  $\mathfrak{r}$  in  $\mathfrak{g}$  are excluded. For exceptional  $\mathfrak{r}$  of types  $G_2, F_4, E_6, E_7, E_8$  the smallest irreducible representation has dimension  $n = 7, 26, 27, 56, 248$  respectively. In all cases, the inequality  $o(\mathfrak{r}) < o_n$  holds.

If  $\mathfrak{r}$  is of type  $A_k$ , then either  $V$  is not self-dual and  $n > k + 1$  (in which case  $o(\mathfrak{r}) = a_k < a_{n-1} = o(V)$ ) or  $V$  is self-dual and  $n > 2(k + 1)$ . Then  $a_k < o_n \leq o(V)$ .

If  $\mathfrak{r}$  is of type  $B_k$  ( $k \geq 2$ ), then all irreducible  $V$  are self-dual and  $n > 2k + 1$ , hence  $b_k < o_n$ . If  $\mathfrak{r}$  is of type  $C_k$  ( $k \geq 3$ ), then all irreducible  $V$  are self-dual and  $n > 4k$ , hence  $c_k < o_n$ . If  $\mathfrak{r}$  is of type  $D_k$  ( $k \geq 4$ ), then for any irreducible  $V$ ,  $n > 3k$  and  $d_k < o_n$ .

From this we conclude that Proposition 27 applies to any simple nonregular  $\mathfrak{r}$  in  $\mathfrak{g}$ , where  $\mathfrak{g}$  is a classical simple Lie algebra.

**Lemma 29.** *Each orbit  $O$  of a fixed point free degenerate  $SL_2$ -action on an affine algebraic variety  $X$  is two dimensional and closed, and the isotropy group of any point  $x \in X$  is either  $\mathbb{C}^*$  or a  $\mathbb{Z}_2$ -extension of  $\mathbb{C}^*$ .*

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*Proof.* In the case of a fixed point free  $\mathrm{SL}_2$ -action the isotropy group  $I_x$  of a point from a closed orbit is either finite or  $\mathbb{C}^*$  or  $\mathbb{Z}_2$ -extension of  $\mathbb{C}^*$  by Lemma 13. Because the action is also degenerate  $I_x$  cannot be finite and, therefore, the closed orbit  $\mathrm{SL}_2/I_x$  is two dimensional. By Proposition 1 (2) the closure of  $O$  must contain a closed orbit. Since  $O$  itself is at most two dimensional it must coincide with this closed orbit.  $\square$

Next we need two lemmas with the proof of the first one being straightforward.

**Lemma 30.** *Let  $G$  be a simple Lie group of dimension  $N$  and rank  $n$ , let  $a$  be an element of  $G$  and let  $C(a)$  be its centralizer. Suppose that  $k$  is the dimension of  $C(a)$ . Then the dimension of the orbit  $O$  of  $a$  under conjugations is  $N - k$ . In particular, when  $a$  is a regular element (i.e.,  $\dim C(a) = n$ ) we have  $\dim O = N - n$  coincides with the codimension of the centralizer of  $a$ .*

**Lemma 31.** *Let  $G$  be a simple Lie group of dimension  $N$  and rank  $n$ , let  $R$  be a proper reductive subgroup of  $G$  whose dimension is  $M$  and rank is  $m$  and let  $\Gamma$  be an  $\mathrm{SL}_2$ -subgroup of  $G$  such that its natural action on  $G/R$  is fixed point free degenerate. Suppose that  $a$  is a semisimple nonidentical element of  $\Gamma$  and  $k$  is the dimension of  $C(a)$ . Then  $M \geq N - k - 1$ . Furthermore, if  $a$  is regular,  $M = N - n + m - 2$ .*

*Proof.* Since the  $\Gamma$ -action on  $G/R$  is fixed point free and degenerate the isotropy group of any element  $gR \in G/R$  is either  $\mathbb{C}^*$  or a  $\mathbb{Z}_2$  extension of  $\mathbb{C}^*$  by Lemma 29. Recall that this isotropy group is  $\Gamma \cap gRg^{-1}$  and, therefore,  $R$  contains a unique subgroup of the form  $g^{-1}L'g$  where  $L'$  is a  $\mathbb{C}^*$ -subgroup of  $\Gamma$ . That is,  $L' = \gamma_0^{-1}L\gamma_0$  for some  $\gamma_0 \in \Gamma$  where  $L$  is the  $\mathbb{C}^*$ -subgroup generated by  $a$ . Furthermore, this  $\gamma_0$  is unique modulo a normalizer of  $L$  in  $\Gamma$  because, otherwise,  $\Gamma^g$  contain another  $\mathbb{C}^*$ -subgroup of  $g^{-1}\Gamma g$  and, therefore, it would be at least two dimensional. The two-dimensional variety  $W_{a,g} = \{(\gamma g)^{-1}a(\gamma g) \mid \gamma \in \Gamma\}$  meets  $R$  exactly at two points  $(\gamma_0 g)^{-1}a(\gamma_0 g)$  and  $(\gamma_0 g)^{-1}a^{-1}(\gamma_0 g)$  (since the normalizer of  $L$  has two components). Varying  $g$  we can suppose that  $W_{a,g}$  contains a general point of the  $G$ -orbit  $O_a$  of  $a$  under conjugations. Since it meets subvariety  $R \cap O_a$  of  $O_a$  at two points we see that  $\dim R \cap O_a = \dim O_a - 2 = N - k - 2$  by Lemma 30. Thus, with  $a$  running over  $L$  we have  $\dim R \geq N - k - 1$ .

For the second statement note that  $b = g^{-1}ag \in R$  is a regular element in  $G$ . Hence the maximal torus in  $G$  (and, therefore, in  $R$ ) containing  $b$  is determined uniquely. Assume that two elements  $b_l = g_l^{-1}ag_l \in R$ ,  $l = 1, 2$ , are contained in the same maximal torus  $T'$  of  $R$  and, therefore, the same maximal torus  $T$  of  $G$ . Then  $g_2g_1^{-1}$  belongs to the normalizer of  $T$ , i.e.  $b_2$  is of the form  $w^{-1}b_1w$  where  $w$  is an element of the Weyl group of  $T$ . Thus  $R \cap O_a$  meets each maximal torus  $T'$  at a finite number of points. The space of the maximal tori of  $R$  is naturally isomorphic to  $R/T'_{\mathrm{norm}}$  where  $T'_{\mathrm{norm}}$  is the normalizer of  $T'$ . Hence  $\dim R \cap O_a = \dim R - \dim T'_{\mathrm{norm}} = M - m$ . We showed already that the last dimension is also  $N - n - 2$  which implies  $M = N - n + m - 2$ .

**Proposition 32.** *Let the assumption of Lemma 31 hold. Then  $a$  cannot be a regular element of  $G$ .*

*Proof.* Assume the contrary, i.e.,  $a$  is regular. Let  $\mathfrak{g}$  be the Lie algebra of  $G$ , let  $\mathfrak{h}$  be its Cartan subalgebra, and let  $\mathfrak{r}_+$  (resp.  $\mathfrak{r}_-$ ) be the linear space generated by positive (resp. negative) root spaces. Set  $\mathfrak{s} = \mathfrak{r}_+ + \mathfrak{r}_-$  and suppose that  $\mathfrak{g}', \mathfrak{h}', \mathfrak{s}'$  are the similar objects for  $R$  with  $\mathfrak{h}' \subset \mathfrak{h}$ . Put  $\mathfrak{r}'_{\pm} = \mathfrak{s}' \cap \mathfrak{r}_{\pm}$ .

Each element  $x'$  of a root space from  $\mathfrak{s}'$  is of the form  $x' = h_0 + x_+ + x_-$  where  $h_0 \in \mathfrak{h}$  and  $x_{\pm} \in \mathfrak{r}_{\pm}$ . Then there exists an element  $h' \in \mathfrak{h}'$  such that the Lie bracket  $[x', h']$  is a nonzero multiple of  $x'$  which implies that  $h_0 = 0$ , since  $[h_0, h'] = 0$ . Thus  $\mathfrak{s}' \subset \mathfrak{s}$ . By assumption  $\mathfrak{h}'$  is a linear subspace of  $\mathfrak{h}$  of codimension  $n - m$ . Hence Lemma 31 implies that  $\mathfrak{s}'$  is of codimension 2 in  $\mathfrak{s}$ . We have two possibilities: (1) either, say,  $\mathfrak{r}'_+ = \mathfrak{r}_+$  and  $\mathfrak{r}'_-$  is of codimension 2 in  $\mathfrak{r}_-$ ; or (2)  $\mathfrak{r}'_{\pm}$  is of codimension 1 in  $\mathfrak{r}_{\pm}$ . In the first case, each element of a root space  $x \in \mathfrak{r}_+$ , being in an eigenspace of  $\mathfrak{h}' \subset \mathfrak{h}$ , is also an element of a root space of  $\mathfrak{g}'$ . However, for each root the negative of it is also contained in  $\mathfrak{g}'$  which implies that  $\mathfrak{r}'_- = \mathfrak{r}_-$ . A contradiction.

In case (2) consider the generators  $x_1, \dots, x_l$  (resp.  $y_1, \dots, y_l$ ) of all root spaces in  $\mathfrak{r}_+$  (resp.  $\mathfrak{r}_-$ ) such that  $h_i = [x_i, y_i]$  is a nonzero element of  $\mathfrak{h}$ . Their linear combination  $\sum_{i=1}^l c_i^+ x_i$  is contained in  $\mathfrak{r}'_+$  if and only if its coefficients satisfy a nontrivial linear equation  $\sum_{i=1}^l d_i^+ c_i^+ = 0$ . Similarly,  $\sum_{i=1}^l c_i^- y_i$  is contained in  $\mathfrak{r}'_-$  if and only if its coefficients satisfy a nontrivial linear equation  $\sum_{i=1}^l d_i^- c_i^- = 0$ . Note that  $d_i^+ = 0$  if and only if  $d_i^- = 0$  since, otherwise, one can find a root of  $\mathfrak{g}'$  whose negative is not a root. Without loss of generality we suppose that the simple roots are presented by  $x_1, \dots, x_n$ , i.e.  $h_1, \dots, h_n$  is a basis of  $\mathfrak{h}$ . Hence at least one coefficient  $d_i^+ \neq 0$  for  $i \leq n$ . Indeed, otherwise  $\mathfrak{r}'_+$  contains  $x_1, \dots, x_n$  which implies that  $\mathfrak{r}'_+ = \mathfrak{r}_+$ , contrary to our assumption. Note that  $[a, x_i] = 2x_i$  for  $i \leq n$  (Proposition 8 in [Bo, Sect. 8.11.4]). Furthermore, since any  $x_j$ ,  $j \geq n+1$ , is a Lie bracket of simple elements one can check via the Jacobi identity that  $[a, x_j] = sx_j$  where  $s$  is an even number greater than 2. If we assume that  $d_j^+ \neq 0$  then a linear combination  $x_i + cx_j$ ,  $c \neq 0$ , is contained in  $\mathfrak{r}'_+$  for some  $x_i$ ,  $i \leq n$ . Taking a Lie bracket with  $a$  we see that  $2x_i + scx_j \in \mathfrak{r}'_+$ . Hence  $x_j \in \mathfrak{r}'_+$ , i.e.,  $d_j^+ = 0$  which is absurd. Thus  $d_k^+ \neq 0$  only for  $k \leq n$ . We can suppose that  $d_i^+ \neq 0$  for  $i \leq l_o \leq n$  and  $d_j^+ = 0$  for any  $j \geq l_o + 1$ . Note that  $h_j \in \mathfrak{h}'$ . If  $l_o \geq 3$  pick any three distinct numbers  $i, j$ , and  $k \leq l_o$ . Then up to nonzero coefficients  $x_i + x_j \in \mathfrak{r}'_+$  and  $y_i + y_k \in \mathfrak{r}'_-$ . Hence  $h_i = [x_i + x_j, y_i + y_k] \in \mathfrak{h}'$ , i.e.,  $\mathfrak{h}' = \mathfrak{h}$ . In this case we can find  $h \in \mathfrak{h}'$  such that  $[h, x_i] = s_i x_i$  and  $[h, x_j] = s_j x_j$  with  $s_i \neq s_j$ . As before this implies that  $x_i \in \mathfrak{r}'_+$ , which is a contradiction. Thus we can suppose that at most  $d_1^+$  and  $d_2^+$  are different from zero.

If  $l_o \leq 2$  and  $n \geq 3$  we can suppose that  $[x_2, x_3]$  is a nonzero nilpotent element. The direct computation shows that up to nonzero coefficients  $[[x_2, x_3], [y_2, y_3]]$  coincides with  $h_2 - h_3$ . Since  $h_3 \in \mathfrak{h}'$ , so is  $h_2$ . The same argument works for  $h_1$ , i.e.,  $\mathfrak{h}' = \mathfrak{h}$  again which leads to a contradiction as before. If  $n = 2$  then the rank  $m$  of  $R$  is 1 (since we do not want  $\mathfrak{h}' = \mathfrak{h}$ ), i.e.,  $R$  is either  $\mathbb{C}^*$  or  $\mathrm{SL}_2$ . In both cases  $\dim R < \dim G - n + m - 2$  contrary to Lemma 31 which yields the desired conclusion.  $\square$

Combining this result with Definition 6 and Proposition 28 we get the following.

**Corollary 33.** *Let  $G$  be a simple Lie group and let  $R$  be its reductive nonprincipal subgroup. Then for any principal  $\mathrm{SL}_2$ -subgroup  $\Gamma < G$  we have a finite  $\Gamma^{g_0}$  for some*

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$g_0 \in G$  and  $\Gamma^g$  is different from  $g^{-1}\Gamma g$  for any  $g \in G$ .

**Lemma 34.** *Let  $R$  be a principal subgroup of  $G$ . Then there exists an  $\mathrm{SL}_2$ -subgroup  $\Gamma < G$  such that  $\Gamma^{g_0}$  is finite for some  $g_0 \in G$  and  $\Gamma^g$  is different from  $g^{-1}\Gamma g$  for any  $g \in G$ .*

*Proof.* Recall that the subregular nilpotent orbit is the unique nilpotent orbit of codimension  $\mathrm{rank} \mathfrak{g} + 2$  in  $\mathfrak{g}$  [CM, Sect. 4.1]. It can be characterized as the unique open orbit in the boundary of the principal nilpotent orbit. The corresponding  $\mathfrak{sl}_2$ -triple  $(X, H, Y)$  in  $\mathfrak{g}$  is also called subregular. The dimension of the centralizer of the semisimple subregular element  $H$  in this triple is  $\mathrm{rank} \mathfrak{g} + 2$  [CM]. We denote the subregular  $\mathrm{SL}_2$  subgroup of  $G$  by  $\Gamma_{\mathrm{sr}}$ .

$G$	$R$	$\mathrm{rank} G + 3$	$\dim G - \dim R$
$B_3$	$G_2$	6	7
$D_4$	$G_2$	7	14
$A_6$	$G_2$	9	34
$E_6$	$F_4$	9	26
$A_{2l-1}$	$C_l$	$2l + 2$	$l(2l - 1) - 1$
$A_{2l}$	$B_l$	$2l + 3$	$2l^2 + 3l$
$D_l$	$B_{l-1}$	$l + 3$	$2l - 1$

We will demonstrate that in the cases listed in the above table, no conjugates of  $\Gamma_{\mathrm{sr}}$  can belong to  $R$ . Then by Lemma 31 the statement of the current lemma follows whenever  $\dim G - \dim R > \mathrm{rank} G + 3$ . From the table above we see that it covers all the principal embeddings from the proof of Proposition 28, with the exceptions of the inclusions  $B_3 \subset D_4$  and  $C_2 \subset A_3$ .

For  $G = A_r$ , the subregular  $\mathfrak{sl}_2$  corresponds to the partition  $(r, 1)$ . If  $r$  is odd, this partition is not symplectic (since in symplectic partitions all odd entries occur with even multiplicity), and if  $r$  is even, this partition is not orthogonal (since in orthogonal partitions all even entries occur with even multiplicity). In other words, the subregular  $\mathrm{SL}_2$ -subgroup  $\Gamma_{\mathrm{sr}}$  in  $A_{2l-1}$  (resp.,  $A_{2l}$ ) does not preserve any nondegenerate symplectic (resp., orthogonal) form on  $\mathbb{C}^{2l}$  ( $\mathbb{C}^{2l+1}$ ) and thus does not belong to  $R = C_l$  (resp.,  $R = B_l$ ). The same is true for any conjugate of  $\Gamma_{\mathrm{sr}}$  in  $G$ .

If  $G = D_l$ , the embedding of  $R = \mathrm{SO}_{2l-1}$  in  $\mathrm{SO}_{2l}$  is defined by the choice of the nonisotropic vector  $v \in \mathbb{C}^{2l}$  which is fixed by  $R$ . The subregular  $\mathfrak{sl}_2$  in  $\mathfrak{so}_{2l}$  corresponds to the partition  $(2l - 3, 3)$ . Thus we see that  $\Gamma_{\mathrm{sr}} \subset \mathrm{SO}_{2l}$  does not fix any one-dimensional subspace in  $\mathbb{C}^{2l}$  (its invariant subspaces have dimensions  $2l - 3$  and  $3$ ) and hence none of its conjugates can belong to  $R$ . Moreover, we can choose  $v$  such that  $xv \neq v$  for  $x \in \Gamma_{\mathrm{sr}}$ ,  $x \neq 1$ . Thus  $\Gamma_{\mathrm{sr}} \cap \mathrm{SO}_{2l-1} = \{e\}$ . This establishes the desired conclusion for the embeddings  $\mathrm{SO}_7 \subset \mathrm{SO}_8$  (i.e., the case of  $B_3 \subset D_4$ ) and  $\mathrm{SO}_5 \subset \mathrm{SO}_6$  (i.e., the case of  $B_2 \simeq C_2 \subset A_3 \simeq D_3$ ), in which the dimension count of Lemma 31 by itself is not sufficient.

The alignments of  $\mathfrak{sl}_2$ -triples in the exceptional cases were analyzed in [LMW]. In particular, it was observed there that any conjugacy class of  $\mathfrak{sl}_2$ -triples in  $\mathfrak{f}_4$  lifts uniquely to a conjugacy class of  $\mathfrak{sl}_2$ -triples in  $\mathfrak{e}_6$ . Consulting the explicit

correspondence given in [LMW, Sect. 2.2], we observe that the largest nonprincipal nilpotent orbit in  $\mathfrak{e}_6$  which has nonempty intersection with  $\mathfrak{f}_4$  has codimension 10. This implies that no  $\mathfrak{sl}_2$ -triple in  $\mathfrak{f}_4$  lifts to a subregular  $\mathfrak{sl}_2$  in  $\mathfrak{e}_6$ . In other words, no conjugate of  $\Gamma_{\text{sr}} \subset \mathbf{E}_6$  belongs to  $F_4$ .

When  $R = \mathbf{G}_2$ , its embedding in  $\mathbf{SO}_8$  is defined by the triality automorphism  $\tau : \mathbf{SO}_8 \rightarrow \mathbf{SO}_8$ , with  $R$  being a fixed point group of this automorphism. Equivalently,  $R = \mathbf{SO}_7 \cap \tau(\mathbf{SO}_7)$ . In particular, only those  $\mathfrak{sl}_2$ -triples in  $\mathfrak{so}_8$  which are fixed under the triality automorphism belong to  $\mathfrak{g}_2$ . Observe that the subregular  $(5, 3)$   $\mathfrak{sl}_2$ -triple is not fixed by triality (cf. Remark 2.6 in [LMW]). Similarly, the subregular  $\mathfrak{sl}_2$ -triple in  $\mathfrak{so}_7$  corresponds to the partition  $(5, 1, 1)$ . Since the  $(5, 1, 1, 1)$   $\mathfrak{sl}_2$ -triple in  $\mathfrak{so}_8$  is not invariant under triality, neither is subregular  $\mathfrak{sl}_2$  in  $\mathfrak{so}_7$ . Thus no conjugates of  $\Gamma_{\text{sr}}$  in  $\mathbf{B}_3$  or  $\mathbf{D}_4$  are fixed by  $\tau$ , and no conjugates of  $\Gamma_{\text{sr}}$  belong to  $\mathbf{G}_2$ .

Finally, when  $G = \mathbf{A}_6$ , the subregular triple in  $\mathbf{A}_6$  does not belong to  $\mathbf{B}_3$  (see above), and thus none of its conjugates lie in  $\mathbf{G}_2 \subset \mathbf{B}_3$ .  $\square$

Now Theorem 24 follows immediately from the combined statements of Corollary 33 and Lemma 34.

*Remark 8.* Note that we proved slightly more than required. Namely, the  $\mathbf{SL}_2$ -subgroup  $\Gamma$  in Theorem 24 can be chosen either principal or subregular.

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