EMBEDDINGS OF $\mathbb{C}^*$-SURFACES INTO WEIGHTED PROJECTIVE SPACES

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Abstract. Let V be a normal affine surface which admits a $\mathbb{C}^*$- and a $\mathbb{C}_+$-action. Such surfaces were classified e.g., in [FlZa_1, FlZa_2], see also the references therein. In this note we show that in many cases V can be embedded as a principal Zariski open subset into a hypersurface of a weighted projective space. In particular, we recover a result of D. Daigle and P. Russell, see Theorem A in [DR]. weighted projective space, $\mathbb{C}^*$-action, $\mathbb{C}_+$-action, affine surface

1. Introduction

If $V = \operatorname{Spec} A$ is a normal affine surface equipped with an effective $\mathbb{C}^*$-action, then its coordinate ring A carries a natural structure of a $\mathbb{Z}$-graded ring $A = \bigoplus_{i\in\mathbb{Z}} A_i$. As was shown in [$\mathrm{FlZa}_1$], such a $\mathbb{C}^*$-action on V has a hyperbolic fixed point if and only if $C = \operatorname{Spec} A_0$ is a smooth affine curve and $A_{\pm 1} \neq 0$. The structure of the graded ring A can be elegantly described in this case in terms of a pair (D_+, D_-) of $\mathbb{Q}$-divisors on C with $D_+ + D_- \le 0$. More precisely, A is the graded subring

$$A = A_0[D_+, D_-] \subseteq K_0[u, u^{-1}]\,, \quad K_0 := \operatorname{Frac} A_0,$$

where for $i \ge 0$

$$(1)\quad A_i = \{f \in K_0 \mid \operatorname{div} f + iD_+ \ge 0\}\, u^i \quad \text{and} \quad A_{-i} = \{f \in K_0 \mid \operatorname{div} f + iD_- \ge 0\}\, u^{-i}\,.$$

This presentation of A (or V) is called in [FlZa_1] a *DPD-presentation*. Furthermore two pairs (D_+, D_-) and (D'_+, D'_-) define equivariantly isomorphic surfaces over C if and only if they are *equivalent* that is,

$$D_+ = D'_+ + \operatorname{div} f \quad \text{and} \quad D_- = D'_- - \operatorname{div} f \quad \text{for some } f \in K_0^\times\,.$$

Our main result (Theorem 2.4) states that if such a surface V admits also a $\mathbb{C}_+$-action then it can be $\mathbb{C}^*$-equivariantly embedded (up to normalization) into a weighted projective space as a hypersurface minus a hyperplane; see also Remark 2.5 and Corollary 2.6 below. In particular we recover the following difficult result of Daigle and Russell (see [DR, Theorem A]; cf. also Remark 3.4 below).

Theorem 1.1. *Let V be a normal Gizatullin surface*[1] *with a finite divisor class group. Then V can be embedded into a weighted projective plane $\mathbb{P}(a, b, c)$ minus a hypersurface. More precisely:*

(a) If $V = V_{d,e}$ is toric[2] *then V is equivariantly isomorphic to the open part*[3] *$\mathbb{D}_+(z)$ of the weighted projective plane $\mathbb{P}(1, e, d)$ equipped with homogeneous coordinates $(x : y : z)$ and with the 2-torus action $(\lambda_1, \lambda_2).(x : y : z) = (\lambda_1 x : \lambda_2 y : z)$.*

[1] That is, V admits a completion by a linear chain of smooth rational curves; see Section 3 below.
[2] See 3.1(a) below.
[3] We use the standard notation $\mathbb{V}_+(f) = \{f = 0\}$ and $\mathbb{D}_+(f) = \{f \neq 0\}$.

(b) If V is non-toric then $V \cong \mathbb{D}_+(xy - z^m) \subseteq \mathbb{P}(a, b, c)$ for some positive integers a, b, c satisfying $a + b = cm$ and $\gcd(a, b) = 1$.

2. Embeddings of $\mathbb{C}^*$-surfaces into weighted projective spaces

According to Proposition 4.8 in [FlZa$_1$] every normal affine $\mathbb{C}^*$-surface V is equivariantly isomorphic to the normalization of a weighted homogeneous surface V' in $\mathbb{A}^4$. In some cases (described in *loc.cit.*) V' can be chosen to be a hypersurface in $\mathbb{A}^3$. Cf. also [Du] for affine embeddings of some other classes of surfaces.

In Theorem 2.4 below (see also Remark 2.5) we show that any normal hyperbolic $\mathbb{C}^*$-surface V with a $\mathbb{C}_+$-action is the normalization of a principal Zariski open subset of some weighted projective hypersurface.

For our purposes it is convenient to consider also weighted projective spaces with any weights in $\mathbb{Z}$ as introduced in [BS]. More precisely, if A is a finitely generated $\mathbb{Z}$-graded algebra over $\mathbb{C}$ then we can form $\operatorname{Proj} A$ to be the scheme covered by the affine pieces $D_+(f) = \operatorname{Spec} A_{(f)}$, where $f \in A$ is homogeneous of non-zero degree and $A_{(f)} = (A_f)_0$. In particular for any $d_0, \dots, d_n \in \mathbb{Z}$ we can form a weighted projective space $\mathbb{P}(d_0, \dots, d_n) = \operatorname{Proj} \mathbb{C}[T_0, \dots, T_d]$, where $\deg T_i = d_i$ for $i = 0, \dots, d$. We note that this space is in general not complete.

In the proofs we use the following observation from [Fl]; this Proposition was formulated in *loc.cit.* only for positively graded algebras. We note that this result – with exactly the same proof – is also valid for $\mathbb{Z}$-graded rings as stated here.

Proposition 2.1. *Let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be a graded R_0-algebra of finite type containing the field of rational numbers $\mathbb{Q}$ and the group $E_d \cong \mathbb{Z}/d\mathbb{Z}$ of dth roots of unity, where $d > 0$. If $z \in R_d$ then E_d acts on R and then also on $R/(z-1)$ via*

$$\zeta.a = \zeta^i \cdot a \quad \text{for} \quad a \in R_i,\ \zeta \in E_d,$$

with ring of invariants $(R/(z-1))^{E_d} \cong (R[1/z])_0$. Consequently

$$(\operatorname{Spec} R/(z-1))/E_d \cong \mathbb{D}_+(z)$$

is isomorphic to the complement of the hypersurface $\{z = 0\}$ in $\operatorname{Proj}(R)$.

We also recall the following result.

Proposition 2.2. *Let $V = \operatorname{Spec} A$ be a normal hyperbolic $\mathbb{C}^*$-surface with DPD-presentation*

$$A = A_0[D_+, D_-] \subseteq \operatorname{Frac}(A_0)[u, u^{-1}],$$

where (D_+, D_-) is a pair of $\mathbb{Q}$-divisors on the curve $C = \operatorname{Spec} A_0$ with $D_+ + D_- \le 0$. Then the following are equivalent.

(a) V carries a $\mathbb{C}_+$-action;

(b) $A_0 \cong \mathbb{C}[t]$, and after interchanging (D_+, D_-), if necessary, the fractional part $\{D_+\}$ of D_+ is supported at one point.

For a proof we refer the reader to [FlZa$_2$], Corollary 3.23.

2.3. We let now $V = \operatorname{Spec} A_0[D_+, D_-]$ be a normal hyperbolic $\mathbb{C}^*$-surface carrying also a $\mathbb{C}_+$-action. Using Proposition 2.2 we can assume that $A_0 = \mathbb{C}[t]$ and that, after interchanging (D_+, D_-) and passing to an equivalent pair, if necessary,

$$\begin{aligned} D_+ &= -\tfrac{e_+}{d}[0] \quad \text{with} \quad 0 < e_+ \le d\,, \\ D_- &= -\tfrac{e_-}{d}[0] - \tfrac{1}{k}D_0 \quad \text{with} \quad k > 0,\ e_+ + e_- \ge 0 \end{aligned} \tag{2}$$

and an integral effective divisor D_0, where $D_0(0) = 0$. We choose a polynomial $Q \in \mathbb{C}[t]$ with $D_0 = \operatorname{div}(Q)$; so $Q(0) \neq 0$.

Theorem 2.4. *Let F be the polynomial*

$$F = x^k y - s^{k(e_+ + e_-)} Q(s^d/z) z^{\deg Q} \in \mathbb{C}[x, y, z, s]\,, \tag{3}$$

which is weighted homogeneous of degree $k(e_+ + e_-) + d \deg Q$ with respect to the weights

$$\deg x = e_+\,, \quad \deg y = ke_- + d \deg Q\,, \quad \deg z = d\,, \quad \deg s = 1\,. \tag{4}$$

Then the surface V as in 2.3 above is equivariantly isomorphic to the normalization of the principal Zariski open subset $\mathbb{D}_+(z)$ of the hypersurface $\mathbb{V}_+(F)$ in the weighted projective 3-space

$$\mathbb{P} = \mathbb{P}(e_+, ke_- + d \deg Q, d, 1)\,. \tag{5}$$

Proof. With $s = \sqrt[d]{t}$ the field $L = \operatorname{Frac}(A)[s]$ is a cyclic extension of $K = \operatorname{Frac}(A)$. Its Galois group is the group of dth roots of unity E_d acting on L via the identity on K and by $\zeta.s = \zeta \cdot s$ if $\zeta \in E_d$. The normalization A' of A in L is stabilized by the action of E_d with invariant ring $A = A'^{E_d}$. According to Proposition 4.12 in [FlZa$_1$]

$$A' = \mathbb{C}[s][D'_+, D'_-] \subseteq \mathbb{C}(s)[u, u^{-1}]$$

with $D'_\pm = \pi_d^*(D_\pm)$, where $\pi_d : \mathbb{A}^1 \to \mathbb{A}^1$ is the covering $s \mapsto s^d$. Thus

$$\big(D'_+,\, D'_-\big) = \left(-e_+[0],\, -e_-[0] - \frac{1}{k}\pi_d^*(D_0)\right) = \left(-e_+[0], -e_-[0] - \frac{1}{k}\operatorname{div}(Q(s^d))\right).$$

The element $x = s^{e_+}u \in A'_1$ is a generator of A'_1 as a $\mathbb{C}[s]$-module. According to Example 4.10 in [FlZa$_1$] the graded algebra A' is isomorphic to the normalization of

$$B = \mathbb{C}[x, y, s]/(x^k y - s^{k(e_+ + e_-)} Q(s^d))\,. \tag{6}$$

More precisely, B can be considered as the subalgebra of L generated over $\mathbb{C}$ by the elements

$$s, \quad x = s^{e_+}u, \quad \text{and} \quad y = x^{-k} s^{k(e_+ + e_-)} Q(s^d). \tag{7}$$

Here the action of E_d is given by

$$\zeta.s = \zeta s\,, \quad \zeta.x = \zeta^{e_+} x\,, \quad \zeta.y = \zeta^{ke_-} y\,.$$

In particular this action stabilizes B. Assigning to x, y, z, s the degrees as in (4), F as in (3) is indeed weighted homogeneous. Since $F(x, y, 1, s) = x^k y - s^{k(e_+ + e_-)} Q(s^d)$, the graded algebra

$$R = \mathbb{C}]x, y, z, s]/(F)$$

satisfies $R/(z-1) \cong B$. Applying Proposition 2.1 $\operatorname{Spec} B^{E_d}$ is isomorphic to $\mathbb{D}_+(z) \cap \mathbb{V}_+(F)$ in the weighted projective space $\mathbb{P}$. Thus the normalizations of $\operatorname{Spec} B^{E_d}$ and $\mathbb{D}_+(z) \cap \mathbb{V}_+(F)$ are isomorphic as well. As normalization commutes with taking invariants the normalization of B^{E_d} is just $A'^{E_d} = A$, proving our result. □

Remark 2.5. In general not all weights of the weighted projective space $\mathbb{P}$ in (5) are positive. Indeed it can happen that $ke_- + d\deg Q \le 0$. In this case we can choose $\alpha \in \mathbb{N}$ with $ke_- + d(\deg Q + \alpha) > 0$ and consider instead of F the polynomial

$$\tilde{F} = x^k y - s^{k(e_+ + e_-)} Q(s^d/z) z^{\deg Q + \alpha} \in \mathbb{C}[x, y, z, s]\,, \tag{8}$$

which is now weighted homogeneous of degree $k(e_+ + e_-) + d(\deg Q + \alpha)$ with respect to the *positive* weights

$$\deg x = e_+\,, \quad \deg y = ke_- + d(\deg Q + \alpha)\,, \quad \deg z = d\,, \quad \deg s = 1\,. \tag{9}$$

As before $V = \operatorname{Spec} A$ is isomorphic to the normalization of the principal open subset $\mathbb{D}_+(z)$ of the hypersurface $\mathbb{V}_+(F)$ in the weighted projective space

$$\mathbb{P} = \mathbb{P}(e_+, ke_- + d(\deg Q + \alpha), d, 1)\,.$$

In certain cases it is unnecessary in Theorem 2.4 to pass to normalization.

Corollary 2.6. *Assume that in (2) one of the following conditions is satisfied.*

(i) $k = 1$;

(ii) $e_+ + e_- = 0$, *and* D_0 *is a reduced divisor.*

Then $V = \operatorname{Spec} A$ *is equivariantly isomorphic to the principal open subset* $\mathbb{D}_+(z)$ *of the weighted projective hypersurface* $\mathbb{V}_+(F)$ *as in (3) in the weighted projective space* $\mathbb{P}$ *from (5).*

Proof. In case (i) the hypersurface in $\mathbb{A}^3$ with equation

$$F(x, y, 1, s) = xy - s^{e_+ + e_-} Q(s^d) = 0$$

is normal. In other words, the quotient $R/(z-1)$ of the graded ring $R = \mathbb{C}[x, y, z, s]/(F)$ is normal and so is its ring of invariants $(R/(z-1))^{E_d}$. Comparing with Theorem 2.4 the result follows.

Similarly, in case (ii)

$$F(x, y, 1, s) = x^k y - Q(s^d)\,.$$

Since the divisor D_0 is supposed to be reduced and $D_0(0) = 0$, the polynomials $Q(t)$ and then also $Q(s^d)$ both have simple roots. Hence the hypersurface $F(x, y, 1, s) = 0$ in $\mathbb{A}^3$ is again normal, and the result follows as before. □

Remark 2.7. The surface V as in 2.3 is smooth if and only if D_0 is reduced and $-m_+ m_-(D_+(0) + D_-(0)) = 1$, where $m_\pm > 0$ are the denominators in the irreducible representation of $D_\pm(0)$, see Proposition 4.15 in [FlZa$_1$]. It can happen, however, that V is smooth but the surface $\mathbb{V}_+(F) \cap \mathbb{D}_+(z) \subseteq \mathbb{P}$ has non-isolated singularities. For instance, if in 2.3 $D_0 = 0$ (and so $Q = 1$), then V is an affine toric surface[4]. In fact, every affine toric surface different from $(\mathbb{A}^1_*)^2$ or $\mathbb{A}^1 \times \mathbb{A}^1_*$ appears in this way, see Lemma 4.2(b) in [FKZ$_1$].

In this case the integer $k > 0$ can be chosen arbitrarily. For any $k > 1$, the affine hypersurface $V_+(F) \cap \mathbb{D}_+(z) \subseteq \mathbb{P}$ with equation $x^k y - s^{k(e_+ + e_-)} = 0$ has non-isolated singularities and hence is non-normal. Its normalization $V = \operatorname{Spec} A$ can be given as the Zariski open part $\mathbb{D}_+(z)$ of the hypersurface $V_+(xy' - s^{e_+ + e_-})$ in $\mathbb{P}' = \mathbb{P}(e_+, e_-, d, 1)$ (which corresponds to the choice $k = 1$). Indeed, the element $y' = s^{e_+ + e_-}/x \in K$ with $y'^k = y$ is integral over A. However cf. Theorem 1.1(a).

[4]See 3.1(a) below.

Example 2.8. (*Danilov-Gizatullin surfaces*) We recall that a Danilov-Gizatullin surface $V(n)$ of index n is the complement to a section S in a Hirzebruch surface Σ_d, where $S^2 = n > d$. By a remarkable result of Danilov and Gizatullin [DaGi, Theorem 5.8.1] up to an isomorphism such a surface only depends on n and neither on d nor on the choice of the section S; see also [CNR], [FKZ$_3$] for alternative proofs.

According to [FKZ$_1$, §5], up to conjugation $V(n)$ carries exactly $(n-1)$ different $\mathbb{C}^*$-actions. They admit DPD-presentations with $A_0 = \mathbb{C}[t]$ and

$$(D_+, D_-) = \left(-\frac{1}{d}[0],\, -\frac{1}{n-d}[1]\right), \quad \text{where} \quad d = 1, \ldots, n-1\,.$$

Applying Theorem 2.4 with $e_+ = 1$, $e_- = 0$, and $k = n - d$, the $\mathbb{C}^*$-surface $V(n)$ is the normalization of the principal open subset $\mathbb{D}_+(z)$ of the hypersurface $\mathbb{V}_+(F_{n,d}) \subseteq \mathbb{P}(1, d, d, 1)$ of degree n, where

$$F_{n,d}(x, y, z, s) = x^{n-d}y - s^{n-d}(s^d - z)\,.$$

Taking here $d = 1$ it follows that $V(n)$ is isomorphic to the normalization of the hypersurface $x^{n-1}y - (s-1)s^{n-1} = 0$ in $\mathbb{A}^3$.

As our next example, let us consider yet another remarkable class of surfaces. These were studied from different viewpoints e.g., in [MM, Theorem 1.1], [FlZa$_3$, Theorem 1.1(iii)], [GMMR, 3.8-3.9], [KK, Theorem 1.1. and Example 1], [Za, Theorem 1(b) and Lemma 7]. Collecting results from *loc.cit.* and from this section, we obtain the following equivalent characterizations.

Theorem 2.9. *For a smooth affine surface V, the following conditions are equivalent.*

(i) V is not Gizatullin and admits an effective $\mathbb{C}^*$-action and an $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$ with exactly one degenerate fiber, which is irreducible[5].

(ii) V is $\mathbb{Q}$-acyclic, $\bar{k}(V) = -\infty$ [6] *and V carries a curve $\Gamma \cong \mathbb{A}^1$ with $\bar{k}(V \setminus \Gamma) \geq 0$.*

(iii) V is $\mathbb{Q}$-acyclic and admits an effective $\mathbb{C}^*$- and $\mathbb{C}_+$-actions. Furthermore, the $\mathbb{C}^*$-action possesses an orbit closure $\Gamma \cong \mathbb{A}^1$ with $\bar{k}(V \setminus \Gamma) \geq 0$.

(iv) The universal cover $\tilde{V} \to V$ is isomorphic to a surface $x^k y - (s^d - 1) = 0$ in $\mathbb{A}^3$, with the Galois group $\pi_1(V) \cong E_d$ acting via $\zeta.(x, y, s) = (\zeta x, \zeta^{-k} y, \zeta^e s)$, where $k > 1$ and $\gcd(e, d) = 1$.

(v) V is isomorphic to the $\mathbb{C}^*$-surface with DPD presentation $\operatorname{Spec}\mathbb{C}[t][D_+, D_-]$, *where*

$$(D_+, D_-) = \left(-\frac{e}{d}[0],\, \frac{e}{d}[0] - \frac{1}{k}[1]\right) \quad \textit{with} \quad 0 < e \leq d,\ \gcd(e, d) = 1, \quad \textit{and} \quad k > 1\,.$$

(vi) V is isomorphic to the Zariski open subset [7]

$$\mathbb{D}_+(x^k y - s^d) \subseteq \mathbb{P}(e, d - ke, 1), \quad \textit{where} \quad 0 < e \leq d,\ \gcd(e, d) = 1, \quad \textit{and} \quad k > 1\,.$$

In view of the references cited above it remains to show that the surfaces in (v) and (vi) are isomorphic. By Corollary 2.6(ii) with $e_+ = -e_- = e$, the surface V as in (v) is

[5] Since V is not Gizatullin there is actually a unique $\mathbb{A}^1$-fibration $V \to \mathbb{A}^1$. A surface V as in (i) is necessarily a $\mathbb{Q}$-homology plane (or $\mathbb{Q}$-acyclic) that is, all higher Betti numbers of V vanish.

[6] As usual, $\bar{k}$ stands for the logarithmic Kodaira dimension.

[7] In the case where $d - ke < 0$, see Remark 2.5.

isomorphic to the principal open subset $\mathbb{D}_+(z)$ in the weighted projective hypersurface

$$V_+(x^k y - (s^d - z)) \subseteq \mathbb{P}(e, d-ke, d, 1)\,.$$

Eliminating z from the equation $x^k y - (s^d - z) = 0$ yields (vi).

These surfaces admit as well a constructive description in terms of a blowup process starting from a Hirzebruch surface, see [GMMR, 3.8] and [KK, Example 1].

An affine line $\Gamma \cong \mathbb{A}^1$ on V as in (ii) is distinguished because it cannot be a fiber of any $\mathbb{A}^1$-fibration of V. There is always a family of such affine lines on V, see [Za].

Some of the surfaces as in Theorem 2.9 can be properly embedded in $\mathbb{A}^3$ as *Bertin surfaces* $x^e y - x - s^d = 0$, see [FlZa$_2$, Example 5.5] or [Za, Example 1].

3. Gizatullin surfaces with a finite divisor class group

A *Gizatullin surface* is a normal affine surface completed by a zigzag i.e., a linear chain of smooth rational curves. By a theorem of Gizatullin [Gi] such a surface can be characterized by the property that it admits two $\mathbb{C}_+$-actions with different general orbits, unless it is isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1_*$.

In this section we give an alternative proof of the Daigle-Russell Theorem 1.1 cited in the Introduction. It will be deduced from the following result proven in [FKZ$_2$, Corollary 5.16].

Proposition 3.1. *Every normal Gizatullin surface with a finite divisor class group is isomorphic to one of the following surfaces.*

(a) The toric surfaces $V_{d,e} = \mathbb{A}^2/E_d$, where the group $E_d \cong \mathbb{Z}/d\mathbb{Z}$ of d-th roots of unity acts on $\mathbb{A}^2$ via

$$\zeta.(x,y) = (\zeta x, \zeta^e y)\,.$$

(b) The non-toric $\mathbb{C}^*$-surfaces $V = \operatorname{Spec} \mathbb{C}[t][D_+, D_-]$, where

$$(D_+, D_-) = \left(-\frac{e}{m}[p],\, \frac{e}{m}[p] - c[q]\right) \quad \text{with} \quad c \ge 1,\ \ p, q \in \mathbb{A}^1,\ \ p \ne q\,, \tag{10}$$

and with coprime integers e, m such that $1 \le e < m$.

Conversely, any normal affine $\mathbb{C}^*$-surface V as in (a) or (b) is a Gizatullin surface with a finite divisor class group.

Let us now deduce Theorem 1.1.

3.2. *Proof of Theorem 1.1.* To prove (a), we note that according to 2.1 the cyclic group E_d acts on the ring $\mathbb{C}[x,y,z]/(z-1) \cong \mathbb{C}[x,y]$ via $\zeta.x = \zeta x$, $\zeta.y = \zeta^e y$, and $\zeta.z = z$, where

$$\deg x = 1\,, \quad \deg y = e, \quad \text{and} \quad \deg z = d\,.$$

Hence $\mathbb{D}_+(z) = \operatorname{Spec} \mathbb{C}[x,y]^{E_d} = V_{d,e}$, as required in (a).

To show (b) we consider $V = \operatorname{Spec} A$ as in 3.1(b), where

$$A = \mathbb{C}[t][D_+, D_-] \subseteq \mathbb{C}(t)[u, u^{-1}]\,.$$

By definition (see (1)) the homogeneous pieces $A_{\pm 1}$ of A are generated as $\mathbb{C}[t]$-modules by the elements

$$u_+ = tu \quad \text{and} \quad u_- = (t-1)^c u^{-1}\,,$$

and similarly $A_{\pm m}$ by

$$v_+ = t^e u^m \quad \text{and} \quad v_- = t^{-e}(t-1)^{cm} u^{-m}\,.$$

Thus

$$u_+^m = t^{m-e}v_+, \quad u_-^m = t^e v_-, \quad \text{and} \quad u_+u_- = t(t-1)^c\,.$$

The algebra A is the integral closure of the subalgebra generated by $u_\pm$, $v_\pm$ and t.

Consider now the normalization A' of A in the field $L = \mathrm{Frac}(A)[u'_+]$, where

$$u'_+ = \sqrt[d]{v_+} \quad \text{with} \quad d = cm\,. \tag{11}$$

Clearly the elements $\sqrt[m]{v_+} = t^{\frac{e-m}{m}}u_+$ and then also $t^{\frac{e-m}{m}}$ both belong to L. Since e and m are coprime we can choose $\alpha, \beta \in \mathbb{Z}$ with $\alpha(e-m)+\beta m = 1$. It follows that the element $\tau := t^{\frac{1}{m}} = t^{\alpha\frac{e-m}{m}}t^\beta$ is as well in L whence being integral over A we have $\tau \in A'$.

The element u'_+ as in (11) and then also $u'_- := \sqrt[d]{v_-} = (t-1)(\sqrt[d]{v_+})^{-1}$ belongs to A'. Now $v_+v_- = (t-1)^{cm}$, so taking dth roots we get for a suitable choice of the root u'_-,

$$u'_+u'_- = \tau^m - 1\,. \tag{12}$$

We note that $u_\pm$, $v_\pm$ and t are contained in the subalgebra $B = \mathbb{C}[u'_+, u'_-, \tau] \subseteq A'$. The equation (12) defines a smooth surface in $\mathbb{A}^3$. Hence B is normal and so

$$A' = B \cong \mathbb{C}[u'_+, u'_-, \tau]/(u'_+u'_- - (\tau^m - 1))\,.$$

By Lemma 3.3 below, for a suitable $\gamma \in \mathbb{Z}$ the integers $a = e - \gamma m$ and d are coprime. We may assume as well that $1 \le a < d$. We let E_d act on A' via $\zeta.u'_+ = \zeta^a u'_+$ and $\zeta|A = \mathrm{id}_A$. Since $\gcd(a,d) = 1$, A is the invariant ring of this action. We claim that the action of E_d on (u'_+, u'_-, τ) is given by

$$\zeta.u'_+ = \zeta^a u'_+, \quad \zeta.u'_- = \zeta^{-a}u'_- = \zeta^b u'_- \quad \text{and} \quad \zeta.\tau = \zeta^c\tau\,, \tag{13}$$

where $b = d - a$. Indeed, the equality $u'^c_+ = t^{\frac{e-m}{m}}u_+ = \tau^{e-m}u_+$ implies that $\zeta.\tau^{e-m} = \zeta^{ac}\tau^{e-m}$. Since $\tau = \tau^{\alpha(e-m)}t^\beta$ the element $\zeta \in E_d$ acts on τ via $\zeta.\tau = \zeta^{\alpha c a}\tau$. In view of the congruence $\alpha a \equiv 1 \mod m$ the last expression equals $\zeta^c\tau$. Now the last equality in (13) follows. In the equation $u'_+u'_- = \tau^m - 1$ the term on the right is invariant under E_d. Hence also the term on the left is. This provides the second equality in (13).

The algebra $B = \mathbb{C}[u'_+, u'_-, \tau]$ is naturally graded via

$$\deg u'_+ = a, \quad \deg u'_- = b, \quad \text{and} \quad \deg \tau = c\,.$$

According to Proposition 2.1 $\mathrm{Spec}\, A = \mathrm{Spec}\, A'^{E_d}$ is the complement of the hypersurface $\mathbb{V}_+(f)$ of degree $d = a + b$ in the weighted projective plane

$$\mathbb{P}(a,b,c), \quad \text{where} \quad f = u'_+u'_- - \tau^m\,,$$

proving (b).

To complete the proof we still have to show the following elementary lemma.

Lemma 3.3. *Assume that $e, m \in \mathbb{Z}$ are coprime. Then for every $c \ge 2$ there exists $\gamma \in \mathbb{Z}$ such that $\gamma m - e$ and c are coprime.*

Proof. Write $c = c'd$ such that c' and m are coprime and every prime factor of d divides m. Then for any $\gamma \in \mathbb{Z}$ the integers $\gamma m - e$ and d are coprime. Hence it is enough to establish the existence of $\gamma \in \mathbb{Z}$ such that $\gamma m - e$ and c' are coprime. However, the latter is evident since the residue classes of γm, $\gamma \in \mathbb{Z}$, in $\mathbb{Z}/c'\mathbb{Z}$ cover this group. □

Remark 3.4. We can also recover the criterion given in Theorem A(3) in [DR] for when two surfaces as in Theorem 1.1 are isomorphic. More precisely we can argue in the cases (a) and (b) of this theorem as follows.

(a) It is a classical fact that two toric surfaces $V_{d,e}$ and $V_{d',e'}$ are isomorphic if and only if $(d,e)=(d',e')$ or $d=d'$ and $ee'\equiv 1 \mod d$, see e.g. [FlZa$_1$, Remark 2.5]. Hence two triples $(1,e,d)$ and $(1,e',d')$ as in Theorem 1.1(a) define isomorphic surfaces if and only if $(d,e)=(d',e')$ or $d=d'$ and $ee'\equiv 1 \mod d$. We note that here the abstract isomorphism type and equivariant isomorphism type amount to the same.

(b) As follows from Theorem 0.2 in [FKZ$_2$], the integers c,m in Theorem 1.1(b) are invariants of the (abstract) isomorphism type of V. Indeed, the fractional parts of both divisors $D_\pm$ as in (10) being nonzero and concentrated at the same point, there is a unique DPD presentation for V up to interchanging D_+ and D_-, passing to an equivalent pair and applying an automorphism of the affine line $\mathbb{A}^1=\operatorname{Spec}\mathbb{C}[t]$.

Furthermore, from the proof of Theorem 1.1 one can easily derive that

$$a\equiv e \mod m \quad\text{and}\quad b=mc-a\equiv -e \mod m\,.$$

Therefore also the pair (a,b) is uniquely determined by the (abstract) isomorphism type of V up to a transposition and up to replacing (a,b) by $(a',b')=(a-sm,b+sm)$, while keeping $\gcd(a',b')=1$.

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