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SHADOWING IN ORDINARY DIFFERENTIAL EQUATIONS

Abstract. Shadowing deals with the existence of true orbits of dynamical systems near approximate orbits with sufficiently small local errors. Although it has roots in abstract dynamical systems, recent developments have made shadowing into a new effective tool for rigorous computer-assisted analysis of specific dynamical systems, especially chaotic ones. For instance, using shadowing it is possible to prove the existence of various unstable periodic orbits, transversal heteroclinic or homoclinic orbits of arguably the most prominent chaotic system—the Lorenz Equations. In this paper we review the current state of the theory and applications of shadowing for ordinary differential equations, with particular emphasis on our own work.

1. Introduction

In this extended introductory section we give a narrative overview of shadowing. Precise mathematical statements of relevant definitions and theorems are presented in the following sections.

Shadowing? An approximate orbit of a dynamical system with small local errors is called a *pseudo orbit*. The subject of shadowing concerns itself with the existence of true orbits near pseudo orbits; in particular, the initial data of the true orbit are near the initial data of the pseudo orbit. Shadowing is a property of hyperbolic sets of dynamical systems. It is akin to a classical result from the theory of ordinary differential equations [23]: if a nonautonomous system has a bounded solution, the variational equation of which admits an exponential dichotomy, then the perturbed system has a bounded solution nearby.

Origins? In its contemporary setting, the first significant result in shadowing, the celebrated *Shadowing Lemma*, was proved by Bowen [7] for the nonwandering sets of Axiom A diffeomorphisms. A similar result for Anosov diffeomorphisms was stated by Anosov [2], which was made more explicit by Sinai [57]. The Shadowing Lemma proved to be a useful tool in the abstract theory of uniformly hyperbolic sets of diffeomorphisms. For example, in [8] Markov partitions for basic sets of Axiom A diffeomorphisms were constructed using the Shadowing Lemma. Also shadowing can be used to give a simple proof of Smale's theorem [59] that the shift can be embedded in the neighbourhood of a transversal homoclinic point as in [47] and [39].

Discrete systems? Since Anosov and Bowen, the Shadowing Lemma has been reproved many times with a multitude of variants. A noteworthy recent development, initiated by Hammel et al. [28], [29], has been a shadowing theory for finite pseudo orbits of non-uniformly hyperbolic sets of diffeomorphisms which proved to be a powerful new paradigm for extracting rigorous results from numerical simulations of discrete chaotic systems. This development also provided new tools for establishing the existence of, for example, transversal homoclinic orbits in specific systems [21], [37]. For these recent shadowing results for discrete dynamical systems, we recommend our review article [18] followed by [21] and the references therein.

ODEs? Developing a useful notion of a pseudo orbit and establishing an appropriate Shadowing Lemma for ordinary differential equations proved to be more difficult because of the lack of hyperbolicity in the direction of the vector field. The first successful attempt in this pursuit and an accompanying Shadowing Lemma was given by Franke and Selgrade [26]. Here we will present our formulation of pseudo orbits and shadowing for ordinary differential equations as initiated in [14], [16]. Our formulation has the advantage that pseudo orbits are taken to be sequences of points and thus can be generated numerically. This permits one to garner rigorous mathematical results with the assistance of numerical simulations. Such computer-assisted shadowing techniques make an attractive complement to classical numerical analysis, especially in the investigation of specific chaotic systems.

Chaotic numerics? The key signature of chaotic systems is the sensitivity of their solutions to initial data. This poses a major challange in numerical anaysis of chaotic systems because such systems tend to amplify, often exponentially, small algorithmic or floating point errors. Here is a gloomy account of this difficulty as given by Hairer et al. [30]:

> "The solution (of the Salzman-Lorenz equations with constants and initial values $\sigma = 10$, $r = 28$, $b = 8/3$; $x(0) = -8$, $y(0) = 8$, $z(0) = 27$) is, for large values of t, *extremely* sensitive to the errors of the first integration steps. For example, at $t = 50$ the solution becomes totally wrong, even if the computations are performed in quadruple precision with $Tol = 10^{-20}$. Hence the numerical results of all methods would be equally useless and no comparison makes any sense. Therefore, we choose $t_{end} = 16$ and check the numerical solution at this point. Even here, all

computations with $Tol > 10^{-7}$, say, fall into a chaotic cloud of meaningless results."

Shadowing reveals a striking silverlining of this "chaotic cloud." While it is true that this chaotic cloud has little to do with the solution having the specified initial data, it is not meaningless: the chaotic cloud is an exceedingly good approximation of another solution whose initial data is very close to the specified initial data. More generally, using the finite-time shadowing theorem in [17], it is possible to shadow numerically generated pseudo orbits of (non-uniformly hyperbolic) chaotic ordinary differential equations for long time intervals.

Chaos? There are many ways chaos can arise in a dynamical system. A common cause, as first observed by Poincaré [51] over a century ago while studying the resticted three-body problem, is the presence of transversal homoclinic points. He called such points "doubly asymptotic" because they are asymptotic both in forward and backward times to a fixed point or a periodic orbit. Birkhoff [4] proved that every homoclinic point of a two-dimensional diffeomorphism is accumulated by periodic orbits. Smale [59] confirmed Poincaré's observation by proving that a transversal homoclinic point of a diffeomorphism in dimension two and higher is contained in a hyperbolic set in which the periodic orbits are infinitely many and dense. Sil'nikov [55] showed that a similar result holds for flows. Recently, it has been conjectured by Palis and Takens [45] that generically chaotic orbits occur if and only if there is a transversal homoclinic orbit. This is indeed the case for continuous interval maps as shown in [6].

In spite of the remarkable mathematical results above, transversal homoclinic orbits are quite difficult to exhibit in specific chaotic flows. Even the periodic orbits, to which the homoclinic orbits are to be doubly asymptotic, are hard to come by. Recently in [22], we have formulated a practical notion of a pseudo homoclinic, more generally pseudo connecting, orbit and proved a shadowing theorem that guarantees the existence of transversal homoclinic, or heteroclinic, orbits to periodic orbits of differential equations. The hypotheses of this theorem can be verified for specific flows with the aid of a computer, thus enabling us to prove the existence of a multitude of periodic orbits and transversal orbits connecting them in, for example, yet again, the chaotic Lorenz Equations.

Contents? Here is a section-by-section description of the contents of the remainder of this paper:

- In Section 2, we first give definitions of an infinite pseudo orbit and its shadowing by a true orbit. Then we present two infinite-time shadowing results, one for pseudo orbits lying in hyperbolic invariant sets, and another for a single pseudo orbit in terms of a certain operator.

- In Section 3, shadowing definitions for finite pseudo orbits and a Finite-time Shadowing Theorem for non-uniformly hyperbolic systems are formulated. This theorem is significant in proving the existence of true orbits near numerically computed ones for long time intervals.
- In Section 4, the shadowing of pseudo periodic orbits is considered. The Periodic Shadowing Theorem stated here is very effective in establishing the existence of periodic orbits, including unstable ones in dimensions three and higher.
- In Section 5, the notion of a pseudo connecting orbit connecting two pseudo periodic orbits is formulated. Then a Connection Orbit Shadowing Theorem that guarantees the existence and tranversality of a true connecting orbit between true periodic orbits is stated. In the particular case when the two periodic orbits coincide we have a Homoclinic Shadowing Theorem, with the aid of which existence of chaos, in the sense of Poincaré, can be rigorously established in specific systems.
- In Section 6, some of the key computational issues such as construction of good pseudo orbits, rigorous bounds on quantities that appear in the hypotheses of the shadowing theorems, and floating point computations are addressed.
- In Section 7, we present several examples to demonstrate the effectiveness of the shadowing results above as a new computer-assisted technique for establishing rigorously finite, periodic, and transversal homoclinic orbits in the quintessential chaotic system—the Lorenz Equations.
- In Section 8, we conclude our review with some parting thoughts.

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2. Infinite-time Shadowing

In this section we first introduce notions of an infinite pseudo orbit and its shadowing by a true orbit of a system of autonomous differential equations. Then we state two general results that guarantee the shadowing of infinite pseudo orbits.

The first result, Theorem 1, in the spirit of the classical Shadowing Lemma, is a shadowing theorem for pseudo orbits lying in a compact hyperbolic set. The second result, Lemma 1, replaces the hyperbolicity assumption with the invertibility of a certain linear operator. The second result is more general; in fact, the classical theorem follows from it. Moreover, the second result has practical applicability in numerical simulations as we shall demonstrate in later sections. Whilst the hyperbolicity assumption on a compact invariant set is not possible to verify in any realistic example (strange attractors are not usually uniformly hyperbolic), the invertibility of the operator associated with a particular pseudo orbit can frequently be established.

Consider a continuous dynamical system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \tag{1}$$

where $f : U \to \mathbb{R}^n$ is a C^2 vector field defined in an open convex subset U of $\mathbb{R}^n$. Let ϕ^t be the associated flow. Throughout this paper we use the Euclidean norm for vectors and the corresponding operator norm for matrices and linear operators, and in product spaces we use the maximum norm.

Definition 1. **Definition of infinite pseudo orbit**. *For a given positive number δ, a sequence of points $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ in U, with $f(\mathbf{y}_k) \neq \mathbf{0}$ for all k, is said to be a δ* pseudo orbit *of Eq. (1) if there is an associated bounded sequence $\{h_k\}_{k=-\infty}^{+\infty}$ of positive times with positive $\inf_{k\in\mathbb{Z}} h_k$ such that*

$$\|\mathbf{y}_{k+1} - \varphi^{h_k}(\mathbf{y}_k)\| \leq \delta \quad \textit{for } k \in \mathbb{Z}.$$

Next, we introduce the notion of shadowing an infinite pseudo orbit by a true orbit.

Definition 2. **Definition of infinite-time shadowing**. *For a given positive number ε, a δ pseudo orbit $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ of Eq. (1) with associated times $\{h_k\}_{k=-\infty}^{+\infty}$ is said to be ε-*shadowed *by a true orbit of Eq. (1) if there are points $\{\mathbf{x}_k\}_{k=-\infty}^{+\infty}$ on the true orbit and positive times $\{t_k\}_{k=-\infty}^{+\infty}$ with $\varphi^{t_k}(\mathbf{x}_k) = \mathbf{x}_{k+1}$ such that*

$$\|\mathbf{x}_k - \mathbf{y}_k\| \leq \varepsilon \quad \textit{and} \quad |t_k - h_k| \leq \varepsilon \quad \textit{for } k \in \mathbb{Z}.$$

In our first Shadowing Theorem we will assume that pseudo orbits lie in a compact hyperbolic set. For completeness, we recall the definition of a hyperbolic set as given in, for example, [47].

Definition 3. **Definition of hyperbolic set**. *A set $S \subset U$ is said to be* hyperbolic *for Eq. (1) if*

(i) $f(\mathbf{x}) \neq 0$ for all $\mathbf{x}$ in S;

(ii) S is invariant under the flow, that is, $\phi^t(S) = S$ for all t;

(ii) there is a continuous splitting

$$\mathbb{R}^n = E^0(\mathbf{x}) \oplus E^s(\mathbf{x}) \oplus E^u(\mathbf{x}) \quad \textit{for } \mathbf{x} \in S$$

such that $E^0(\mathbf{x})$ is the one-dimensional subspace spanned by $\{f(\mathbf{x})\}$, and the subspaces $E^s(\mathbf{x})$ and $E^u(\mathbf{x})$ have constant dimensions; moreover, these subspaces have the invariance property

$$D\phi^t(\mathbf{x})(E^s(\mathbf{x})) = E^s(\phi^t(\mathbf{x})), \quad D\phi^t(\mathbf{x})(E^u(\mathbf{x})) = E^u(\phi^t(\mathbf{x}))$$

under the linearized flow and the inequalities

$$\begin{aligned} \|D\phi^t(\mathbf{x})\,\xi\| &\leq K_1 e^{-\alpha_1 t}\|\xi\| \quad \textit{for } t \geq 0,\ \xi \in E^s(\mathbf{x}), \\ \|D\phi^t(\mathbf{x})\,\xi\| &\leq K_2 e^{\alpha_2 t}\|\xi\| \quad \textit{for } t \leq 0,\ \xi \in E^u(\mathbf{x}) \end{aligned}$$

are satisfied for some positive constants K_1, K_2, α_1, and α_2.

Now, we can state our first shadowing theorem for infinite pseudo orbits of ordinary differential equations.

THEOREM 1. **Infinite-time Shadowing Theorem**. *Let S be a compact hyperbolic set for Eq. (1). For a given sufficiently small $\varepsilon > 0$, there is a $\delta > 0$ such that any δ pseudo orbit $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ of Eq. (1) lying in S is ε-shadowed by a true orbit $\{\mathbf{x}_k\}_{k=-\infty}^{+\infty}$. Moreover, there is only one such orbit satisfying*

$$f(\mathbf{y}_k)^*(\mathbf{x}_k - \mathbf{y}_k) = 0 \quad \textit{for } k \in \mathbb{Z}.$$

In preparation for our second infinite shadowing result, we next introduce various mathematical entities. Take a fixed pseudo orbit $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ with associated times $\{h_k\}_{k=-\infty}^{+\infty}$. Let Y_k be the subspace of $\mathbb{R}^n$ consisting of the vectors orthogonal to $f(\mathbf{y}_k)$. Then let Y be the Banach space of bounded sequences $\mathbf{v} = \{\mathbf{v}_k\}_{k\in\mathbb{Z}}$ with $\mathbf{v}_k \in Y_k$, and equip Y with the norm

$$\|\mathbf{v}\| = \sup_{k\in\mathbb{Z}} \|\mathbf{v}_k\|.$$

Also, let $\tilde{Y}$ be a similar Banach space except that $\mathbf{v}_k \in Y_{k+1}$. Then let

$$L_{\mathbf{y}} : Y \to \tilde{Y}$$

be the linear operator defined by

$$(L_{\mathbf{y}}\mathbf{v})_k = \mathbf{v}_{k+1} - P_{k+1} D\phi^{h_k}(\mathbf{y}_k)\mathbf{v}_k,$$

where $P_k : \mathbb{R}^n \to \mathbb{R}^n$ is the orthogonal projection defined by

$$P_k\mathbf{v} = \mathbf{v} - \frac{f(\mathbf{y}_k)^*\mathbf{v}}{\|f(\mathbf{y}_k)\|^2}\, f(\mathbf{y}_k).$$

So $L_{\mathbf{y}}$ is a linear operator associated with the derivative of the flow along the pseudo orbit, but restricted to the subspaces orthogonal to the vector field. This operator plays a key role in what follows. We assume that the operator is invertible with a bounded inverse:

$$\|L_{\mathbf{y}}^{-1}\| \leq K.$$

Next we define various constants. We begin with

$$M_0 = \sup_{\mathbf{x}\in U} \|f(\mathbf{x})\|, \qquad M_1 = \sup_{\mathbf{x}\in U} \|Df(\mathbf{x})\|, \qquad M_2 = \sup_{\mathbf{x}\in U} \|D^2 f(\mathbf{x})\|,$$

and

$$h_{\min} = \inf_{k\in \mathbb{Z}} h_k, \quad h_{\max} = \sup_{k\in \mathbb{Z}} h_k.$$

Next, we choose a positive number $\varepsilon_0 \leq h_{\min}$ such that for all k and $\|\mathbf{x}-\mathbf{y}_k\| \leq \varepsilon_0$ the solution $\varphi^t(\mathbf{x})$ is defined and remains in U for $0 \leq t \leq h_k+\varepsilon_0$. Continuing, we define

$$\Delta = \inf_{k\in \mathbb{Z}} \|f(\mathbf{y}_k)\|, \quad \overline{M}_0 = \sup_{k\in \mathbb{Z}} \|f(\mathbf{y}_k)\|, \quad \overline{M}_1 = \sup_{k\in \mathbb{Z}} \|Df(\mathbf{y}_k)\|.$$

Now, we define the following constants in terms of the ones already given:

$$\begin{aligned} C &= \max\left\{K,\ \Delta^{-1}(e^{M_1 h_{\max}} K + 1)\right\}, \\ \overline{C} &= (1 - M_1\delta C)^{-1} C, \end{aligned}$$

and

$$\begin{aligned} N_1 &= 8\left[\overline{M}_0\overline{M}_1 + 2\overline{M}_1 e^{M_1(h_{\max}+\varepsilon_0)} + M_2(h_{\max}+\varepsilon_0)e^{2M_1(h_{\max}+\varepsilon_0)}\right], \\ N_2 &= 8\left[1 + 4C\left(M_0 + e^{M_1(h_{\max}+\varepsilon_0)}\right)\right]\left[M_1\overline{M}_1 + M_2\overline{M}_0 + 2M_2 e^{M_1(h_{\max}+\varepsilon_0)}\right], \\ N_3 &= 8\left[1 + 4C\left(M_0 + e^{M_1(h_{\max}+\varepsilon_0)}\right)\right]^2 M_1 M_2. \end{aligned}$$

Now, we can state our second shadowing result for an infinite pseudo orbit in terms of the associated operator $L_{\mathbf{y}}$ and the constants introduced above.

LEMMA 1. **Infinite-time Shadowing Lemma**. *Let $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ be a bounded δ pseudo orbit of Eq. (1) with associated times $\{h_k\}_{k=-\infty}^{+\infty}$ such that $L_{\mathbf{y}}$ is invertible with $\|L_{\mathbf{y}}^{-1}\| \leq K$. Then if*

$$4C\delta < \varepsilon_0, \qquad 2M_1 C\delta \leq 1, \qquad C^2(N_1\delta + N_2\delta^2 + N_3\delta^3) < 1,$$

the pseudo orbit $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ is ε-shadowed by a true orbit $\{\mathbf{x}_k\}_{k=-\infty}^{+\infty}$ of Eq. (1) with associated times $\{t_k\}_{k=-\infty}^{+\infty}$ and with

$$\varepsilon \leq 2\bar{C}\delta.$$

Moreover this is the unique such orbit satisfying

$$f(\mathbf{y}_k)^*(\mathbf{x}_k - \mathbf{y}_k) = 0 \quad \textit{for } k \in \mathbb{Z}.$$

Notes on infinite-time shadowing: The details of the proof of the Infinite-time Shadowing Lemma are given in [22]. The idea of the proof is to set up the problem of finding a true orbit near the pseudo orbit as the solution of a nonlinear equation in a Banach space of sequences. The invertibility of the linear operator $L_{\mathbf{y}}$ implies the invertibility of another linear operator associated with the abstract problem and this enables one to apply a Newton-Kantorovich type theorem [33] to obtain the existence and uniqueness of the true orbit.

One uses the Infinite-time Shadowing Lemma to prove the Infinite-time Shadowing Theorem. The main problem is to show that hyperbolicity implies that the operator $L_{\mathbf{y}} : Y \to \tilde{Y}$ is invertible with a uniform bound on its inverse. A slightly different proof of the Infinite-time Shadowing Theorem is in an earlier publication [16].

For flows, unlike diffeomophisms, there are various alternatives for the definition of a pseudo orbit. Should it be a sequence of points or solution segments that are functions of time? Here we have elected to use sequences of points. These choices of definitions have obvious advantages when we consider finite pseudo orbits that come from numerical computations. With such considerations in mind, the definitions of a pseudo orbit and shadowing for ordinary differential equations as given here first appeared in [14] and [16]. The problem in proving the shadowing theorem for flows is the lack of hyperbolicity in the direction of the vector field. To compensate for this, we allow a rescaling of time in our definition of shadowing.

A shadowing theorem for flows, with a somewhat different notion of shadowing, was first proved in [26]. Different versions of the shadowing theorem have been proved in [42], [34], [35], [16], and [49]. For a "continuous" shadowing theorem, see [47] and also [49]. In the latter book the author also proves a shadowing theorem for structurally stable systems, which includes that for hyperbolic systems.

3. Finite-time Shadowing

Since chaotic systems exhibit sensitive dependence on initial conditions, a numerically generated orbit will diverge quickly from the true orbit with the same initial condition. However, we observe that a computed orbit is a pseudo orbit and so, according to our Infinite-time Shadowing Lemma, would be shadowed by a true orbit in the presence of hyperbolicity, albeit with slightly different initial condition. It turns out chaotic systems are seldom uniformly hyperbolic but still exhibit enough hyperbolicity that pseudo orbits can still be shadowed for long times. To this end, we formulate a finite-time shadowing theorem in this section. As in the Infinite-time Shadowing Lemma, our condition involves a linear operator associated with the pseudo orbit but now the aim is to choose a right inverse with small norm. First we recall a precise notion of shadowing of a finite pseudo orbit by an associated nearby true orbit. Then we present the Finite-time Shadowing Theorem.

Definition 4. **Definition of finite pseudo orbit.** *For a given positive number δ, a sequence of points $\{\mathbf{y}_k\}_{k=0}^{N}$ is said to be a δ* pseudo orbit *of Eq. (1) if $f(\mathbf{y}_k) \neq \mathbf{0}$ and there is an associated sequence $\{h_k\}_{k=0}^{N-1}$ of positive times such that*

$$\|\mathbf{y}_{k+1} - \varphi^{h_k}(\mathbf{y}_k)\| \leq \delta \quad \textit{for } k = 0, \ldots, N-1.$$

Definition 5. **Definition of finite-time shadowing.** *For a given positive number ε, an orbit of Eq. (1) is said to ε-*shadow *a δ pseudo orbit $\{\mathbf{y}_k\}_{k=0}^{N}$ with associated times $\{h_k\}_{k=0}^{N-1}$ if there are points $\{\mathbf{x}_k\}_{k=0}^{N}$ on the true orbit and times $\{t_k\}_{k=0}^{N-1}$ with $\varphi^{t_k}(\mathbf{x}_k) = \mathbf{x}_{k+1}$ such that*

$$\|\mathbf{x}_k - \mathbf{y}_k\| \leq \varepsilon \quad \textit{for } k = 0, \ldots, N \quad \textit{and} \quad |t_k - h_k| \leq \varepsilon \quad \textit{for } k = 0, \ldots, N-1.$$

To state our theorem we need to develop a bit of notation and introduce certain relevant mathematical constructs. Let $\{\mathbf{y}_k\}_{k=0}^{N}$ be a δ pseudo orbit of Eq. (1) with associated times $\{h_k\}_{k=0}^{N-1}$. With the subspaces Y_k and the projections P_k defined as in Section 2, we define a linear operator

$$L_{\mathbf{y}} \colon Y_0 \times \cdots \times Y_N \to Y_1 \times \cdots \times Y_N$$

in the following way: If $\mathbf{v} = \{\mathbf{v}_k\}_{k=0}^{N}$ is in $Y_0 \times \cdots \times Y_N$, then we take $L_{\mathbf{y}}\mathbf{v} = \{[L_{\mathbf{y}}\mathbf{v}]_k\}_{k=0}^{N-1}$ to be

$$[L_{\mathbf{y}}\mathbf{v}]_k = \mathbf{v}_{k+1} - P_{k+1} D\phi^{h_k}(\mathbf{y}_k)\mathbf{v}_k \qquad \textit{for } k = 0, \ldots, N-1.$$

The operator $L_{\mathbf{y}}$ has right inverses and we choose one such right inverse $L_{\mathbf{y}}^{-1}$ with

$$\|L_{\mathbf{y}}^{-1}\| \leq K.$$

Also we define constants as before Lemma 1 with the range of k being appropriately adjusted. Now we can state our theorem.

Theorem 2. **Finite-time Shadowing Theorem.** *Let $\{\mathbf{y}_k\}_{k=0}^{N}$ be a δ pseudo orbit of Eq. (1) with associated times $\{h_k\}_{k=0}^{N-1}$ and let $L_{\mathbf{y}}^{-1}$ be a right inverse of the operator $L_{\mathbf{y}}$ with $\|L_{\mathbf{y}}^{-1}\| \leq K$. Then if*

$$4C\delta < \varepsilon_0, \qquad 2M_1 C\delta \leq 1, \qquad C^2(N_1\delta + N_2\delta^2 + N_3\delta^3) < 1,$$

the pseudo orbit $\{\mathbf{y}_k\}_{k=0}^{N}$ is ε-shadowed by a true orbit $\{\mathbf{x}_k\}_{k=0}^{N}$ with

$$\varepsilon \leq 2\bar{C}\delta.$$

Notes on finite-time shadowing: The proof of the Finite-time Shadowing Theorem above is quite similar to that of Lemma 1 except that here we use Brouwer's fixed point theorem rather than the contraction mapping principle.

It was first observed in [28], [29] that pseudo orbits of certain chaotic maps could be shadowed for long times by true orbits, despite the lack of uniform hyperbolicity. Others, [9], [10], and [52] realised that these observations could be generalized using shadowing techniques. Here the key idea is the construction of a right inverse of small norm for a linear operator similar to the one used for infinite-time shadowing. The choice of this right inverse is guided by the infinite-time case—one takes the formula for the inverse in the infinite-time case and truncates it appropriately (see [18]). However, the ordinary differential equation case is somewhat more complicated. It is not simply a matter of looking at the time-one map and applying the theory for the map case. One must somehow "quotient-out" the direction of the vector field and allow for rescaling of time as is done in Theorem 2; this needs to be done because of lack of hyperbolicity in the direction of the vector field. That this leads to much better shadowing results is shown in [17]. For other finite-time shadowing theorems in the context of autonomous ordinary differential equations, see [11], [12], [32], [62].

4. Periodic Shadowing

In simulations of differential equations apparent periodic orbits, usually asymptotically stable, are often calculated. In this section we show how shadowing can be used to verify that there do indeed exist true periodic orbits near the computed orbits. Our method can be applied even to unstable periodic orbits which are ubiqitous in chaotic systems. We first recall the notions of pseudo periodic orbit and periodic shadowing for autonomous ordinary differential equations. Then, we state a Periodic Shadowing Theorem which guarantees the existence of a true periodic orbit near a pseudo periodic orbit.

DEFINITION 6. **Definition of pseudo periodic orbit.** *For a given positive number δ, a sequence of points $\{\mathbf{y}_k\}_{k=0}^N$, with $f(\mathbf{y}_k) \neq \mathbf{0}$ for all k, is said to be a δ* pseudo periodic orbit *of Eq. (1) if there is an associated sequence $\{h_k\}_{k=0}^N$ of positive times such that*

$$\|\mathbf{y}_{k+1} - \varphi^{h_k}(\mathbf{y}_k)\| \leq \delta \quad \text{for } k = 0, \ldots, N-1,$$

and

$$\|\mathbf{y}_0 - \varphi^{h_N}(\mathbf{y}_N)\| \leq \delta.$$

DEFINITION 7. **Definition of periodic shadowing.** *For a given positive number ε, a δ pseudo periodic orbit $\{\mathbf{y}_k\}_{k=0}^N$ with associated times $\{h_k\}_{k=0}^N$ is said to be ε-*shadowed *by a true periodic orbit if there are points $\{\mathbf{x}_k\}_{k=0}^N$ and positive times $\{t_k\}_{k=0}^N$ with $\varphi^{t_k}(\mathbf{x}_k) = \mathbf{x}_{k+1}$ for $k = 0, \ldots, N-1$, and $\mathbf{x}_0 = \varphi^{t_N}(\mathbf{x}_N)$ such that*

$$\|\mathbf{x}_k - \mathbf{y}_k\| \leq \varepsilon \quad \text{and} \quad |t_k - h_k| \leq \varepsilon \quad \text{for } k = 0, \ldots, N.$$

To decide if a pseudo periodic orbit is shadowed by a true periodic orbit we need to compute certain other quantities. Let $\{\mathbf{y}_k\}_{k=0}^N$ be a δ pseudo periodic orbit of Eq. (1) with associated times $\{h_k\}_{k=0}^N$. With the subspaces Y_k and the projections P_k defined as in Section 2, we define a linear operator

$$L_{\mathbf{y}}\colon Y_0 \times Y_1 \times \cdots \times Y_N \to Y_1 \times \cdots \times Y_N \times Y_0$$

as follows: if $\mathbf{v} = \{\mathbf{v}_k\}_{k=0}^N$ then

$$\begin{aligned} (L_{\mathbf{y}}\mathbf{v})_k &= \mathbf{v}_{k+1} - P_{k+1} D\phi^{h_k}(\mathbf{y}_k)\mathbf{v}_k, \quad for\ k = 0, \ldots, N-1 \\ (L_{\mathbf{y}}\mathbf{v})_N &= \mathbf{v}_0 - P_N D\phi^{h_N}(\mathbf{y}_N)\mathbf{v}_N. \end{aligned}$$

We assume the operator $L_{\mathbf{y}}$ is invertible with $\|L_{\mathbf{y}}^{-1}\| \leq K$. Also we define constants as before Lemma 1 with the range of k being appropriately adjusted. Now, we can state our main theorem.

THEOREM 3. **Periodic Shadowing Theorem**. *Let $\{\mathbf{y}_k\}_{k=0}^N$ be a δ pseudo periodic orbit of the autonomous system Eq. (1) such that the operator $L_{\mathbf{y}}$ is invertible with $\|L_{\mathbf{y}}^{-1}\| \leq K$. Then if*

$$4C\delta < \varepsilon_0, \qquad 2M_1C\delta \leq 1, \qquad C^2(N_1\delta + N_2\delta^2 + N_3\delta^3) < 1,$$

the pseudo periodic orbit $\{\mathbf{y}_k\}_{k=0}^N$ is ε-shadowed by a true periodic orbit $\{\mathbf{x}_k\}_{k=0}^N$ of Eq. (1) with associated times $\{t_k\}_{k=0}^N$ and with

$$\varepsilon \leq 2\bar{C}\delta.$$

Moreover, this is the unique such orbit satisfying

$$f(\mathbf{y}_k)^*(\mathbf{x}_k - \mathbf{y}_k) = 0 \quad for\ 0 \leq k \leq N.$$

Notes on periodic shadowing: The proof of the Periodic Shadowing Theorem above is in [15]. Computation of periodic orbits with long periods requires special care; this situation is addressed in [19].

Normally one expects periodic orbits to be plentiful in chaotic invariant sets. One would like to be able to prove that periodic orbits are dense in attractors like those for the Lorenz Equations. This we have not been able to do. However, in [15], [19] we were able to use shadowing techniques to prove the existence of various periodic orbits of the Lorenz Equations, including orbits with long periods. We will display some of these periodic orbits in Section 7.

The idea of using computer assistance for rigorously establishing the existence of periodic orbits occurred to other people before us. For instance, Franke and Selgrade [27] gave a computer-assisted method to rigorously prove the existence of a periodic orbit of a two-dimensional autonomous system. Other relevant studies are [1], [43], [44], [54], [58], and [61].

Using periodic shadowing techniques, one can also provide rigorous estimates for the Lyapunov exponents of periodic orbits. Details of such computations, along with the Lyapunov exponents of several periodic orbits of the Lorenz Equations, are given in [19].

5. Homoclinic Shadowing

General theorems proving the existence of a transversal homoclinic orbit to a hyperbolic periodic orbit, or a transversal heteroclinic orbit connecting one periodic orbit to another periodic orbit, of a flow are few. In this section we present two such theorems. In Section 7 we will show the effective use of these theorems on specific systems.

We first introduce the definition of an infinite pseudo periodic orbit, which is just the finite pseudo periodic orbit in Definiton 6 extended periodically. This is more convenient for our purposes in this section.

Definition 8. **Definition of infinite pseudo periodic orbit.** *A sequence* $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ *with associated times* $\{\ell_k\}_{k=-\infty}^{+\infty}$ *is said to be a* δ pseudo periodic orbit *of period* $N \geq 1$ *of Eq. (1) if* $\inf_{k\in\mathbb{Z}} \ell_k > 0$, $f(\mathbf{y}_k) \neq \mathbf{0}$ *for all* k *and*

$$\|\mathbf{y}_{k+1} - \phi^{\ell_k}(\mathbf{y}_k)\| \leq \delta \quad \text{and} \quad \mathbf{y}_{k+N} = \mathbf{y}_k, \ \ \ell_{k+N} = \ell_k \quad for \ k \in \mathbb{Z}.$$

Now we define shadowing of such a pseudo periodic orbit.

Definition 9. **Definition of periodic shadowing.** *For a given positive number* ε*, an infinite* δ *pseudo periodic orbit* $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ *with associated times* $\{\ell_k\}_{k=-\infty}^{+\infty}$ *is said to be* ε-shadowed *by a true periodic orbit if there are points* $\{\mathbf{x}_k\}_{k=-\infty}^{+\infty}$ *and positive times* $\{t_k\}_{k=-\infty}^{+\infty}$ *such that* $\mathbf{x}_{k+N} = \mathbf{x}_k$ *and* $t_{k+N} = t_k$ *and* $\varphi^{t_k}(\mathbf{x}_k) = \mathbf{x}_{k+1}$ *for all* $k \in \mathbb{Z}$ *and*

$$\|\mathbf{x}_k - \mathbf{y}_k\| \leq \varepsilon \quad and \quad |t_k - h_k| \leq \varepsilon \quad for \ k \in \mathbb{Z}.$$

We next formalize the definition of a pseudo connecting orbit, homoclinic or heteroclinic, connecting one pseudo periodic orbit to another.

Definition 10. **Definition of pseudo connecting orbit.** *Consider two* δ *pseudo periodic orbits* $\{\bar{\mathbf{y}}_k\}_{k=-\infty}^{+\infty}$ *and* $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ *with associated times* $\{\bar{\ell}_k\}_{k=-\infty}^{+\infty}$ *and* $\{\ell_k\}_{k=-\infty}^{+\infty}$*. An infinite sequence* $\{\mathbf{w}_k\}_{k=-\infty}^{+\infty}$ *with associated times* $\{h_k\}_{k=-\infty}^{+\infty}$ *is said to be a* δ pseudo connecting orbit *connecting* $\{\bar{\mathbf{y}}_k\}_{k=-\infty}^{+\infty}$ *to* $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ *if* $f(\mathbf{w}_k) \neq \mathbf{0}$ *for all* k *and*

(i) $\|\mathbf{w}_{k+1} - \phi^{h_k}(\mathbf{w}_k)\| \leq \delta$ *for* $k \in \mathbb{Z}$,

(ii) $\mathbf{w}_k = \bar{\mathbf{y}}_k$, $h_k = \bar{\ell}_k$ *for* $k \leq p$ *and* $\mathbf{w}_k = \mathbf{y}_k$, $h_k = \ell_k$ *for* $k \geq q$ *for some integers* $p < q$.

In particular, $\{\mathbf{w}_k\}_{k=-\infty}^{+\infty}$ *is said to be a* pseudo homoclinic orbit *if there exists* τ, $0 \leq \tau < N$*, such that* $\bar{\mathbf{y}}_k = \mathbf{y}_{k+\tau}$ *and* $\bar{\ell}_k = \ell_{k+\tau}$ *for all* k.

Shadowing of a pseudo connecting orbit is defined as in Definition 2 for infinite-time shadowing.

Let $L_{\mathbf{w}} : Y \to \tilde{Y}$ be the linear operator defined by

$$(L_{\mathbf{w}}\mathbf{v})_k = \mathbf{v}_{k+1} - P_{k+1} D\phi^{h_k}(\mathbf{w}_k)\mathbf{v}_k,$$

where P_k, Y and $\tilde{Y}$ are defined as in Section 2 with $\mathbf{w}$ replacing $\mathbf{y}$. We assume the operator is invertible and that $\|L_{\mathbf{w}}^{-1}\| \leq K$. Also we define constants as before Lemma 1 with $\mathbf{y}$ replaced by $\mathbf{w}$.

With the definitions and notations above, here we state our main shadowing theorems for connecting pseudo orbits. The first theorem guarantees the existence of a true hyperbolic connecting orbit near a pseudo connecting orbit. Note that an orbit is *hyperbolic* if and only if the invariant set defined by it is hyperbolic as in Definition 3.

THEOREM 4. **Connecting Orbit Shadowing Theorem**. *Suppose that $\{\bar{\mathbf{y}}_k\}_{k=-\infty}^{+\infty}$ and $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ are two δ pseudo periodic orbits with periods $\bar{N}$ and N, respectively, of Eq. (1). Let $\{\mathbf{w}_k\}_{k=-\infty}^{+\infty}$ be a δ pseudo connecting orbit of Eq. (1) with associated times $\{h_k\}_{k=-\infty}^{+\infty}$ connecting $\{\bar{\mathbf{y}}_k\}_{k=-\infty}^{+\infty}$ to $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$. Suppose that the operator $L_{\mathbf{w}}$ is invertible with*

$$\|L_{\mathbf{w}}^{-1}\| \leq K.$$

Then if

$$4C\delta < \varepsilon_0, \quad 4M_1C\delta \leq \min\{2,\ \Delta\}, \quad C^2\left(N_1\delta + N_2\delta^2 + N_3\delta^3\right) < 1,$$

(i) the pseudo periodic orbits $\{\bar{\mathbf{y}}_k\}_{k=-\infty}^{+\infty}$ and $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ are ε-shadowed by true periodic orbits $\{\bar{\mathbf{x}}_k\}_{k=-\infty}^{\infty}$ of period $\bar{N}$ and $\{\mathbf{x}_k\}_{k=-\infty}^{\infty}$ of period N where

$$\varepsilon \leq 2\bar{C}\delta,$$

moreover, $\phi^t(\bar{\mathbf{x}}_0)$ and $\phi^t(\mathbf{x}_0)$ are hyperbolic (non-equilibrium) periodic orbits;

(ii) the pseudo connecting orbit $\{\mathbf{w}_k\}_{k=-\infty}^{+\infty}$ above is also ε-shadowed by a true orbit $\{\mathbf{z}_k\}_{k=-\infty}^{\infty}$. Moreover, $\phi^t(\mathbf{z}_0)$ is hyperbolic and there are real numbers $\bar{\alpha}$ and α such that $\|\phi^t(\mathbf{z}_0) - \phi^{t+\bar{\alpha}}(\bar{\mathbf{x}}_0)\| \to 0$ as $k \to -\infty$ and $\|\phi^t(\mathbf{z}_0) - \phi^{t+\alpha}(\mathbf{x}_0)\| \to 0$ as $t \to \infty$.

In the special case of the theorem above when the two periodic orbits coincide, we obtain a transversal homoclinic orbit as hyperbolicity of the connecting orbit implies its transversality. Now, an additional condition is required to ensure that the connecting orbit does not coincide with the periodic orbit. The condition given in the theorem below is that there be a point on the pseudo homoclinic orbit sufficiently distant from the pseudo periodic orbit. With the setting as in the previous theorem, we state the following theorem.

THEOREM 5. **Homoclinic Orbit Shadowing Theorem**. *Suppose that $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ is a δ pseudo periodic orbit with period N of Eq. (1). Let $\{\mathbf{w}_k\}_{k=-\infty}^{+\infty}$*

be a δ pseudo homoclinic orbit of Eq. (1) with associated times $\{h_k\}_{k=-\infty}^{+\infty}$ connecting $\{\bar{\mathbf{y}}_k\}_{k=-\infty}^{+\infty} = \{\mathbf{y}_{k+\tau}\}_{k=-\infty}^{+\infty}$ to $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$, where $0 \le \tau < N$. Suppose that the operator $L_{\mathbf{w}}$ is invertible with

$$\|L_{\mathbf{w}}^{-1}\| \le K.$$

Then if

$$4C\delta < \varepsilon_0, \quad 4M_1C\delta \le \min\{2,\ \Delta\}, \quad C^2\left(N_1\delta + N_2\delta^2 + N_3\delta^3\right) < 1,$$

(i) the pseudo periodic orbit $\{\mathbf{y}_k\}_{k=-\infty}^{+\infty}$ above is ε-shadowed by a true periodic orbit $\{\mathbf{x}_k\}_{k=-\infty}^{\infty}$ of period N where

$$\varepsilon \le 2\bar{C}\delta,$$

moreover, $\phi^t(\mathbf{x}_0)$ is a hyperbolic (non-equilibrium) periodic orbit;

(ii) the pseudo homoclinic orbit $\{\mathbf{w}_k\}_{k=-\infty}^{+\infty}$ above is also ε-shadowed by a true orbit $\{\mathbf{z}_k\}_{k=-\infty}^{\infty}$. Moreover, $\phi^t(\mathbf{z}_0)$ is hyperbolic and there are real numbers $\bar{\alpha}$ and α such that $\|\phi^t(\mathbf{z}_0) - \phi^{t+\bar{\alpha}}(\mathbf{x}_0)\| \to 0$ as $k \to -\infty$ and $\|\phi^t(\mathbf{z}_0) - \phi^{t+\alpha}(\mathbf{x}_0)\| \to 0$ as $t \to \infty$. Furthermore, provided there exists r with $p < r < q$ such that

$$\|\mathbf{w}_r - \mathbf{y}_k\| > (\|f(\mathbf{y}_k)\| + 2M_1\bar{C}\delta)\frac{e^{M_1(h_{\mathbf{max}}+\varepsilon_0)-1}}{M_1} + 4\bar{C}\delta$$

for $0 \le k \le N-1$, then $\mathbf{z}_r$ does not lie on the orbit of $\mathbf{x}_0$ and so we may conclude that $\mathbf{z}_0$ is a transversal homoclinic point associated with the periodic orbit $\phi^t(\mathbf{x}_0)$.

Notes on homoclinic shadowing: We prove the theorems above partly using earlier theorems. First we prove the existence of the periodic orbits using the Periodic Shadowing Theorem. Then we use the Infinite-time Shadowing Lemma to show the existence of a unique orbit shadowing the pseudo homoclinic orbit. However, here we have to show this orbit is asymptotic to the periodic orbits. It turns out that this can be proved by a compactness argument using the uniqueness. Moreover, we need to show the transversality as well. This follows from the hyperbolicity which is proved by a rather involved argument. Detailed proofs of these theorems are available in our forthcoming paper [22].

Examples of flows with transversal connecting orbits are scarce. The papers [36], [38], and [64] have examples of connecting orbits in celestial mechanics. Note that [36] also employs shadowing methods but applies a theorem for a sequence of maps (see also [49] for such a theorem) rather than a theorem specifically for differential equations. There are also studies using shooting methods combined with interval arithmetic that attempt to establish the existence of connecting orbits; see, for example, [60] where such orbits in the Lorenz Equations are computed. In Section 7 we will exhibit the existence of a transversal homoclinic orbit in the Lorenz Equations for the classical parameter values.

There are a number of studies for effectively computing accurate approximations to finite segments of orbits of flows connecting two periodic orbits. For example, [25] and [48], inspired by [3], approximate a connecting orbit by the solution of a certain boundary value problem and they derive estimates for the error in the approximation. However, all existing work is carried out on the assumption that a true connecting orbit exists. In contrast, our Connecting and Homoclinic Shadowing Theorems provide a new computer-assisted method for rigorously establishing the existence of such orbits.

According to Sil'nikov's theorem [55], [56], and [46], the existence of a transversal homoclinic orbit implies chaos. A single transversal heteroclinic orbit does not imply chaos. However, a cycle of transversal heteroclinic orbits does imply chaos. Our Connecting Orbit Shadowing Theorem can be used to prove the existence of such cycles in, for example, the Lorenz Equations, as demonstrated in [22].

6. Implementation Issues

There are two main computational issues in applying the shadowing theorems we have presented in the previous sections:

(i) Small δ: finding a suitable pseudo orbit with sufficently small rigorous local error bound;

(ii) $||L^{-1}|| \leq K$: verifying the invertibility of the operator L (or finding a suitable right inverse) and calculating a rigorous upper bound K on the norm of the inverse.

In this section we highlight certain key ideas regarding these computational considerations, at suitable points directing the reader to references where further details can be found.

(i) Finding a suitable pseudo orbit: In the case of finite-time shadowing, one could be tempted to use a sophisticated numerical integration method with local error tolerance control to generate a good pseudo orbit. However, in order to claim the existence of a true orbit near the computed approximate orbit, we need a *rigorous* bound on the local discretization error δ. We have found a high-order Taylor Method to be the most effective numerical integration method for this purpose. To get a rigorous δ, one must also account for the floating point errors in the calculation of $\phi^{h_k}(\mathbf{y}_k)$, which we handle using the techniques of Wilkinson [65]. Details of the implementation of the Taylor Method with floating error estimates for the Lorenz Equations are given in [17].

In the case of periodic or homoclinic shadowing, it is very difficult to find a pseudo periodic orbit or pseudo homoclinic orbit with δ small enough by a routine use of a numerical integrator using simple shooting, that is, various initial conditions are tried until one is found with a small δ. Usually the δ found in this way is not small enough to apply our theorems. To get a pseudo orbit with a smaller δ, we refine the "crude" pseudo orbit with a suitable global Newton's method. "Global" means we work with the whole pseudo orbit, not just its initial point since it turns out that working with just the initial point is not effective.

Now we describe what we do in the periodic case. Let $\{\mathbf{y}_k\}_{k=0}^N$ be a δ pseudo periodic orbit of Eq. (1), found perhaps by simple shooting, or by concatenating segments of several orbits. In general, the δ associated with such a crude pseudo orbit will not be sufficiently small to apply our Periodic Orbit Shadowing Theorem. We want to replace this pseudo periodic orbit by a nearby one with a smaller δ. Ideally there would be a nearby sequence of points $\{\mathbf{x}_k\}_{k=0}^N$ and a sequence of times $\{t_k\}_{k=0}^N$ such that

$$\begin{aligned} \mathbf{x}_{k+1} &= \phi^{t_k}(\mathbf{x}_k) \quad \text{for } k = 0, \ldots, N-1 \\ \mathbf{x}_0 &= \varphi^{t_N}(\mathbf{x}_N). \end{aligned}$$

We write $\mathbf{x}_k = \mathbf{y}_k + \mathbf{z}_k$, where $\mathbf{z}_k$ is orthogonal to $f(\mathbf{y}_k)$, and $t_k = h_k + s_k$. So we need to solve the equations

$$\begin{aligned} \mathbf{z}_{k+1} &= \phi^{h_k+s_k}(\mathbf{y}_k + \mathbf{z}_k) - \mathbf{y}_{k+1} \quad \text{for } k = 0, \ldots, N-1 \\ \mathbf{z}_0 &= \varphi^{h_N+s_N}(\mathbf{y}_N + \mathbf{z}_N) - \mathbf{y}_0. \end{aligned}$$

As in Newton's method, we linearize:

$$\phi^{h_k+s_k}(\mathbf{y}_k + \mathbf{z}_k) - \mathbf{y}_{k+1} \approx f(\phi^{h_k}(\mathbf{y}_k))s_k + D\phi^{h_k}(\mathbf{y}_k)\mathbf{z}_k + \phi^{h_k}(\mathbf{y}_k) - \mathbf{y}_{k+1}.$$

Next we write

$$\mathbf{z}_k = S_k \mathbf{u}_k,$$

where $\mathbf{u}_k \in \mathbb{R}^{n-1}$ and $\{S_k\}_{k=0}^N$ is a sequence of $n \times (n-1)$ matrices chosen so that $[f(\mathbf{y}_k)/\|f(\mathbf{y}_k)\|\, S_k]$ is orthogonal. So now we solve the linear equations

$$\begin{aligned} S_{k+1}\mathbf{u}_{k+1} &= f(\mathbf{y}_{k+1})s_k + D\phi^{h_k}(\mathbf{y}_k)S_k\mathbf{u}_k + \mathbf{g}_k \quad \text{for } k = 0, \ldots, N-1 \\ S_0\mathbf{u}_0 &= f(\mathbf{y}_N)s_N + D\phi^{h_N}(\mathbf{y}_N)S_N\mathbf{u}_N + \mathbf{g}_N \end{aligned}$$

for s_k and $\mathbf{u}_k$, where $\mathbf{g}_k = \phi^{h_k}(\mathbf{y}_k) - \mathbf{y}_{k+1}$. Multiplying each equation in the first set by S_{k+1}^* and $f(\mathbf{y}_{k+1})^*$ and multiplying the last equation by S_0^* and $f(\mathbf{y}_0)$, under the assumption of no floating point errors, we obtain

$$\begin{aligned} \mathbf{u}_{k+1} - A_k\mathbf{u}_k &= S_{k+1}^*\mathbf{g}_k \quad \text{for } k = 0, \ldots, N-1 \\ \mathbf{u}_0 - A_N\mathbf{z}_N &= S_0^*\mathbf{g}_N \end{aligned} \tag{2}$$

and

$$\begin{aligned} s_k &= -\|f(\mathbf{y}_{k+1})\|^{-2} f(\mathbf{y}_{k+1})^* \{D\phi^{h_k}(\mathbf{y}_k)S_k\mathbf{u}_k + \mathbf{g}_k\} \quad \text{for } k = 0, \ldots, N-1 \\ s_N &= -\|f(\mathbf{y}_0)\|^{-2} f(\mathbf{y}_0)^* \{D\phi^{h_N}(\mathbf{y}_N)S_N\mathbf{u}_N + \mathbf{g}_N\}, \end{aligned}$$

where $A_k = S_{k+1}^* D\phi^{h_k}(\mathbf{y}_k)S_k$ for $k = 0, \dots, N-1$ and $A_N = S_0^* D\phi^{h_N}(\mathbf{y}_N)S_N$. So the main problem is to solve Eq. (2). This is solved by exploiting the local hyperbolicity along the pseudo orbit which implies the existence of contracting and expanding directions. We use a triangularization procedure which enables us to solve forward first along the contracting directions and then backwards along the expanding directions. Note that it is numerically impossible to solve the whole system forwards because of the expanding directions.

Once the new pseudo periodic orbit $\{\mathbf{y}_k + \mathbf{z}_k\}_{k=0}^N$ with associated times $\{h_k + s_k\}_{k=0}^N$ is found, we check if its delta is small enough. If it is not, we repeat the procedure for further refinement. For complete details see [19]. A similar method to refine a crude pseudo homoclinic orbit is given in [22].

(ii) Verifying the invertibility of the operator (or finding a suitable right inverse) and calculating an upper bound on the norm of the inverse: Again we just look at the periodic case. A similar procedure is used in the finite-time and homoclinic cases but in the homoclinic case it is rather more complicated since the sequence spaces are infinite-dimensional; however, we can handle it due to the periodicity at both ends. First we outline the procedure which would be used in the case of exact computations.

To construct $L_{\mathbf{y}}^{-1}$, we need to find the unique solution $\mathbf{z}_k \in Y_k$ of

$$\begin{aligned} \mathbf{z}_{k+1} &= P_{k+1} D\phi^{h_k}(\mathbf{y}_k)\mathbf{z}_k + \mathbf{g}_k, \quad \text{for } k = 0, \dots, N-1 \\ \mathbf{z}_0 &= P_N D\phi^{h_k}(\mathbf{y}_k)\mathbf{z}_N + \mathbf{g}_N. \end{aligned}$$

whenever $\mathbf{g}_k$ is in Y_{k+1} for $k = 0, \dots, N-1$ and in Y_0 for $k = N$. We use the $n \times (n-1)$ matrices S_k as defined in *(i)* and make the transformation

$$\mathbf{z}_k = S_k \mathbf{u}_k,$$

where $\mathbf{u}_k$ is in $\mathbb{R}^{n-1}$. Making this transformation, our equations become

$$\begin{aligned} \mathbf{u}_{k+1} - A_k \mathbf{u}_k &= S_{k+1}^* \mathbf{g}_k, \quad \text{for } k = 0, \dots, N-1 \\ \mathbf{u}_0 - A_N \mathbf{u}_N &= S_0^* \mathbf{g}_N, \end{aligned}$$

where A_k is the $(n-1) \times (n-1)$ matrix $A_k = S_{k+1}^* D\phi^{h_k}(\mathbf{y}_k)S_k$ for $k = 0, \dots, N-1$ and $A_N = S_0^* D\phi^{h_N}(\mathbf{y}_N)S_N$. As in *(i)*, these equations are solved by exploiting the hyperbolicity which implies the existence of contracting and expanding directions. We use a triangularization procedure which enables us to solve forward first along the contracting directions and then backwards along the expanding directions. Thus we are able to obtain a computable criterion for invertibility and a formula for the inverse from which an upper bound for the inverse can be obtained.

Of course, we have to take into account round-off and discretization error. For example, we have to work with a computed approximation to $D\phi^{h_k}(\mathbf{y}_k)$. However, this only induces a small perturbation in the operator. Another problem is that the columns of S_k, as computed, are not exactly orthonormal. Next

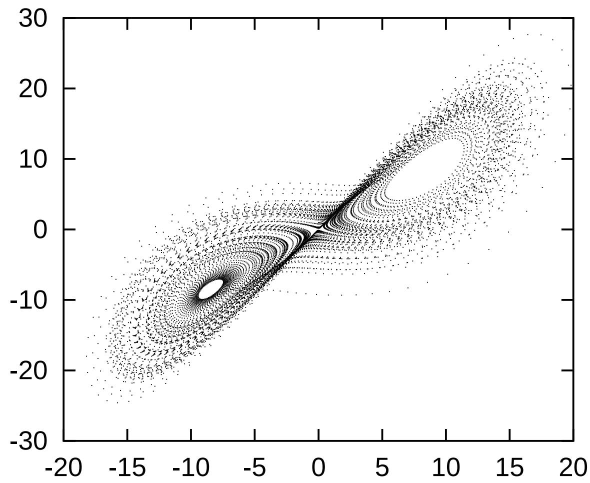

Figure 1.1: A pseudo orbit, with $\delta \leq 1.978 \times 10^{-12}$, of the Lorenz Equations for the classical parameter values $\sigma = 10$, $\beta = 8/3$, $\rho = 28$ and initial data (0, 1, 0) projected onto the $(x,\, y)$-plane. There exists a true orbit within $\varepsilon \leq 2.562 \times 10^{-9}$ of this pseudo orbit. For clarity, we have plotted only the first 120 time units of the pseudo orbit. Shadowing of this pseudo orbit for much longer time is possible.

A_k cannot be computed precisely. Finally the difference equation cannot be solved exactly but we are still able to obtain rigorous upper bounds, using standard perturbation theory of linear operators and the techniques in [65] to obtain rigorous upper bounds on the error due to floating point operations. The necessary interval arithmetic can be implemented by using IEEE-754 compliant hardware with a compiler that supports rounding mode control. For more details of these and other issues see [19] and for the finite-time and homoclinic cases see [17] and [22], respectively.

7. Examples

In this section, we offer representative applications of the shadowing theorems from Sections 3, 4, and 5, using the Lorenz Equations [41]

$$\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= \rho x - y - xz \\
\dot{z} &= xy - \beta z
\end{aligned}$$

with the classical parameter values $\sigma = 10$, $\rho = 28$, $\beta = 8/3$. Using the Lyapunov function $V(x, y, z) = \rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2$, it is not difficult to

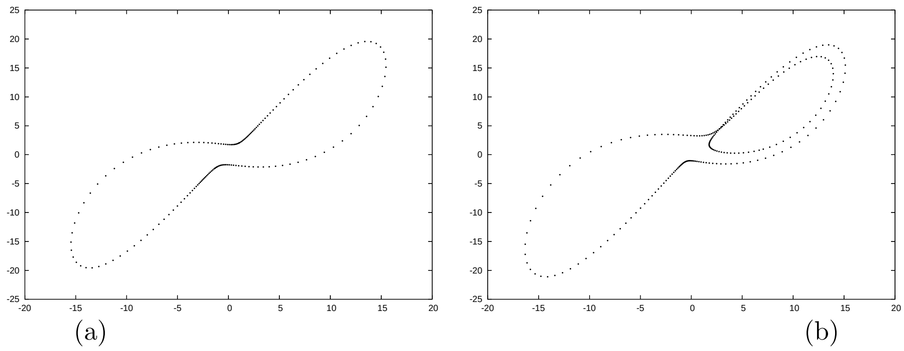

Figure 1.2: Two short pseudo periodic orbits of the Lorenz Equations for the classical parameter values projected onto the $(x,\, y)$-plane. The pseudo orbit in (a) has approximate period 1.559 time units; it is shadowed by a true periodic orbit within $\varepsilon \leq 1.800 \times 10^{-12}$. It is interesting to observe that these periodic orbits coexist with the nonperiodic orbit in Fig. 1.1.

establish that the set

$$U = \{(x, y, z) : \rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2 \leq \sigma\rho^2\beta^2/(\beta - 1)\}$$

is forward invariant under the flow of the Lorenz Equations for $\sigma \geq 1$, $\rho > 0$, and $\beta > 1$. Each pseudo orbit $\{\mathbf{y}_k\}_{k=0}^N$ of the Lorenz Equations we calculate below lies inside this forward invariant ellipsoid U.

First we give an example of finite-time shadowing for the initial data $(0, 1, 0)$ used by Lorenz. The pseudo orbit $\{\mathbf{y}_k\}_{k=0}^N$ of the Lorenz Equations in Fig. 1.1 and the sequence of matrices approximating $D\phi^{h_k}(\mathbf{y}_k)$ are generated by applying a Taylor series method of order 31 with initial value $\mathbf{y}_0$ at $t = 0$ and with constant time step h_k. The Taylor method has the advantages that a bound for the local discretization error is easily calculated and it also allows us to use relatively large step sizes h_k. This pseudo orbit is depicted in Fig.1.1. It is a δ pseudo orbit with $\delta \leq 1.978 \times 10^{-13}$ and we are able to show that it is ε-shadowed by a true orbit with $\varepsilon \leq 2.562 \times 10^{-9}$ for at least $850,000$ time units. This example is taken from [17] where all the implementation details can be found. A pseudo orbit from almost all initial data of the Lorenz Equations are shadowable for reasonably long time intervals. In the same reference shadowing times of many other pseudo orbits are tabulated.

Next, we present examples of periodic shadowing. Periodic orbits of the Lorenz Equations exhibit a great deal of geometric variations and they have been the subject of many publications; see, for example, [1], [5], [63]. Most of these works are either of numerical nature or pertain to a model system which is believed to capture the essential features of the actual equations. One of the first rigorous demonstrations of a specific periodic orbit of the Lorenz Equations was given in [15] using the Periodic Shadowing Theorem of Section 4.

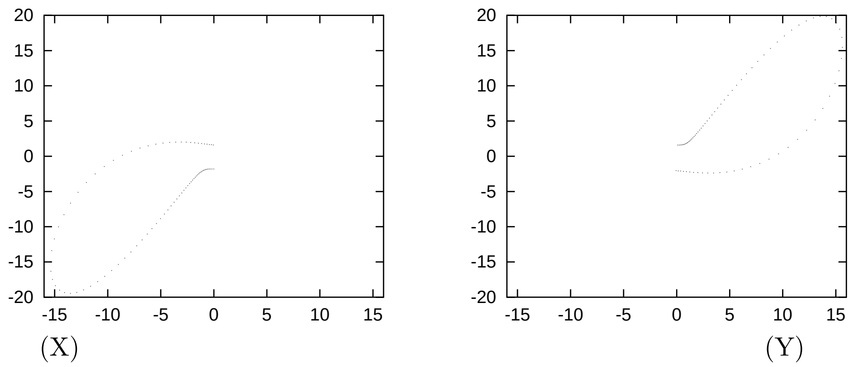

Figure 1.3: "X" and "Y" pieces of a crude pseudo periodic orbit of the Lorenz Equations for the classical parameter values projected onto the (x, y)-plane. Concatenated copies in the order XY and XYY are used as initial guesses for the Global Newton's method to generate the refined pseudo periodic orbits in Fig. 1.2.

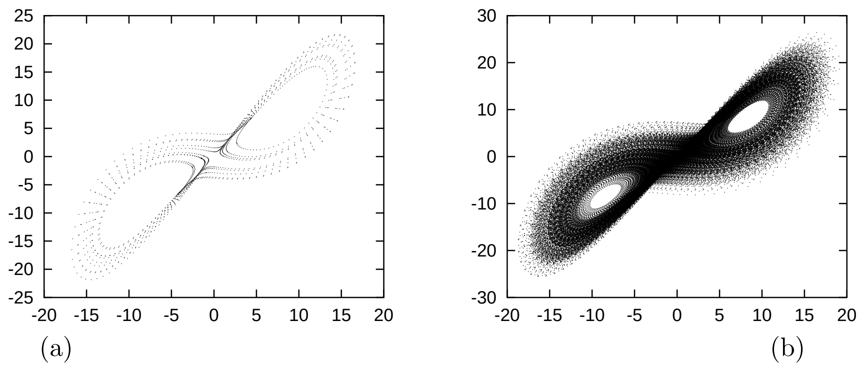

Figure 1.4: Two long pseudo periodic orbits of the Lorenz Equations for the classical parameter values projected onto the $(x,\ y)$-plane. The pseudo periodic orbit in (a) is of type XXYYXXXYYYXYXYXXXYY. The pseudo periodic orbit in (b) has approximate period 1100.787 time units; it is shadowed by a true periodic orbit within $\varepsilon \leq 5.239 \times 10^{-12}$.

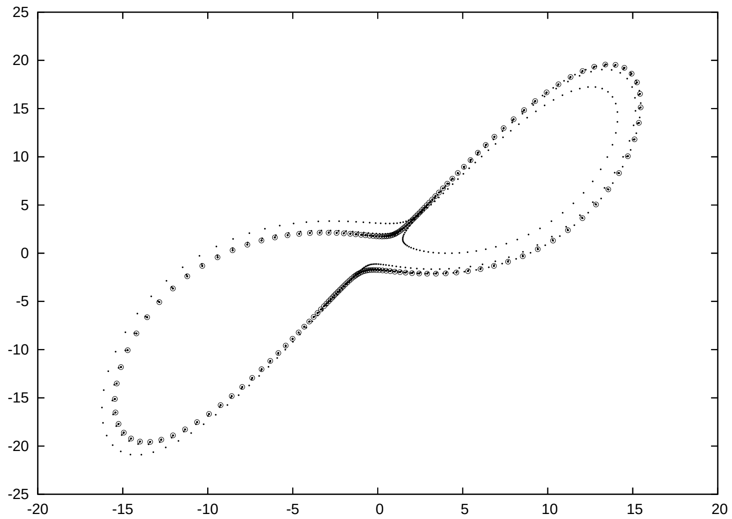

Figure 1.5: The pseudo periodic orbit of the Lorenz Equations from Fig. 1.2(a) (plotted in small circles) and a pseudo homoclinic orbit (plotted in dots) doubly asymptotic to this pseudo periodic orbit with $\delta \leq 2.012 \times 10^{-12}$. There exist a true hyperbolic periodic orbit and a true transversal homoclinic orbit within $\varepsilon \leq 2.266 \times 10^{-9}$ of the pseudo ones.

In Fig. 1.2 two short pseudo periodic orbits of the Lorenz Equations are plotted. The pseudo periodic orbit in Fig. 1.2(a) has approximate period 1.559 and is readily visible on the computer screen by following the solution from the initial data

$$(-12.78619065852397642,\ -19.36418793711800464,\ 24.0)$$

with a decent integrator. We found the periodic orbit in Fig. 1.2(a) as follows: first we computed the two orbit segments X and Y in Fig. 1.3 whose endpoints were fairly close and concatenated them together to form a pseudo periodic orbit XY with a quite large δ. Then we used the global Newton's method as outlined in Section 6 to find a refined pseudo periodic orbit with sufficiently small δ to apply the Periodic Shadowing Theorem. With a similar concatenation method, it is possible to generate pseudo periodic orbits with various geometries. Additional pseudo periodic orbits are depicted in Figs. 1.2 and 1.4. Further computational and mathematical details of the analysis of these pseudo periodic orbits are available in [15], [19], [22].

Lastly, we give an example of homoclinic shadowing. First we took the pseudo periodic orbit in Fig. 1.2(a). Using our concatenation technique followed by a global Newton's method, we next found a pseudo homoclinic orbit connecting the pseudo periodic orbit to itself. This pseudo connecting orbit, which is pictured in Fig. 1.5, has $\delta \le 2.012 \times 10^{-12}$. The values of the significant constants associated with this pseudo connecting orbit and the requisite inequalities for the Homoclinic Orbit Shadowing Theorem in Section 5 are tabulated in Fig. 1.6. It is evident from these numbers that the hypotheses of our Homoclinic Orbit Shadowing Theorem are fulfilled. Thus we conclude: there exists a true transversal homoclinic orbit within $\varepsilon \le 2.266 \times 10^{-9}$ of this pseudo one. As outlined in the Introduction, the existence of such an orbit implies chaotic behavior in the vicinity.

Parameters:

$$\sigma = 10.0, \quad \beta = 8.0/3.0, \quad \rho = 28.0$$

Convex set:

$$U = \{(x,\, y,\, z) : \rho x^2 + \sigma y^2 + \sigma(z - 2\rho)^2 \le \sigma\rho^2\beta^2/(\beta - 1)\}$$

Significant quantities:

$M_0 \le 5546.1807694948548$	$K = 567.89771838622869$
$M_1 \le 87.040362193445489$	$\Delta \ge 37.640569857537358$
$M_2 \le 1.4142135623730951$	$\overline{M}_0 \le 243.91779884309116$
$h_{\max} = 0.011826110602228697$	$\overline{M}_1 \le 28.802758744306345$
$h_{\min} = 0.0019715050733614117$	$C \le 567.89771838622869$
$\varepsilon_0 = 1.0000000000000001 \times 10^{-5}$	$\overline{C} \le 567.89777438966746$
$N = 168$	$N_1 \le 57496.233185753386$
$p = 167$	$N_2 \le 2.8839060646841870 \times 10^{11}$
$q = 3940$	$N_3 \le 1.5646336856150979 \times 10^{17}$
$\tau = 76$	$\varepsilon = 2.2659685392584887 \times 10^{-9}$
$\delta = 2.0118772701071166 \times 10^{-12}$	$r = 3277$

Inequalities:

$$\begin{aligned} 4C\delta &\le 4.5701620454677841 \times 10^{-9} < \epsilon_0 \\ 4M_1 C\delta &\le 3.9778855972025359 \times 10^{-7} \le \min(2, \Delta) = 2 \\ C^2((N_3\delta + N_2)\delta + N_1)\delta &\le 3.7306585982472891 \times 10^{-2} < 1 \end{aligned}$$

Figure 1.6: Significant constants and the requisite inequalities for the pseudo homoclinic orbit in Fig. 1.5. From the Homoclinic Shadowing Theorem, there exist a true hyperbolic periodic orbit and a true transversal homoclinic orbit within $\varepsilon \le 2.266 \times 10^{-9}$ of the pseudo ones.

8. Closing Remarks

In the preceding pages, we have tried to demonstrate the utility and the promise of shadowing as a new development in the rigorous computer-assisted analysis of ordinary differential equations. Shadowing can be used to show that computer simulations of chaotic systems do indeed represent true states of the system. Moreover, shadowing is able to establish the existence of orbits with various dynamical properties. In contrast to classical numerical analysis which tries to compute known orbits as accurately as possible, shadowing can show their existence rigorously.

One limitation of the shadowing methods presented here is that they will only work with systems which display some hyperbolicity, even if not uniform. Limits to shadowing due to fluctuating Lyapunov exponents or unstable dimension variability have been investigated in [24], [53].

In this review article we tried to give the highlights of our work in shadowing during the past decade. Due to the inevitable time and space constraints, we could not include many other developments in shadowing; we apologize to the authors of many excellent publications to which we could not refer. We hope we have convinced you of the utility of shadowing and that you will be interested enough to look at the details of our work and the works of the others listed in the references.

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