Shilnikov Saddle-Focus Homoclinic Orbits from Numerics: Higher Dimensions

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Abstract
In a previous paper we studied parametrized autonomous systems and gave a computable criterion that an approximate orbit connecting hyperbolic equilibria is shadowed by a true connecting orbit. This criterion was used to give rigorously verified examples of Shilnikov saddle-focus homoclinic orbits in three dimensions. This involved verifying a condition on the eigenvalues of the linearization at the equilibrium. In dimensions greater than three, there are three more conditions which must be established: general position, asymptotic tangency and a transversality condition. In this paper we give computable criteria for verifying these three conditions. An example in four dimensions, in which detailed rigorous computations are carried out, is given.

Key words: Shilnikov chaos; Homoclinic; Shadowing; Rigorous numerics.

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1. Introduction

Chaos cannot occur in two dimensional autonomous systems of ordinary differential equations. It does occur for two dimensional diffeomorphisms which have a transversal homoclinic orbit to a hyperbolic fixed point and hence for periodic nonautonomous systems in two dimensions. In Silnikov [1965] it was shown that chaos can occur for autonomous systems in three dimensions in the neighborhood of a homoclinic orbit to a hyperbolic equilibrium — not any hyperbolic equilibrium but only one for which the linearization has a pair of complex conjugate eigenvalues with positive real part less than the absolute value of the other real negative eigenvalue. Silnikov proved the existence of chaotic behavior in the neighborhood of such a homoclinic orbit. His theorem is stated later in this introduction.

Later Silnikov [1967a] generalized his theorem to four dimensions and in Silnikov [1970] generalized it to arbitrarily high dimensions. For this the homoclinic orbit, besides the eigenvalue condition called (D1) below, had to satisfy three other conditions, (D2)–(D4) below, These three conditions, as originally given, are not well adapted to computation. The conditions given below are purely in terms of bounded solutions of the variational system along the homoclinic orbit and are better adapted to computation. Their relation to the original conditions and their geometric meaning are discussed later in this introduction.

We consider the differential equation

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \tag{1}$$

where f is C^r $(r \geq 1)$, and denote by ϕ^t its associated flow. We suppose z is a hyperbolic equilibrium and that $q(t)$ is an associated homoclinic orbit. We will call $q(t)$ a *Shilnikov saddle-focus homoclinic orbit* if the following four conditions are satisfied:

(D1) the eigenvalues of $f'(z)$ having the (strictly) smallest positive real part are $\beta \pm i\omega$ with $\omega > 0$, each having algebraic multiplicity one, and satisfying

$$0 < \beta < -\mathrm{Re}(\lambda)$$

for all eigenvalues λ with negative real parts;

(D2) up to a scalar multiple, $\dot{q}(t)$ is the only bounded solution of

$$\dot{y} = f'(q(t))y;$$

(D3) $\dot{q}(t)e^{-\nu t}$ is not bounded on $\mathbb{R}_-$, where $\beta < \nu < \mathrm{Re}(\lambda)$ for all eigenvalues λ with $\mathrm{Re}(\lambda) > \beta$;

(D4) there does not exist $\xi \neq 0$ such that solution $y(t)$ of

$$\dot{y} = -f'(q(t))^T y, \quad y(0) = \xi$$

is bounded on $\mathbb{R}_-$ and the solution $y(t)$ of

$$\dot{y} = -[f'(q(t))^T - \nu]y, \quad y(0) = \xi$$

is bounded on $\mathbb{R}_+$.

Note that (D2) is automatic when the dimension of the stable manifold of z is one and (D3), (D4) are not needed when the dimension of the unstable manifold of z is 2 so that none of these conditions is needed in three dimensions. Also note that if (D3) and (D4) hold for some ν satisfying the condition $\beta < \nu < \mathrm{Re}(\lambda)$, then they hold for all such ν.

As mentioned earlier, in three dimensions Shilnikov studied this kind of homoclinic orbit in 1965 and proved the existence of complicated dynamics in its neighbourhood. He extended his investigations to the $n-$dimensional case in 1970, where he proved the following theorem, which was extended by Deng [1993] under the assumption that f is only C^5.

Theorem. *Suppose f in Eq. (1) is analytic and conditions (D1)–(D4) hold. Let $\Omega(\rho)$ be the set of doubly infinite sequences $(\ldots, j_i, j_{i+1}, \ldots)$ consisting of the symbols $0, 1, 2, \ldots$ and satisfying the condition that $j_{i+1} < \rho j_i$ for all i for some ρ satisfying $1 < \rho < -\mathrm{Re}\,\lambda/\beta$ for all eigenvalues λ with negative real part. Then in an arbitrary neighborhood of the homoclinic orbit $q(t)$, there exists a subsystem of trajectories which is in one-to-one correspondence with the set $\Omega(\rho)$.*

As noted earlier, in previous papers, conditions (D2)–(D4) were described in a different way from what we have given here. Now we describe these differences. First regarding condition (D2), we note that it is equivalent to *general position*, that is, the one-dimensionality of the intersection of the tangent spaces to the stable and unstable manifolds along $q(t)$.

As shown in Battelli-Palmer [2011], (D3) is equivalent to the *asymptotic tangency* of $q(t)$ to the linear span V of the eigenvectors corresponding to $\beta \pm i\omega$ as $t \to -\infty$, which in turn is equivalent to $q(t)$ not lying in the strong unstable manifold, the latter being the invariant manifold corresponding to the unstable eigenvalues apart from $\beta \pm i\omega$. These are the conditions used in Shilnikov [1970], Shilnikov et al. [2001] and Deng [1993].

As shown in Battelli-Palmer [2011], (D4) is equivalent to the transversality of the intersection of the *extended stable manifold* and the unstable manifold along $q(t)$, which is the condition used in Shilnikov et al. [2001]. Shilnikov's original condition in [1970] was that a certain quantity δ not vanish. In Shilnikov et al. [2001], δ was replaced by A and there it was indicated that the condition that A not vanish was equivalent to the transversality condition just mentioned. Note that the local extended stable manifold is an invariant manifold containing the equilibrium z such that its tangent space at z is the sum of the stable subspace and the two-dimensional eigenspace corresponding to the pair of complex eigenvalues. Such a manifold is not unique but each such manifold contains the homoclinic orbit and all these manifolds share the same tangent space along the homoclinic orbit.

Instead of (D4), Deng [1993] gave a different condition which he calls the *strong inclination condition: there is a submanifold $\mathcal{M}_0$ of the unstable manifold containing $q(0)$ with dim $\mathcal{M}_0$ = dim W^{uu} such that $\lim_{t\to\infty} T_{q(t)}\mathcal{M}_t = T_0W^{uu}$, where W^{uu} is the strong unstable manifold and $\mathcal{M}_t = \phi^t(\mathcal{M}_0)$.*(Note this condition depends only on the tangent space to the submanifold $\mathcal{M}_0$ at $q(0)$, not on the submanifold itself.) Again this was shown in Battelli-Palmer [2011] to be equivalent to (D4) as given above.

In Battelli-Palmer [2011], an analytical example of a Shilnikov saddle-focus homoclinic orbit in four dimensions was given and all conditions (D1)–(D4) were verified. Another analytical example of a Shilnikov saddle-focus homoclinic orbit in four dimensions was given in Belykh and Pankratova [2014]. However, conditions (D2)–(D4) were not explicitly verified. Numerical examples of such an orbit in a four dimensional Lotka-Volterra system are given in Vano et al. [2006] and Wang and Xiao [2010].

In this paper we study the rigorous numerical verification of the existence of this kind of homoclinic orbit when an approximate homoclinic orbit is given. We already did this in Coomes et al. [2016] for the three-dimensional case. Other examples of Shilnikov orbits in three dimensions, obtained by rigorous computations, are given in Ambrosi et al. [2012] and Capiński, M.J. and Waisieczko-Zając, A. [2017]. Here we want to consider the higher dimensional case.

The content of this paper is as follows:

- In Section 2, we recall the theorem from Coomes et al. [2016] ensuring the existence of homoclinic orbits in parametrized autonomous systems, given an approximate such orbit. The main condition here was that a certain matrix L associated with the approximate orbit was invertible.

- In Section 3, we show how it can be verified that this orbit satisfies the four conditions (D1)–(D4). It turns out that the invertibility of L together with additional smallness conditions implies (D2). (D3) and (D4) are implied by the existence of left inverses for two other matrices together with additional smallness conditions.
- In Section 4, we detail rigorous computations for specific parameter values of the four dimensional example depending on two parameters in Battelli-Palmer [2011] which is obtained from a pair of linearly coupled oscillators. We also report further investigations of the connecting orbits in this system for additional parameter values.
- The example in Section 4 was obtained by perturbing a system satisfying (D1), (D2) and (D4). In Section 5, we show it is also possible to write down systems which satisfy only (D1) and (D2) or only (D1), (D2) and (D3).
- In Appendix 1, we give the explicit conditions needed to be verified in order to apply the Existence Theorem in Section 2; in Appendix 2 we derive some technical estimates needed in Section 3; in Appendix 3, we prove the lemmas needed for Section 3; in Appendix 4 we give computational details for the example in Section 4.

The name Shilnikov has been spelled in different ways in the literature, variously as Shilnikov, Shil'nikov, Šilnikov and Sil'nikov. We have decided to use the form Shilnikov but in the bibliography we have used the forms as they are given in the journals.

2. Homoclinic orbits in parametrized autonomous systems

In this section we recall the theorem from Coomes et al. [2016], ensuring the existence of homoclinic orbits in parametrized autonomous systems, given an approximate such orbit. First we give the precise definition of an approximate homoclinic orbit.

Definition. *Let z be an equilibrium for Eq. (1). Then we say a sequence y_k, $k \in \mathbb{Z}$, with associated times $h_k > 0$ is a δ approximate homoclinic orbit if*

$$|\phi^{h_k}(y_k) - y_{k+1}| \le \delta h_k, \quad k \in \mathbb{Z}$$

and $y_k = z$ and h_k is constant for large $|k|$.

In what follows, we consider a system with real parameter a

$$\dot{x} = f(x, a) \tag{2}$$

in $\mathbb{R}^n$, where f is a C^2 function and we denote the corresponding flow by $\phi^t(x, a)$. In preparation for the statement of our theorem, we next record the three requisite hypotheses:

Hypothesis 1: When $a = a_0$, we assume that the system in Eq. (2) has an equilibrium z such that the matrix $A = f_x(z, a_0)$ is hyperbolic. Let H be a matrix, the first r columns of which form a basis for the stable subspace of A and the remaining columns a basis for the unstable subspace of A.

Also note that by the implicit function theorem, the equation $f(x, a) = 0$ has a C^2 solution $z(a)$ for a near a_0 such that $z(a_0) = z$.

Hypothesis 2: We assume y_k is a δ approximate homoclinic orbit of Eq. (2) for $a = a_0$ with associated times h_k such that there exist positive integers N_1, N_2 and a positive number $\delta_1 > 0$ such that

$$|y_k - z| \leq \delta_1 \text{ for } k \geq N_1 \quad \text{and} \quad |y_k - z| \leq \delta_1 \text{ for } k < -N_2. \tag{3}$$

Hypothesis 3: Let $Y_k = \phi_x^{h_k}(y_k, a_0)$, $z_k = \phi_a^{h_k}(y_k, a_0)$ and set

$$v = H^{-1}A^{-1}P_r f_a(z, a_0) + H^{-1}A^{-1}(I_n - P_r) f_a(z, a_0),$$

where

$$P_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix},$$

I_r being the $r \times r$ identity matrix. Then we assume the $[(N_1 + N_2 + 1)n + 1] \times [(N_1 +$

$N_2+1)n+1]$ matrix L given below is invertible:

$$\begin{bmatrix}
-Y_{-N_2}H & I_n & 0 & \cdot & \cdot & \cdot & \cdot & 0 & -z_{-N_2} \\
0 & -Y_{-N_2+1} & I_n & 0 & \cdot & \cdot & \cdot & 0 & -z_{-N_2+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdot & -Y_{-1} & I_n & 0 & \cdot & \cdot & 0 & -z_{-1} \\
0 & \cdot & 0 & -Y_0 & I_n & 0 & \cdot & 0 & -z_0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdot & \cdot & \cdot & 0 & -Y_{N_1-2} & I_n & 0 & -z_{N_1-2} \\
0 & \cdot & \cdot & \cdot & \cdot & 0 & -Y_{N_1-1} & H & -z_{N_1-1} \\
P_r & 0 & \cdot & \cdot & \cdot & \cdot & 0 & I_n-P_r & v \\
0 & \cdot & 0 & f(y_0,a_0)^T & 0 & \cdot & \cdot & \cdot & 0
\end{bmatrix}$$

Now we are ready to state the theorem.

Existence Theorem for Homoclinic Orbits. *Consider the parametrized system $\dot{x}=f(x,a)$ in $\mathbb{R}^n$, as in Eq. (2), with flow $\phi^t(x,a)$. Let z be a hyperbolic equilibrium as in Hypothesis 1 and y_k an associated δ approximate homoclinic orbit of Eq. (2) for $a=a_0$ and associated times h_k such that the inequalities in Eq. (3) hold for some N_1, N_2 and δ_1, as in Hypothesis 2.*

Then if L is invertible as in Hypothesis 3, there exists a constant C such that if δ_1 and δ are sufficiently small, there is a unique parameter value a^* and a unique sequence x_k such that

$$f(y_0,a_0)^T(x_0-y_0)=0, \tag{4}$$

$$|a^*-a_0|\le 2C\delta, \quad |x_k-y_k|\le 2C\delta, \quad x_{k+1}=\phi^{h_k}(x_k,a^*) \quad \text{for all} \quad k\in\mathbb{Z} \tag{5}$$

and

$$\phi^t(x_0,a^*)\to z(a^*) \quad \text{as} \quad t\to\pm\infty. \tag{6}$$

The homoclinic orbit $\phi^t(x_0,a^*)$ is distinct from the equilibrium $z(a^*)$ provided that there exists k such that $|y_k-z(a^*)|>2C\delta$.

In Appendix 1, we give more information about the constant C and the conditions which δ and δ_1 need to satisfy.

3. Verification that the homoclinic orbit is Shilnikov saddle-focus

In this section we describe how it can be verified that a homoclinic orbit, found using the Existence Theorem for Homoclinic Orbits, satisfies the conditions (D1)–(D4). First we list the assumptions needed to apply the theorem.

We assume that when $a = a_0$ the system in Eq. (2) has an equilibrium z such that the matrix $A = f_x(z, a_0)$ is hyperbolic and the eigenvalues of A satisfy the condition (D1). So Hypothesis 1 is satisfied. Note in Hypothesis 1, we choose H as a matrix $[H_1\ H_2\ H_3]$, where the first r columns H_1 form a basis for the stable subspace of A, the next two columns $H_2 = [u\ v]$ a basis for the two-dimensional subspace corresponding to the eigenvalues $\beta \pm i\omega$ so that $Au = \beta u - \omega v$, $Av = \omega u + \beta v$, and the remaining columns H_3 a basis for the subspace corresponding to the other eigenvalues.

Next we assume further that y_k is an associated δ approximate homoclinic orbit of Eq. (2) for $a = a_0$ with associated times h_k such that the inequalities in Eq. (3) hold for some N_1, N_2 and δ_1. So Hypothesis 2 holds.

Next we assume L from Hypothesis 3 is invertible. Finally we assume δ and δ_1 are sufficiently small (see Appendix 1 for the precise conditions) so that the Existence Theorem for Homoclinic Orbits can be applied to deduce that there is a unique parameter value a^* and a unique sequence x_k such that Eq. (4), Eq. (5) and Eq. (6) hold. We assume also that there exists k such that $|y_k - z(a^*)| > 2C\delta$ so that the homoclinic orbit $q(t) = \phi^t(x_0, a^*)$ is distinct from the equilibrium $z(a^*)$.

We define t_k to be the sequence such that $t_0 = 0$ and $t_{k+1} = t_k + h_k$ for all integers k so that $x_k = \phi^{t_k}(x_0, a^*)$. In particular, we define $T_1 = t_{N_1}$ and $T_2 = -t_{-N_2}$. Now we proceed with the verification of (D1), (D2), (D3) and (D4).

3.1. Verification of (D1) for $z(a^*)$

It follows from Eq. (44) in Appendix 2 that

$$|f_x(z(a^*), a^*) - f_x(z, a_0)| \le 2(M_2 + M_4)C\delta, \tag{7}$$

where M_2 and M_4 are as in Appendix 1. So to verify (D1), we would need to show that if a matrix B satisfies $|B - f_x(z, a_0)| \le 2(M_2 + M_4)C\delta$, then the eigenvalues of B have similar properties to those of $A = f_x(z, a_0)$. Actually, in the example in Section 4, we are able to directly calculate $|f_x(z(a^*), a^*) - f_x(z, a_0)|$ and so can verify (D1) more easily than in the general case.

3.2. Some preliminary estimates

Before verifying (D2)–(D4), we need to introduce some notation and make some preliminary estimates. This subsection is divided into three parts. The purpose of part (i) is to characterize the dichotomy properties and thus locate the bounded solutions of $\dot{y} = Ay$ and $\dot{y} = [A - \nu]y$ and their adjoints, where $A = f_x(z, a_0)$ and ν is as in (D3) and (D4). In the second part (ii), we derive estimates for $|\phi^t(x_0, a^*) - z(a^*)|$ for large $|t|$. The conditions (D2)–(D4) for our homoclinic orbit $\phi^t(x_0, a^*)$ involve bounded solutions of $\dot{y} = f_x(q(t), a^*)y$ and $\dot{y} = (f_x(q(t), a^*) - \nu)y$ and their adjoints, where $q(t) = \phi^t(x_0, a^*)$. These bounded solutions will be close to those of $\dot{y} = Ay$ and $\dot{y} = [A - \nu]y$ and their adjoints, since for $|t|$ large, $\phi^t(x_0, a^*)$ is close to the equilibrium $z(a^*)$. In (ii) we quantify this closeness precisely. In the third part (iii), we estimate $|f(y_0, a_0) - f(x_0, a^*)|$ and $|\phi_x^{h_k}(x_k, a^*) - \phi_x^{h_k}(y_k, a_0)|$. These are needed for approximating matrices depending on x_k and a^* with matrices depending on y_k and a_0. We observe that the estimates in (ii) and (iii) follow from the Existence Theorem for Homoclinic Orbits and do not require additional assumptions.

(i) *Dichotomy properties*: We are assuming that equation Eq. (1) has an equilibrium at z such that the eigenvalues of $A = f_x(z, a_0)$ satisfy (D1), where the real part of the pair of complex eigenvalues is β. Now let α be a positive number such that $-\alpha$ exceeds the real parts of the eigenvalues with negative real parts and let σ be a positive number which is less than the real parts of the eigenvalues with positive real parts apart from β. In view of (D1), we can assume that $\beta < \alpha$, $\beta < \sigma$. Then we choose ν so that

$$\beta < \nu < \sigma. \tag{8}$$

Next let P be the projection with range the stable subspace and nullspace the unstable subspace. Note that $AP = PA$. Let Q be the projection with range the sum of the stable subspace and the eigenspace corresponding to the pair of complex eigenvalues and nullspace the sum of the generalized eigenspaces corresponding to the remaining eigenvalues with positive real parts. Note that $AQ = QA$. Then, from the eigenvalue properties, we see that there exists a positive constant K such that for $t \geq 0$

$$|e^{tA}P| \leq Ke^{-\alpha t}, \quad |e^{-tA}(I_n - P)| \leq Ke^{-\beta t}$$

and

$$|e^{t(A-\nu I_n)}Q| \leq Ke^{-(\nu-\beta)t}, \quad |e^{-t(A-\nu I_n)}(I_n - Q)| \leq Ke^{-(\sigma-\nu)t}.$$

Now we show how to compute K. With $H = [H_1\ H_2\ H_3]$ as defined above, we have

$$AH_1 = H_1 D^s, \quad AH_2 = H_2 D^c, \quad AH_3 = H_3 D^u,$$

where D^s is $r \times r$, $D^c = \begin{bmatrix} \beta & -\omega \\ \omega & \beta \end{bmatrix}$ and D^u is $(n-r-2) \times (n-r-2)$, so that

$$H^{-1}AH = \begin{bmatrix} D^s & 0 & 0 \\ 0 & D^c & 0 \\ 0 & 0 & D^u \end{bmatrix}.$$

Then in view of the assumptions on α, β and σ, there exists a positive constant $\bar{K}$ such that for $t \geq 0$

$$|e^{tD^s}| \leq \bar{K} e^{-\alpha t}, \quad |e^{\pm tD^c}| = e^{\pm \beta t}, \quad |e^{-tD^u}| \leq \bar{K} e^{-\sigma t}, \tag{9}$$

where here, and in the sequel unless otherwise indicated, $|\cdot|$ denotes the Euclidean norm. Note if $P_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, then

$$HP_r H^{-1} = P \tag{10}$$

and it follows that for $t \geq 0$

$$|e^{tA}P| = |He^{tD}P_r H^{-1}| \leq Ke^{-\alpha t}, \quad |e^{-tA}(I_n - P)| = |He^{-tD}(I_n - P_r)H^{-1}| \leq Ke^{-\beta t}, \tag{11}$$

where

$$K = |H^{-1}|\,|H|\bar{K}. \tag{12}$$

The matrix $A - \nu I_n$ is also hyperbolic and if $P_s = \begin{bmatrix} I_s & 0 \\ 0 & 0 \end{bmatrix}$ with $s = r + 2$, then

$$HP_s H^{-1} = Q \tag{13}$$

and for $t \geq 0$ we have

$$|e^{t(A - \nu I_n)}Q| \leq Ke^{-(\nu - \beta)t}, \quad |e^{-t(A - \nu I_n)}(I_n - Q)| \leq Ke^{-(\sigma - \nu)t}. \tag{14}$$

(ii) *Convergence of* $\phi^t(x_0, a^*)$ *to* $z(a^*)$: Here we derive estimates for $|\phi^t(x_0, a^*) - z(a^*)|$ and $|f_x(\phi^t(x_0, a^*), a^*) - f_x(z, a_0)|$ for large $|t|$. In fact, we show for $t \geq T_1 = t_{N_1}$ and $t \leq -T_2 = t_{-N_2}$ that

$$\begin{aligned} |\phi^t(x_0, a^*) - z(a^*)| &\leq \mu_1 = (4C\delta + \delta_1)e^{M_1 h_{\max}}, \\ |f_x(\phi^t(x_0, a^*), a^*) - f_x(z, a_0)| &\leq \rho = M_2\mu_1 + 2(M_2 + M_4)C\delta, \end{aligned} \tag{15}$$

where the quantities C, M_i and $h_{\max}$ are as defined in Appendix 1. The proof can be found in Appendix 2.

(iii) *Estimates for* $|f(y_0, a_0) - f(x_0, a^*)|$ *and* $|\phi_x^{h_k}(x_k, a^*) - \phi_x^{h_k}(y_k, a_0)|$: Here we derive the inequalities

$$|f(y_0, a_0) - f(x_0, a^*)| \leq 2C(M_1 + M_3)\delta \tag{16}$$

and

$$|\phi_x^{h_k}(x_k, a^*) - \phi_x^{h_k}(y_k, a_0)| \leq \delta_2 = 2C(M_6 + M_7)h_{\max}e^{M_1 h_{\max}}\delta, \tag{17}$$

for $k = -N_2 + 1, \ldots, N_1 - 1$. The quantities C, M_i and $h_{\max}$ are as in Appendix 1. The proofs can be found in Appendix 2.

3.3. Verification of (D2)

To verify (D2), we need to show $\dot{q}(t)$, where $q(t) = \phi^t(x_0, a^*)$ is our homoclinic solution, is up to a scalar multiple, the unique bounded solution of

$$\dot{x} = f_x(q(t), a^*)x, \quad q(t) = \phi^t(x_0, a^*) \tag{18}$$

First we find the subspace V_1 of initial values at $t = T_1$ of solutions which are bounded for $t \geq T_1$ (see Lemma 1 below) and also the subspace V_2 of initial values at $t = -T_2$ of solutions which are bounded for $t \leq -T_2$ (see Lemma 2 below). Then the problem of finding bounded solutions independent of $\dot{q}(t)$ is reduced to the boundary value problem of finding the solutions $x(t)$ with $x(-T_2) \in V_2$ and $x(T_1) \in V_1$ such that $x(0)$ is orthogonal to $\dot{q}(0) = f_x(x_0, a^*)$. It turns out that $x(t)$ is such a solution if and only if the sequence $x(t_k)$, $k = -N_2, \ldots N_1$ is in the nullspace of the $[(N_1 + N_2 + 1)n + 1] \times [(N_1 + N_2 + 1)n]$ matrix $\hat{L}$ given by

$$\begin{bmatrix}
-\tilde{Y}_{-N_2} & I_n & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & -\tilde{Y}_{-N_2+1} & I_n & 0 & \cdot & \cdot & \cdot & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdot & -\tilde{Y}_{-1} & I_n & 0 & \cdot & \cdot & 0 \\
0 & \cdot & 0 & -\tilde{Y}_0 & I_n & 0 & \cdot & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdot & \cdot & \cdot & 0 & -\tilde{Y}_{N_1-2} & I_n & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & 0 & -\tilde{Y}_{N_1-1} & I_n \\
H^{-1}P_- & 0 & \cdot & \cdot & \cdot & \cdot & 0 & H^{-1}(I_n - P_+) \\
0 & \cdot & 0 & f(q(0), a^*)^T & 0 & \cdot & \cdot & 0
\end{bmatrix},$$

where $\tilde{Y}_k = \phi_x^{h_k}(q(t_k), a^*) = \phi_x^{h_k}(x_k, a^*)$, and the projections P_+ and P_- define the subspaces V_1 and V_2. So we need to show $\hat{L}$ is one to one or, equivalently, has a left inverse. However we can only approximate this matrix by L with the last column removed. Then the invertibility of L implies that this matrix has a left inverse and then we need additional smallness conditions to conclude that $\hat{L}$ has a left inverse.

Proposition 1. *(D2) holds provided that*

$$\lambda_1 = K(\alpha^{-1} + \beta^{-1} + K^2\beta^{-1})\rho < 1 \tag{19}$$

and

$$[4\sqrt{n}K^4 \max\{1, \sqrt{n}|H|\}|H^{-1}|\beta^{-1}(1-\lambda_1)^{-1}\rho + \delta_3]\|L^{-1}\|_\infty < 1, \tag{20}$$

where ρ is as in Eq. (15) and

$$\delta_3 = \sqrt{n}\max\{\delta_2, |H|\delta_2, 2C(M_1+M_3)\delta\}, \qquad \delta_2 \text{ as in Eq. (17)}, \tag{21}$$

where here and in Propositions 2 and 3, $\|\cdot\|_\infty$ means the maximum row sum of the absolute values of the entries in each row.

For the proof we need the following two lemmas, the proofs of which are deferred to Appendix 3. In these lemmas we find the bounded solutions of (18) on the intervals $[T_1, \infty)$ and $(-\infty, -T_2$, where in these and the following Lemmas, we take $T_1 = t_{N_1}$ and $T_2 = -t_{-N_2}$. Note it follows from Eq. (15) that for $t \geq T_1$ and $t \leq -T_2$,

$$|f_x(q(t), a^*) - A| = |f_x(q(t), a^*) - f_x(z, a_0)| \leq \rho.$$

Lemma 1. *Suppose ρ in Eq. (15) satisfies*

$$\rho < [K(\alpha^{-1} + \beta^{-1} + K^2\beta^{-1})]^{-1}.$$

Then if $\xi \in \mathcal{R}(P)$, there exists a unique solution $x(t) = x(t,\xi)$ of Eq. (18) bounded on $t \geq T_1$ such that $Px(T_1) = \xi$. Moreover

$$\sup_{t \geq T_1} |x(t)| \leq (1 - K(\alpha^{-1} + \beta^{-1})\rho)^{-1}K|\xi|$$

and the set of initial values $\{x(T_1,\xi) : \xi \in \mathcal{R}(P)\}$ is the range of a projection P_+ which has the same nullspace as P and satisfies

$$|P_+ - P| \leq 2K^4\beta^{-1}\left(1 - K(\alpha^{-1} + \beta^{-1} + K^2\beta^{-1})\rho\right)^{-1}\rho.$$

Lemma 2. *Suppose ρ in Eq. (15) satisfies*

$$\rho < [K(\alpha^{-1} + \beta^{-1} + K^2\alpha^{-1})]^{-1}.$$

Then if $\xi \in \mathcal{N}(P)$, there exists a unique solution $x(t) = x(t, \xi)$ of Eq. (18) bounded on $t \leq -T_2$ such that $(I_n - P)x(-T_2) = \xi$. Moreover

$$\sup_{t \leq -T_2} |x(t)| \leq (1 - K(\alpha^{-1} + \beta^{-1})\rho)^{-1} K|\xi|$$

and the set of initial values $\{x(-T_2, \xi) : \xi \in \mathcal{N}(P)\}$ is the nullspace of a projection P_- which has the same range as P and satisfies

$$|P_- - P| \leq 2K^4\alpha^{-1}\left(1 - K(\alpha^{-1} + \beta^{-1} + K^2\alpha^{-1})\rho\right)^{-1}\rho.$$

Proof of Proposition 1. First we approximate the matrix L from the Existence Theorem for Homoclinic Orbits by another matrix $\bar{L}$, which we then show to be invertible and use this to show that $\hat{L}$ has a left inverse, finally showing that this implies (D2).

Approximation of L by $\bar{L}$: The information we have is that L as given in the Existence Theorem for Homoclinic Orbits is invertible. Replace $Y_k = \phi_x^{h_k}(y_k, a_0)$ in L by $\phi_x^{h_k}(x_k, a^*)$ and $f(y_0, a_0)$ by $f(x_0, a^*)$ to get a new matrix $\bar{L}$. We first show that $\bar{L}$ is invertible by showing it is close to L.

Proof that $\bar{L}$ is invertible: From Eq. (17), we have

$$|Y_k - \tilde{Y}_k| \leq \delta_2$$

and from Eq. (16) we have

$$|f(y_0, a_0) - f(x_0, a^*)| \leq 2C(M_1 + M_3)\delta.$$

It follows that $\|\bar{L} - L\|_\infty \leq \delta_3$. (Note that the $\sqrt{n}$ in δ_3 comes from the fact that the ℓ^1 norm of an n-vector is bounded by $\sqrt{n}$ times its Euclidean norm.) Then, since Eq. (20) implies that $\delta_3\|L^{-1}\|_\infty < 1$, $\bar{L}$ is invertible and

$$\|\bar{L}^{-1}\|_\infty \leq (1 - \delta_3\|L^{-1}\|_\infty)^{-1}\|L^{-1}\|_\infty.$$

Proof that $\hat{L}$ has a left inverse: Define the matrix $\tilde{L}$ as $\bar{L}$ minus the last column. It follows that $\tilde{L}$ has a left inverse $\tilde{L}^{-1}$ (which is just $\bar{L}^{-1}$ minus its last row) and

$$\|\tilde{L}^{-1}\|_\infty \leq \|\bar{L}^{-1}\|_\infty \leq (1 - \delta_3\|L^{-1}\|_\infty)^{-1}\|L^{-1}\|_\infty.$$

$\tilde{L}$ is the same as $\hat{L}$ except that $\tilde{Y}_{-N_2}$ is multiplied by H, it has $P_r = H^{-1}PH$ instead of $H^{-1}P_-$ and $I_n - P_r = H^{-1}(I_n - P)H$ instead of $H^{-1}(I_n - P_+)$. First we note, using Eq. (10), that

$$\tilde{L} = L_1 D,$$

where L_1 is

$$\begin{bmatrix}
-\tilde{Y}_{-N_2} & I_n & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & -\tilde{Y}_{-N_2+1} & I_n & 0 & \cdot & \cdot & \cdot & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdot & -\tilde{Y}_{-1} & I_n & 0 & \cdot & \cdot & 0 \\
0 & \cdot & 0 & -\tilde{Y}_0 & I_n & 0 & \cdot & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdot & \cdot & \cdot & 0 & -\tilde{Y}_{N_1-2} & I_n & 0 \\
0 & \cdot & \cdot & \cdot & \cdot & 0 & -\tilde{Y}_{N_1-1} & I_n \\
H^{-1}P & 0 & \cdot & \cdot & \cdot & \cdot & 0 & H^{-1}(I_n - P) \\
0 & \cdot & 0 & f(q(0), a^*)^T & 0 & \cdot & \cdot & 0
\end{bmatrix},$$

with $D = \mathrm{diag}(H, I_n, \ldots, I_n, H)$. Then if $\tilde{L}^{-1}$ is the left inverse of $\tilde{L}$ determined above, $L_1^{-1} = D\tilde{L}^{-1}$ is a left inverse of L_1 and

$$\|L_1^{-1}\|_\infty \leq \|D\|_\infty \|\tilde{L}^{-1}\|_\infty \leq \max\{1, \sqrt{n}|H|\}\|\tilde{L}^{-1}\|_\infty.$$

Since $\lambda_1 < 1$ and $\beta < \alpha$, ρ satisfies the conditions of both Lemmas 1 and 2. We replace $H^{-1}P$ in L_1 by $H^{-1}P_-$, where P_- is from Lemma 2, and $H^{-1}(I_n - P)$ by $H^{-1}(I_n - P_+)$, where P_+ is from Lemma 1, to get the matrix $\hat{L}$. From the lemmas,

$$|H^{-1}P_+ - H^{-1}P|,\ |H^{-1}P_- - H^{-1}P| \leq \theta,$$

where

$$\theta = 2K^4|H^{-1}|\beta^{-1}(1 - \lambda_1)^{-1}\rho$$

so that

$$\|\hat{L} - L_1\|_\infty \leq 2\sqrt{n}\theta.$$

Since, using Eq. (20), $2\sqrt{n}\theta\|L_1^{-1}\|_\infty \leq 2\sqrt{n}\theta \max\{1, n|H|\}\|\tilde{L}^{-1}\|_\infty < 1$, it follows that $\hat{L}$ has the left inverse $L_1^{-1}(I + (\hat{L} - L_1)L_1^{-1})^{-1}$ and so is one to one.

Proof that existence of a left inverse for $\hat{L}$ implies (D2): First note that $\phi_x^{t-s}(q(s), a^*)$ is the transition matrix for Eq. (18). Suppose there is a nonzero bounded solution $x(t)$ of Eq. (18) which is not a multiple of $\dot{q}(t) = f(q(t), a^*)$. Then we may assume $f(q(0), a^*)^T x(0) = 0$. Then if we set $\xi_k = x(t_k)$ for $k = -N_2, \ldots, N_1$, we see that $\xi_{k+1} = x(t_{k+1}) = \phi_x^{h_k}(q(t_k), a^*))x(t_k) = \tilde{Y}_k \xi_k$ for $k = -N_2, \ldots, N_1 - 1$. Also since $x(t)$ is bounded on $[T_1, \infty)$ and T_1 satisfies the conditions of Lemma 1, we have that $H^{-1}(I_n - P_+)\xi_{N_1} = H^{-1}(I_n - P_+)x(T_1) = 0$ and since $x(t)$ is bounded on $(-\infty, -T_2]$ and T_2 satisfies the conditions of Lemma 2, we have $H^{-1}P_-\xi_{-N_2} = H^{-1}P_-x(-T_2) = 0$. That is, $\hat{L}\xi = 0$, where $\xi = (\xi_{-N_2}, \ldots, \xi_{N_1})$. So $\xi = 0$. This implies $x(t) = 0$, a contradiction. Thus a nonzero bounded solution of Eq. (18) must be a multiple of $\dot{q}(t)$. This completes the proof that (D2) holds.

3.4. Verification of (D3)

To verify (D3), we need to show $\dot{q}(t)e^{-\nu t}$ is not bounded on $\mathbb{R}_-$, where $q(t) = \phi^t(x_0, a^*)$ is our homoclinic solution. Note that $\dot{q}(t)e^{-\nu t}$ is a solution of

$$\dot{x} = [f_x(q(t), a^*) - \nu]x. \tag{22}$$

We find the subspace V of initial values at $t = -T_2$ of solutions of (22) which are bounded for $t \le -T_2$ (see Lemma 3 below). Then $\dot{q}(t)e^{-\nu t}$ is bounded on $\mathbb{R}_-$ if and only if there exists a solution $x(t)$ of (22) such that $x(-T_2) \in V$ and $x(0) = \dot{q}(0)$. It turns out that this happens if and only if the sequence $x(t_k)$, $k = -N_2, \ldots, 0$ is in the nullspace of the $[(N_2+2)n] \times [(N_2+1)n]$ matrix $\hat{C}$ given by

$$\hat{C} = \begin{bmatrix} -\tilde{Y}_{-N_2}e^{-\nu h_{-N_2}} & I_n & 0 & 0 & 0 \\ 0 & -\tilde{Y}_{-N_2+1}e^{-\nu h_{-N_2+1}} & I_n & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -\tilde{Y}_{-1}e^{-\nu h_{-1}} & I_n \\ H^{-1}Q_- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n - \tilde{e}\tilde{e}^T \end{bmatrix},$$

where $\tilde{Y}_k = \phi_x^{h_k}(x_k, a^*)$, $\tilde{e} = f(x_0, a^*)/|f(x_0, a^*)|$ and Q_- is a projection which determines V. This means we just have to show that the nullspace of $\hat{C}$ is 0 or, equivalently, that $\hat{C}$ has a left inverse. We approximate $\hat{C}$ by $\mathcal{C}$ in the Proposition below and so need some smallness conditions to conclude that $\hat{C}$ has a left inverse from the fact that $\mathcal{C}$ has a left inverse.

Proposition 2. *(D3) holds provided the matrix*

$$
\mathcal{C} = \begin{bmatrix}
-Y_{-N_2}e^{-\nu h_{-N_2}}H & I_n & 0 & 0 & 0 \\
0 & -Y_{-N_2+1}e^{-\nu h_{-N_2+1}} & I_n & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & -Y_{-1}e^{-\nu h_{-1}} & I_n \\
P_s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_n - e_0 e_0^T
\end{bmatrix}, \tag{23}
$$

where $Y_k = \phi_x^{h_k}(y_k, a_0)$ *and* $e_0 = \frac{f(y_0,a_0)}{|f(y_0,a_0)|}$, *has a left inverse* $\mathcal{C}^{-1}$, *and the following inequalities hold:*

$$\lambda_2 = K\{(\nu-\beta)^{-1} + (\sigma-\nu)^{-1} + K^2(\nu-\beta)^{-1}\}\rho < 1, \tag{24}$$

$$[2\sqrt{n}K^4 \max\{1, \sqrt{n}|H|\}|H^{-1}|(\nu-\beta)^{-1}(1-\lambda_2)^{-1}\rho + \delta_4]\|\mathcal{C}^{-1}\|_\infty < 1, \tag{25}$$

where ρ *is as in Eq. (15) and*

$$\delta_4 = \sqrt{n}\max\{\delta_2, \delta_2|H|, 8(M_1+M_3)C\delta/|f(y_0,a_0)|\}, \quad \delta_2 \text{ as in Eq. (17).} \tag{26}$$

For the proof we need the following lemma, in which we find the solutions of Eq. (22) bounded on $\mathbb{R}_-$. The proof is in Appendix 3.

Lemma 3. *Suppose* ρ *in Eq. (15) satisfies*

$$\rho < [K\{(\sigma-\nu)^{-1} + (\nu-\beta)^{-1} + K^2(\nu-\beta)^{-1}\}]^{-1}.$$

Then if $\xi \in \mathcal{N}(Q)$, *there exists a unique solution* $x(t) = x(t,\xi)$ *of Eq. (22) bounded on* $t \le -T_2$ *such that* $(I_n - Q)x(-T_2) = \xi$. *Moreover*

$$\sup_{t \le -T_2} |x(t)| \le (1 - K\{(\sigma-\nu)^{-1} + (\nu-\beta)^{-1}\}\rho)^{-1}K|\xi|$$

and the set of initial values $\{x(-T_2,\xi) : \xi \in \mathcal{N}(Q)\}$ *is the nullspace of a projection* Q_- *which has the same range as* Q *and satisfies*

$$|Q_- - Q| \le 2K^4(\nu-\beta)^{-1}\left(1 - K\{(\nu-\beta)^{-1} + (\sigma-\nu)^{-1} + K^2(\nu-\beta)^{-1}\}\rho\right)^{-1}\rho.$$

Proof of Proposition 2. First we approximate $\mathcal{C}$ by another matrix $\tilde{C}$, prove that $\tilde{C}$ has a left inverse, then use this to prove that $\hat{C}$ has a left inverse from which we show that (D3) follows.

Approximation of $\mathcal{C}$ by $\tilde{C}$ and proof that $\tilde{C}$ has a left inverse: Consider the matrix $\tilde{C}$ where we replace $Y_k = \phi_x^{h_k}(y_k, a_0)$ in $\mathcal{C}$ by $\tilde{Y}_k = \phi_x^{h_k}(x_k, a^*)$ and $f(y_0, a_0)$ by $f(x_0, a^*)$. From Eq. (17)

$$|Y_k - \tilde{Y}_k| \leq \delta_2.$$

Also, using Eq. (16), note that with $\tilde{e} = f(x_0, a^*)/|f(x_0, a^*)|$

$$|e_0 e_0^T - \tilde{e}\tilde{e}^T| \leq 2|e_0 - \tilde{e}| \leq 4\frac{|f(y_0, a_0) - f(x_0, a^*)|}{|f(y_0, a_0)|} \leq \frac{8(M_1 + M_3)C\delta}{|f(y_0, a_0)|}.$$

This means that $\|\tilde{C} - \mathcal{C}\|_\infty \leq \delta_4$. Since Eq. (25) implies that $\delta_4\|\mathcal{C}^{-1}\| < 1$, $\tilde{C}$ also has the left inverse $\mathcal{C}^{-1}(I + (\tilde{C} - \mathcal{C})\mathcal{C}^{-1})^{-1}$ and

$$\|\tilde{C}^{-1}\|_\infty \leq (1 - \delta_4\|\mathcal{C}^{-1}\|_\infty)^{-1}\|\mathcal{C}^{-1}\|_\infty.$$

Proof that $\hat{C}$ has a left inverse: Next note, using Eq. (13), that

$$\tilde{C} = \bar{C}D,$$

where

$$\bar{C} = \begin{bmatrix} -\tilde{Y}_{-N_2}e^{-\nu h_{-N_2}} & I_n & 0 & 0 & 0 \\ 0 & -\tilde{Y}_{-N_2+1}e^{-\nu h_{-N_2+1}} & I_n & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -\tilde{Y}_{-1}e^{-\nu h_{-1}} & I_n \\ H^{-1}Q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_n - \tilde{e}\tilde{e}^T \end{bmatrix},$$

and

$$D = \mathrm{diag}(H, I_n, \ldots, I_n).$$

Then if $\tilde{C}^{-1}$ is the left inverse of $\tilde{C}$ determined above, $\bar{C}^{-1} = D\tilde{C}^{-1}$ is a left inverse of $\bar{C}$ and

$$\|\bar{C}^{-1}\|_\infty \leq \|D\|_\infty\|\tilde{C}^{-1}\|_\infty \leq \max\{1, \sqrt{n}|H|\}\|\tilde{C}^{-1}\|_\infty.$$

Now since Eq. (24) implies that $\lambda_2 < 1$, ρ satisfies the condition in Lemma 3. We replace $H^{-1}Q$ in $\bar{C}$ by $H^{-1}Q_-$ to get the matrix $\hat{C}$. From Lemma 3

$$|H^{-1}Q_- - H^{-1}Q| \leq \theta = 2|H^{-1}|K^4(\nu - \beta)^{-1}(1 - \lambda_2)^{-1}\rho$$

so that

$$\|\hat{C} - \bar{C}\|_\infty \le \sqrt{n}\theta.$$

Since, by Eq. (25), $\sqrt{n}\theta\|\bar{C}^{-1}\|_\infty \le \sqrt{n}\theta \max\{1, \sqrt{n}|H|\}\|\tilde{C}^{-1}\|_\infty < 1$, it follows that $\hat{C}$ has the left inverse $\bar{C}^{-1}(I + (\hat{C} - \bar{C})\bar{C}^{-1})^{-1}$ and so is one to one.

Proof that existence of a left inverse for $\hat{C}$ implies (D3): Now $x(t) = \dot{q}(t)e^{-\nu t} = f(q(t), a^*)e^{-\nu t}$ is a solution of Eq. (22). Set $\xi_k = x(t_k)$ for $k = -N_2, \ldots, 0$. Then, noting that the transition matrix of (22) is $\phi_x^{t-s}(q(s), a^*)e^{-\nu(t-s)}$, we see that

$$\xi_{k+1} = x(t_{k+1}) = \phi_x^{h_k}(q(t_k), a^*)e^{-\nu h_k}x(t_k) = \tilde{Y}_k e^{-\nu h_k}\xi_k$$

for $k = -N_2, \ldots, -1$ and

$$\tilde{e}\tilde{e}^T\xi_0 = \tilde{e}\tilde{e}^T f(q(0), a^*) = |f(q(0), a^*)|\tilde{e}\tilde{e}^T\tilde{e} = |f(q(0), a^*)|\tilde{e} = f(q(0), a^*) = x(0) = \xi_0.$$

Suppose $x(t)$ is bounded on $\mathbb{R}_-$. Then since T_2 satisfies the condition in Lemma 3, $H^{-1}Q_-\xi_{-N_2} = H^{-1}Q_-x(-T_2) = 0$. Then $\hat{C}\xi = 0$ where $\xi = (\xi_{-N_2}, \ldots, \xi_0)$. Since $\hat{C}$ is one to one, ξ must be 0 and hence $x(t) = 0$. This is a contradiction. So $f(q(t), a^*)e^{-\nu t}$ is unbounded on $\mathbb{R}_-$ and (D3) follows.

3.5. Verification of (D4)

To verify (D4), we need to show that if $x(t)$ is a continuous function which is a bounded solution of the equation

$$\dot{x} = -[f_x(q(t), a^*)^T - \nu]x, \quad q(t) = \phi^t(x_0, a^*) \tag{27}$$

on $\mathbb{R}_+$ and a bounded solution of the equation

$$\dot{x} = -f_x(q(t), a^*)^T x \tag{28}$$

on $\mathbb{R}_-$, then $x(0) = 0$. First we find the subspace V_1 of initial values at $t = T_1$ of solutions of Eq. (27) which are bounded for $t \ge T_1$ (see Lemma 4 below) and also the subspace V_2 of initial values at $t = -T_2$ of solutions of Eq. (28) which are bounded for $t \le -T_2$ (see Lemma 5 below). Then the problem of finding bounded continuous functions $x(t)$ which solve Eq. (27) on $\mathbb{R}_+$ and Eq. (28) on $\mathbb{R}_-$ is reduced to the boundary value problem of finding the solutions $x(t)$ with $x(-T_2) \in V_2$ and $x(T_1) \in V_1$. It turns out that $x(t)$ is such a function if and only if the sequence $x(t_k)$, $-N_2 \le k \le N_1$ is in the nullspace of the $[(N_1+N_2+2)n] \times [(N_1+N_2+1)n]$ matrix

$$\hat{B} = \begin{bmatrix} -\tilde{Z}_{-N_2} & I_n & 0 & 0 & 0 \\ 0 & -\tilde{Z}_{-N_2+1} & I_n & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -\tilde{Z}_{N_1-1} & I_n \\ H^T R_- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H^T(I_n - R_+) \end{bmatrix},$$

where we obtain $\tilde{Z}_k$ from $\tilde{Y}_k = \phi_x^{h_k}(x_k, a^*)$ in the same way we obtain Z_k in $\mathcal{B}$ from $Y_k = \phi_x^{h_k}(y_k, a_0)$ in the matrix $\mathcal{B}$ in the Proposition below; R_+ and R_- are projections determining the subspaces V_1 and V_2. So to verify (D4), we just need to show this matrix has zero nullspace or, equivalently, has a left inverse. We approximate $\hat{B}$ by $\mathcal{B}$ in the Proposition below and so need some smallness conditions to conclude that $\hat{B}$ has a left inverse from the fact that $\mathcal{B}$ has a left inverse.

Proposition 3. *(D4) holds if the matrix*

$$\mathcal{B} = \begin{bmatrix} -Z_{-N_2}(H^{-1})^T & I_n & 0 & 0 & 0 \\ 0 & -Z_{-N_2+1} & I_n & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -Z_{N_1-1} & (H^{-1})^T \\ I_n - P_r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_s \end{bmatrix}, \tag{29}$$

where $Z_k = (Y_k^{-1})^T$ *for* $k < 0$ *and* $Z_k = e^{\nu h_k}(Y_k^{-1})^T$ *for* $k \ge 0$, *has a left inverse* $\mathcal{B}^{-1}$, *and the following inequalities are satisfied:*

$$\lambda_1 < 1, \quad \lambda_2 < 1, \quad M_9\delta_2 < 1, \tag{30}$$

$$[2nK^4|H|\,|H^{-1}|\max\{(\nu-\beta)^{-1}(1-\lambda_2)^{-1}, \beta^{-1}(1-\lambda_1)^{-1}\}\rho + \delta_5]\|\mathcal{B}^{-1}\|_\infty < 1, \tag{31}$$

where λ_1 *and* λ_2 *are as in Eq. (19) and Eq. (24),* ρ *is as in Eq. (15) and*

$$\delta_5 = \sqrt{n}\max\{e^{\nu h_{\max}}, |H^{-1}|\}(1-M_9\delta_2)^{-1}M_9^2\delta_2,\ M_9 = \sup|Y_k^{-1}|,\ \delta_2 \text{ as in Eq. (17).} \tag{32}$$

In the proof of Proposition 3, for $q(t) = \phi^t(x_0, a^*)$, we need to find the bounded solutions of Eq. (27) on $\mathbb{R}_+$ and the bounded solutions of Eq. (28) on $\mathbb{R}_-$. This we do in the two lemmas below. The proofs are in Appendix 3.

Lemma 4. *Suppose ρ in Eq. (15) satisfies*

$$\rho < [K\{(\sigma-\nu)^{-1} + (\nu-\beta)^{-1} + K^2(\nu-\beta)^{-1}\}]^{-1}.$$

Then if $\xi \in \mathcal{R}(I_n - Q^T)$, there exists a unique solution $x(t) = x(t,\xi)$ of Eq. (27) bounded on $t \geq T_1$ such that $(I_n - Q^T)x(T_1) = \xi$. Moreover

$$\sup_{t\geq T_1} |x(t)| \leq (1 - K\{(\sigma-\nu)^{-1} + (\nu-\beta)^{-1}\}\rho)^{-1}K|\xi|$$

and the set of initial values $\{x(T_1,\xi) : \xi \in \mathcal{R}(I_n - Q^T)\}$ is the range of a projection R_+ which has the same nullspace as $I_n - Q^T$ and satisfies

$$|R_+ - (I_n - Q^T)| \leq 2K^4(\nu-\beta)^{-1}\left(1 - K\{(\nu-\beta)^{-1} + (\sigma-\nu)^{-1} + K^2(\nu-\beta)^{-1}\}\rho\right)^{-1}\rho.$$

Lemma 5. *Suppose ρ in Eq. (15) satisfies*

$$\rho < [K(\alpha^{-1} + \beta^{-1} + K^2\beta^{-1})]^{-1}.$$

Then if $\xi \in \mathcal{N}(I_n - P^T)$, there exists a unique solution $x(t) = x(t,\xi)$ of Eq. (28) bounded on $t \leq -T_2$ such that $P^T x(-T_2) = \xi$. Moreover

$$\sup_{t\leq -T_2} |x(t)| \leq (1 - K(\alpha^{-1} + \beta^{-1})\rho)^{-1}K|\xi|$$

and the set of initial values $\{x(-T_2,\xi) : \xi \in \mathcal{N}(I_n - P^T)\}$ is the nullspace of a projection R_- which has the same range as $I_n - P^T$ and satisfies

$$|R_- - (I_n - P^T)| \leq 2K^4\beta^{-1}\left(1 - K(\alpha^{-1} + \beta^{-1} + K^2\beta^{-1})\rho\right)^{-1}\rho.$$

Proof of Proposition 3. First we approximate $\mathcal{B}$ by another matrix $\tilde{B}$, then prove that $\tilde{B}$ has a left inverse, use this to show $\hat{B}$ has a left inverse, from which we show (D4) follows.

Approximation of $\mathcal{B}$ by $\tilde{B}$ and proof that $\tilde{B}$ has a left inverse: Consider the matrix $\tilde{B}$ where we replace $Y_k = \phi_x^{h_k}(y_k, a_0)$ in $\mathcal{B}$ by $\tilde{Y}_k = \phi_x^{h_k}(x_k, a^*)$ (with corresponding $\tilde{Z}_k$). From Eq. (17), $|Y_k - \tilde{Y}_k| \le \delta_2$. Since $M_9\delta_2 < 1$, it follows that

$$|Y_k^{-1} - \tilde{Y}_k^{-1}| \le (1 - M_9\delta_2)^{-1} M_9^2 \delta_2$$

so that

$$|Z_k - \tilde{Z}_k| \le \begin{cases} (1 - M_9\delta_2)^{-1} M_9^2 \delta_2 & (-N_2 \le k < 0) \\ e^{\nu h_{\max}} (1 - M_9\delta_2)^{-1} M_9^2 \delta_2 & (0 \le k < N_1). \end{cases}$$

This means that $\|\tilde{B} - \mathcal{B}\|_\infty \le \delta_5$. Since Eq. (31) implies that $\delta_5 \|\mathcal{B}^{-1}\|_\infty < 1$, $\tilde{B}$ also has the left inverse $\tilde{B}^{-1} = \mathcal{B}^{-1}(I + (\tilde{B} - \mathcal{B})\mathcal{B}^{-1})^{-1}$ and

$$\|\tilde{B}^{-1}\|_\infty \le (1 - \delta_5 \|\mathcal{B}^{-1}\|_\infty)^{-1} \|\mathcal{B}^{-1}\|_\infty.$$

Proof that $\hat{B}$ has a left inverse: Next note, using Eqs. (10) and (13), that

$$\tilde{B} = \bar{B} D,$$

where

$$\bar{B} = \begin{bmatrix} -\tilde{Z}_{-N_2} & I_n & 0 & 0 & 0 \\ 0 & -\tilde{Z}_{-N_2+1} & I_n & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & -\tilde{Z}_{N_1-1} & I_n \\ H^T(I_n - P^T) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & H^T Q^T \end{bmatrix},$$

and

$$D = \mathrm{diag}((H^{-1})^T, I_n, \ldots, I_n, (H^{-1})^T).$$

Then if $\tilde{B}^{-1}$ is the left inverse of $\tilde{B}$ determined above, $\bar{B}^{-1} = D\tilde{B}^{-1}$ is a left inverse of $\bar{B}$ and

$$\|\bar{B}^{-1}\|_\infty \le \|D\|_\infty \|\tilde{B}^{-1}\|_\infty \le \sqrt{n}|H^{-1}| \|\tilde{B}^{-1}\|_\infty.$$

Next we replace H^TQ^T in $\bar{B}$ by $H^T(I_n - R_+)$ and $H^T(I_n - P^T)$ by H^TR_- to get the matrix $\hat{B}$. Now from Lemmas 4 and 5, since $\lambda_1, \lambda_2 < 1$ by Eq. (30),

$$|H^T(I_n - R_+) - H^TQ^T| = |H^TR_+ - H^T(I_n - Q^T)| \le \theta_1 = 2K^4|H|(\nu - \beta)^{-1}(1 - \lambda_2)^{-1}\rho$$

and

$$|H^TR_- - H^T(I_n - P^T)| \le \theta_2 = 2K^4|H|\beta^{-1}(1 - \lambda_1)^{-1}\rho.$$

Hence

$$\|\hat{B} - \bar{B}\|_\infty \leq \sqrt{n}\theta = \sqrt{n}\max\{\theta_1, \theta_2\}.$$

Since, by Eq. (31), $\sqrt{n}\theta\|\bar{B}^{-1}\|_\infty \leq n\theta|H^{-1}|\|\tilde{B}^{-1}\|_\infty < 1$, it follows that $\hat{B}$ has the left inverse $\bar{B}^{-1}(I + (\hat{B} - \bar{B})\bar{B}^{-1})^{-1}$ and so is one to one.

Proof that existence of a left inverse for $\hat{B}$ implies (D4): Note that $Z(t, s) = (\phi_x^{t-s}(q(s), a^*)^T)^{-1}$ is the transition matrix for Eq. (28) and $Z(t, s)e^{\nu(t-s)}$ is the transition matrix for Eq. (27). Let ξ be such that the solution $x(t)$ of Eq. (28) with $x(0) = \xi$ is bounded on $\mathbb{R}_-$ and the solution $x(t)$ of Eq. (27) with $x(0) = \xi$ is bounded on $\mathbb{R}_+$. Set $\eta_k = x(t_k)$ for $k = -N_2, \ldots, N_1$. Then $\eta_{k+1} = x(t_{k+1}) = Z(t_{k+1}, t_k)x(t_k) = \tilde{Z}_k\eta_k$ for $k = -N_2, \ldots, -1$ and $\eta_{k+1} = x(t_{k+1}) = e^{\nu h_k}Z(t_{k+1}, t_k)x(t_k) = \tilde{Z}_k\eta_k$ for $k = 0, \ldots, N_1 - 1$. Also since $|x(t)|$ is bounded on $\mathbb{R}_+$ and T_1 satisfies the condition in Lemma 4, we have $H^T(I_n - R_+)\eta_{N_1} = H^T(I_n - R_+)x(t_{N_1}) = H^T(I_n - R_+)x(T_1) = 0$ and since $|x(t)|$ is bounded on $\mathbb{R}_-$ and $-T_2$ satisfies the condition in Lemma 5, we have $H^T R_-\eta_{-N_2} = H^T R_- x(t_{-N_2}) = H^T R_- x(-T_2) = 0$. That is, $\hat{B}\eta = 0$, where $\eta = (\eta_{-N_2}, \ldots, \eta_{N_1})$. Since $\hat{B}$ is one to one, η must be 0 and hence $\xi = \eta_0 = 0$. So there does not exist $\xi \neq 0$ such that the solution $x(t)$ of Eq. (28) with $x(0) = \xi$ is bounded on $\mathbb{R}_-$ and the solution $x(t)$ of Eq. (27) with $x(0) = \xi$ is bounded on $\mathbb{R}_+$. Hence $q(t)$ satisfies condition (D4).

4. Example

In this section we carry out the detailed rigorous computations for a particular example. As in Battelli and Palmer [2011], consider the following 4−dimensional system depending on two parameters κ and γ:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + x_2 + \kappa x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -g(x_3) + x_1 + \gamma x_4,
\end{aligned} \tag{33}$$

with $g(y) = 2y^3 - y$, which corresponds to the system of coupled oscillators

$$\begin{aligned}
\ddot{x} - \dot{x} + x &= \kappa y \\
\ddot{y} - \gamma\dot{y} + g(y) &= x.
\end{aligned}$$

The second order equation $\ddot{y} + g(y) = 0$ has the homoclinic orbit $(\zeta_0(t), \dot{\zeta}_0(t))$ with $\zeta_0(t) = \text{sech}(t)$ associated with the saddle point (0, 0). The equilibrium (0, 0, 0, 0) is a saddle-focus for the unperturbed system (that is, $\kappa = \gamma = 0$) satisfying (D1) and $(0,\, 0,\, \zeta_0(t),\, \dot{\zeta}_0(t))$ is a homoclinic orbit satisfying (D2) and (D4) but not (D3). Note that the origin is an equilibrium for all values of the parameters κ and γ.

Battelli and Palmer [2011] showed the existence of a smooth curve $\gamma(\kappa)$ with $\gamma(0) = 0$ along which for $\kappa \neq 0$ small, the system possesses a saddle-focus homoclinic orbit satisfying all the conditions (D1)–(D4). We increase κ and use numerical continuation to follow the evolution of this orbit until we reach $\kappa = 1.0$ and obtain a value $\gamma = \gamma_0 = -0.53237259116071756$ for which a homoclinic orbit appears to exist. The graph of the corresponding parameter curve $\gamma(\kappa)$ for $\kappa \in [0,\, 1.0]$ is shown in Figure 1.

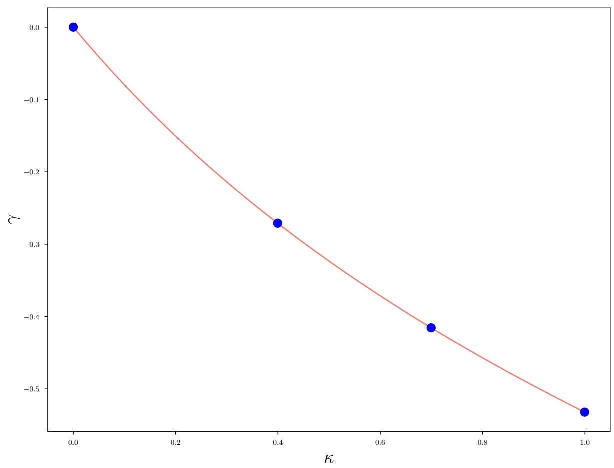

Figure 1. *Curve in the parameter space* $(\kappa,\, \gamma)$ *for which there is a numerically generated approximate homoclinic orbit. This curve is obtained by a continuation of the homoclinic orbit at the origin for* $\kappa \in [0,\, 1]$*. We will refer to this curve as the "standard curve."*

The projections of the approximate homoclinic orbits for four values of κ set to 0, 0.4, 0.7, 1, with the corresponding γ values 0, -0.271336, -0.415845, -0.532373, into the $(x_3,\, x_4)$-plane are plotted in Figure 2. The picture on the right in Figure 2 depicts a three-dimensional view of these four orbits in the $(x_2,\, x_3,\, x_4)$-space.

Two three-dimensional views of the approximate homoclinic orbit for $\kappa = 1.0$ are shown in Figure 3. Note that the existence of a true homoclinic orbit near this approximate homoclinic orbit does not follow from Battelli and Palmer [2011] as there the existence was only verified for small κ. What we want to do now is to use the theory de-

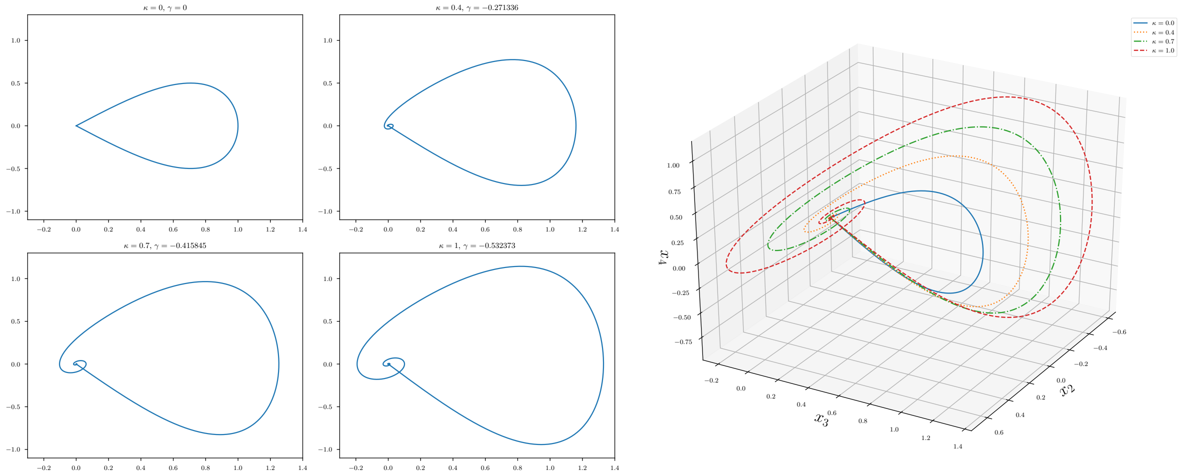

Figure 2. *Evolution of the approximate homoclinic orbits as the parameters are varied along the standard curve in Fig 1 from* $(\kappa, \gamma) = (0, 0)$ *to* $(\kappa, \gamma) = (1, -0.53237259116071756)$. *The four pictures on the left are the projections of the approximate homoclinic orbits onto the* (x_3, x_4)*-plane. Their parameter values on the standard curve are marked in Fig 1. The picture on the right depicts the three dimensional views of the same four approximate homoclinic orbits in the* (x_2, x_3, x_4)*-space.*

veloped in this paper to verify rigorously that for $\kappa = 1.0$, a true homoclinic orbit exists for some value of γ near -0.53237259116071756 and that the orbit satisfies (D1)–(D4).

4.1. Verification of the existence of the homoclinic orbit

First we verify the existence of the homoclinic orbit using the Existence Theorem for Homoclinic Orbits given in Section 2. The definitions of the relevant quantities can be found in Appendix 1 and more computational details are provided in Appendix 4. With κ fixed at 1.0, the system under consideration is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_2 + x_3 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= x_3 - 2x_3^3 + x_1 + \gamma x_4. \end{aligned} \tag{34}$$

We write this vector field in (33) as $f(x, \gamma)$ and the corresponding flow as $\phi^t(x, \gamma)$.

From the preliminary calculations made earlier, we determine for $\gamma = \gamma_0$, an approximate homoclinic orbit y_k, $k = -N_2, \ldots, N_1$, of (34), to the origin with associated

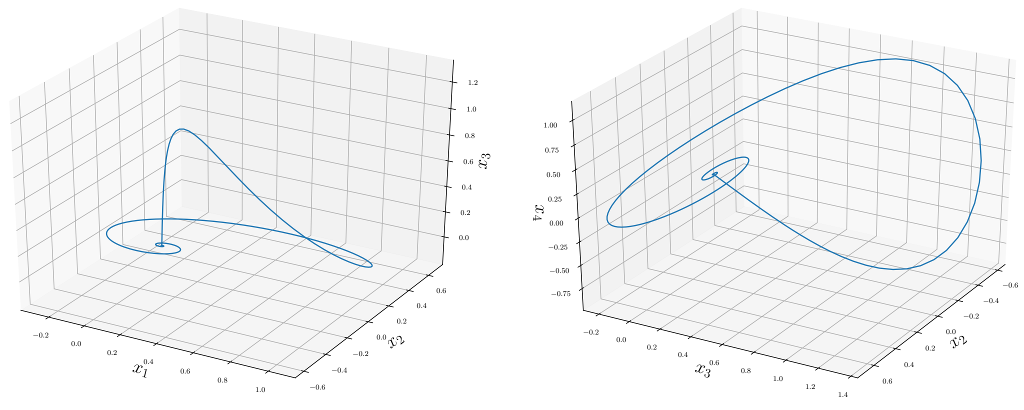

Figure 3. *Three-dimensional views of the approximate homoclinic orbit for* $(\kappa,\, \gamma) = (1,\, -0.53237259116071756)$*. The figure on the left is the projection into the* $(x_1,\, x_2,\, x_3)$*-space, and the figure on the right is the projection into the* $(x_2,\, x_3,\, x_4)$*-space. A rigorous verification of the existence a Shilnikov homoclinic orbit near this approximate orbit along with the conditions (D1)–(D4) is demonstrated.*

times h_k, $k = -N_2, \ldots, N_1 - 1$, and determine δ_1. The projections of the y_k into the $(x_2,\, x_3,\, x_4)$-space are depicted in Figure 4. More information about the computations involved here and in the calculation of δ can be found in Appendix 4 (i).

We calculate a rigorous containment region $U = \bigcup_k B_{2R_k}(y_k)$ around the homoclinic orbit, in which we obtain upper bounds for the constants M_i and M and which we also employ in order to calculate δ. Here the issues regarding rigor in the calculation of the M_i are handled as in subsection 6.4 in Coomes et al. [2016]. The projection of U into the three-dimensional (x_2, x_3, x_4)-space is depicted in Figure 5.

Next we determine the eigenvalues and eigenvectors of the linearization $f_x(0, \gamma_0)$ of (34) with $\gamma = \gamma_0$ at the origin and thus are able to determine the dichotomy constants K, α and β and bounds on $|H|$ and $|H^{-1}|$, where H is the matrix of eigenvectors. More information about the computations involved can be found in Appendix 4 (ii).

Then we find that the matrix L defined in Hypothesis 3 of the Existence Theorem is invertible and obtain an upper bound for $\|L^{-1}\|$. Here $v = 0$. We determine the errors in the calculation of Y_k and z_k as in the first part of subsection 6.7 in Coomes et al. [2016]. Next we follow the procedure in subsection 6.7 in Coomes et al. [2016] to verify

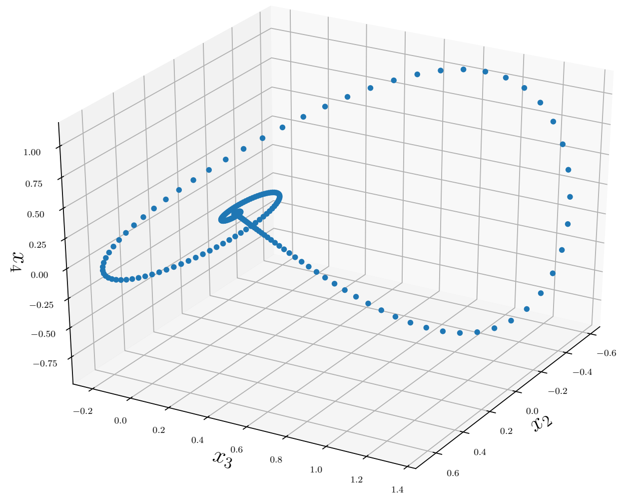

Figure 4. *The numerically computed approximate homoclinic orbit in Fig 3 for* $(\kappa, \gamma) = (1, -0.53237259116071756)$ *actually consists of 1562 points. There exists a true homoclinic orbit to the origin within* $4.4566093099211481 \times 10^{-11}$ *of this approximate orbit for a pair of parameter values* $\kappa = 1.0$ *and* $\gamma \in [-0.53237259120528369, -0.53237259111615143]$.

the invertibility of L. We find that L is invertible and obtain an upper bound for $\|L^{-1}\|$. It is important to note here that we regard L as a linear operator mapping a vector $(\xi_{-N_2}, \ldots, \xi_{N_1}, b)$ on to a vector $(g_{-N_2}, \ldots, g_{N_1-1}, q, \alpha)$, where b and α are scalars and q, the ξ's and g's are vectors in $\mathbb{R}^n$. The norm in the domain space is $\max\{\sup|\xi_k|, |b|\}$ and the norm in the range space is $\max\{\sup(|g_k|/h_k), |q|, |\alpha|\}$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. $\|L^{-1}\|$ means the corresponding operator norm. We list all the quantities and the inequalities needed to verify the conditions of the Theorem in Table 1; further details are given in Appendix 1.

The inequalities are all satisfied and the Existence Theorem for Homoclinic Orbits applies. So there exists γ^* within a distance

$$2C\delta = 4.4566093099211481 \times 10^{-11}$$

from γ_0 and hence in the interval $[-0.53237259120528369, -0.53237259111615143]$, and a point x_0 such that $\phi^t(x_0, \gamma^*) \to z^*$ as $|t| \to \infty$, where $f(z^*, \gamma^*) = 0$. However, the system (34) has three equilibria $(0, 0, 0, 0)$, $(1, 0, 1, 0)$ and $(-1, 0, -1, 0)$. Now, we

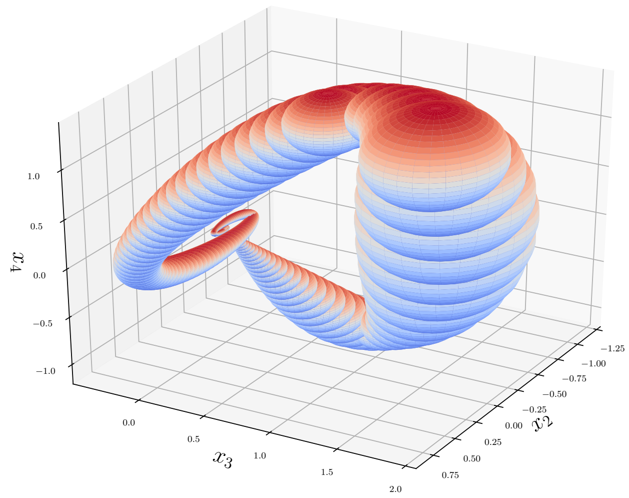

Figure 5. *The rigorous containment region U about the approximate homoclinic orbit in Fig 4 that contains the true homoclinic orbit.*

show that $z^* = (0, 0, 0, 0)$. Observe that $y_k = (0, 0, 0, 0)$ for large k and also

$$|y_k - z^*| \le |y_k - x_k| + |x_k - z^*| \le 2C\delta + |x_k - z^*| \to 2C\delta \quad \text{as } k \to \infty.$$

So $|z^*| \le 2C\delta$. Since $2C\delta < 2$, we must have $z^* = (0, 0, 0, 0)$. We easily verify there is a k such that $|y_k| > 2C\delta$ so that the homoclinic orbit does not coincide with the equilibrium. In fact, $y_0 = (0.810316, -0.540278, 0.999121, 1.08659)$. Hence we have verified the existence of a γ^* in the interval $[-0.53237259120528369, -0.53237259111615143]$, and a point x_0 such that $\phi^t(x_0, \gamma^*)$ is distinct from $z^* = (0, 0, 0, 0)$ and $\to z^*$ as $|t| \to \infty$.

4.2. Verification of (D1)–(D4)

Here we verify that the homoclinic orbit whose existence was established above satisfies all the Silnikov conditions (D1)–(D4). More computational details of these verifications are given in Appendix 4 (ii) and (iii).

The four eigenvalues of $f_x(z^*, \gamma^*)$ are $-\alpha_1$, σ_1, $\beta_1 \pm i\omega_1$, where $\alpha_1 \ge 1.4059364945327910$, $\sigma_1 \ge 1.1282487656074565$, $0.37265756884438972 \le \beta_1 \le 0.37265756895644164$, and $\omega_1 \ge 1.0592283643324152$. So these eigenvalues satisfy (D1). (D2) is automatic here because the dimension of the stable manifold is one.

For (D3) and (D4), we first verify that $\mathcal{C}$ (see Eq. (23)) has a left inverse $\mathcal{C}^{-1}$ and $\mathcal{B}$ (see Eq. (29)) has a left inverse $\mathcal{B}^{-1}$, and we determine upper bounds for their norms.

$$
\begin{aligned}
\gamma_0 &= -0.53237259116071756 & & \\
N_2 &= 1221 & N_1 &= 340 \\
h_{\max} &= 0.075000000000002842 & & \\
\delta_1 &= 2.4647560945643079 \times 10^{-15} & & \\
R_k &\leq 0.29428927321473447 & & \\
\delta &= 3.3237220276132601 \times 10^{-14} & & \\
M_0 &= 12.777479198229464 & M_1 &= 30.411711504022588 \\
M_2 &= 22.951480250413660 & M_3 &= 1.4129950679529282 \\
M_4 &= 1.0 & M_5 &= 0.0 \\
M_6 &= 224.58646912291854 & M_7 &= 12.900233994987197 \\
M_8 &= 0.53073919008341075 & M &= 2455.2976255375606 \\
K &= 2.2608982925794296 & \alpha &= 1.4059364946668425 \\
\beta &= 0.37265756890039831 & & \\
|H| &\leq 1.1608562757978054 & |H^{-1}| &\leq 1.9476125853957360 \\
\|L^{-1}\| &\leq 8.1541017311549702 & & \\
C &= 670.42449291726803 & f_\gamma(0,\gamma_0) &= 0 \\
\Delta_0 &= 0.02 & \mu &= 1.4192114116877699 \times 10^{-3} \\
\Delta_1 &= 6.5432609319166310 \times 10^{-5} & \Delta_2 &= 9.2402122741189533 \times 10^{-4}
\end{aligned}
$$

The inequalities to be verified:

$$
\begin{aligned}
&\text{(i)} \quad 2MC^2\delta = 7.3359874417271812 \times 10^{-5} < 1 \\
&\text{(ii)} \quad \delta_1 \leq \Delta_2 \\
&\text{(iii)} \quad 2C\delta = 4.4566093099211481 \times 10^{-11} < \min\{\Delta_1, \Delta_2\} \\
&\text{(iv)} \quad 2Ce^{M_1 h_{\max}}(1 + M_3 h_{\max})\delta = 4.8230594468091724 \times 10^{-10} \leq \mu \\
&\text{(v)} \quad C(M_6 + M_7)e^{M_1 h_{\max}}\delta_1 = 3.8400422852203796 \times 10^{-9} < 1
\end{aligned}
$$

Table 1. *The requisite constants and inequalities for the verification of the homoclinic orbit using the Existence Theorem for Homoclinic Orbits. Definitions of the constants are given in Appendix 1.*

Next, we determine the constants listed in Table 2, where H is the matrix of eigenvectors of $f_x(z, \gamma_0)$, σ is a lower bound for σ_1 and ν is a number between β_1 and σ_1. Finally, we establish the inequalities in Propositions 2 and 3, as listed in Table 2, thus verifying that our homoclinic orbit satisfies the Shilnikov conditions (D3) and (D4).

4.3. Further study of connecting orbits

We study Eq. (33) in the parameter range $\kappa > 0$ and $-1 < \gamma < 0$. It turns out that the orbit we have studied is just one among many.

First we make some general remarks about this system. Since the vector field is an odd function, system Eq. (33) with $g(y) = 2y^3 - y$, has reflection symmetry, that is, if $x(t)$ is a solution so also is $-x(t)$.

When $\kappa > 0$, the system has three equilibria

$$(0,\,0,\,0,\,0), \quad (\kappa\sqrt{(\kappa+1)/2},\,0,\,\sqrt{(\kappa+1)/2},\,0), \quad -(\kappa\sqrt{(\kappa+1)/2},\,0,\,\sqrt{(\kappa+1)/2},\,0).$$

When $-1 < \gamma < 0$, the derivative of the vector field at the origin always has a pair of complex eigenvalues with positive real parts and two nonzero real eigenvalues of opposite sign. At the other two equilibria, the vector field has the same derivative and there are two pairs of complex eigenvalues with real parts of opposite sign.

As mentioned earlier, we found the orbit studied in detail in sections 4.1 and 4.2 by starting with the homoclinic orbit in the system when $\gamma = \kappa = 0$ and increasing κ until we reached $\kappa = 1$. If we continue the standard curve in Figure 1 further in the parameter plane (κ, γ), we obtain the curve depicted in Figure 6.

Figure 7 shows ten representative homoclinic orbits along the extended standard curve as the curve converges towards a certain point. We have rigorously verified that all these homoclinic orbits exist, although not all the conditions (D1)-(D4) have been verified. As the point of convergence is approached, these homoclinic orbits appear to evolve into a heteroclinic cycle, that is, two heteroclinic connections between the origin and one of the other equilibria going opposite ways. Indeed, we have found numerically such a heteroclinic cycle between the origin and one of the other equilibria for parameter values near the point of convergence. However, the existence of this cycle, which is depicted in Figure 8, is yet to be rigorously verified. The point of convergence resembles a T-point as studied by Glendinning and Sparrow [1986], Kokubu [1993] and Knobloch et al. [2018]. We have also found several other parameter values (κ, γ), not on the extended standard curve, at which heteroclinic cycles appear to exist; for example, $(1.3028800206343, -0.721055813230436)$, $(1.00514637752089, -0.425294310240785)$, $(1.19897676718663, -0.70363368227532)$.

$$\begin{aligned}
\nu &= 0.75045316730285982 \\
\sigma &= 1.1282487657052955 \\
|H| &\le 1.1608562757978054 \\
|H^{-1}| &\le 1.9476125853957360 \\
\|\mathcal{C}^{-1}\|_\infty &\le 99.573180337289486 \\
\|\mathcal{B}^{-1}\|_\infty &\le 194.92669801482472 \\
\rho &= 2.1085860433082737 \times 10^{-8} && \text{(see (15))} \\
\delta_2 &= 7.7674418729513060 \times 10^{-9} && \text{(see (17))} \\
\lambda_1 &= 8.1575516263319998 \times 10^{-7} && \text{(see (19))} \\
\lambda_2 &= 8.9740092255911929 \times 10^{-7} && \text{(see (24))} \\
\delta_4 &= 1.5534883745902612 \times 10^{-8} && \text{(see (26) with } n = 4) \\
M_9 &= 1.4021856000798298 \\
\delta_5 &= 2.7230347931865971 \times 10^{-5} && \text{(see (32) with } n = 4))
\end{aligned}$$

The inequalities to be verified for (D3):

(i) $\lambda_2 < 1$

(ii) $\left[4K^4 \max\{1, 2|H|\}|H^{-1}|(\nu - \beta)^{-1}(1 - \lambda_2)^{-1}\rho + \delta_4\right] \|\mathcal{C}^{-1}\|_\infty < 1$

The inequalities to be verified for (D4):

(i) $\lambda_1 < 1$

(ii) $\lambda_2 < 1$

(iii) $M_9\delta_2 < 1$

(iv) $\left[8K^4|H|\,|H^{-1}| \max\{(\sigma - \nu)^{-1}(1 - \lambda_2)^{-1}, \beta^{-1}(1 - \lambda_1)^{-1}\}\rho + \delta_5\right] \|\mathcal{B}^{-1}\|_\infty < 1$

Table 2. *The requisite constants and the inequalities for the verification of Shilnikov conditions (D3) and (D4) using Propositions 2 and 3.*

We have identified other curves in the parameter space along which homoclinic orbits exist. These curves are depicted in Figure 9. At several points on these curves we have

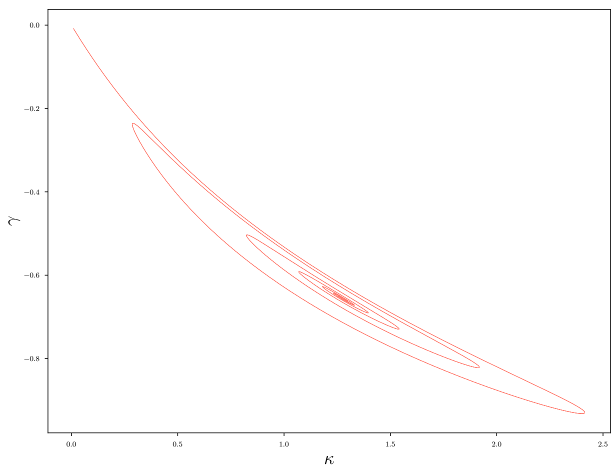

Figure 6. *Extension of the "standard curve" from Figure 1 in the parameter space* (κ, γ) *for which there is an approximate homoclinic orbit.*

been able to verify rigorously that a homoclinic orbit to the origin exists satisfying (D1). (D2) is automatic here and mostly also we have been able to verify (D3). However, in many cases we cannot verify (D4). One of the orbits for which everything has been verified is shown in Figure 10.

On one of the other parameter curves, at a point very close to the parameter values for the orbit we have studied in detail, we found a double-pulse homoclinic orbit. We could verify rigorously that this orbit is shadowed by a true orbit satisfying (D1)-(D3) but not (D4). We suspect that (D4) could be verified with calculations in higher order precision. This orbit is depicted in Figure 11.

5. Systems not satisfying all of (D1)–(D4)

Note that in our detailed example the unperturbed system satisfied (D1), (D2) and (D4) hold but not (D3). We can also easily construct systems where (D1) and (D2) hold but neither (D3) nor (D4) and also systems where (D1), (D2) and (D3) hold but not (D4).

First, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ be C^1 functions such that $f(x_0) = 0$, $A = f'(x_0)$ has eigenvalues $\beta \pm i\omega$; $g(y_0) = 0$, $g'(y_0)$ has eigenvalues $-\lambda_1 < 0 < \lambda_2$ with $\omega > 0, \quad 0 < \beta < \min\{\lambda_1, \lambda_2\}$; also $\dot{y} = g(y)$ has a solution $\zeta(t) \neq y_0$ such that $\zeta(t) \to y_0$ as $|t| \to \infty$. Then $z = (x_0, y_0)$ is an equilibrium for

$$\dot{x} = f(x), \quad \dot{y} = g(y)$$

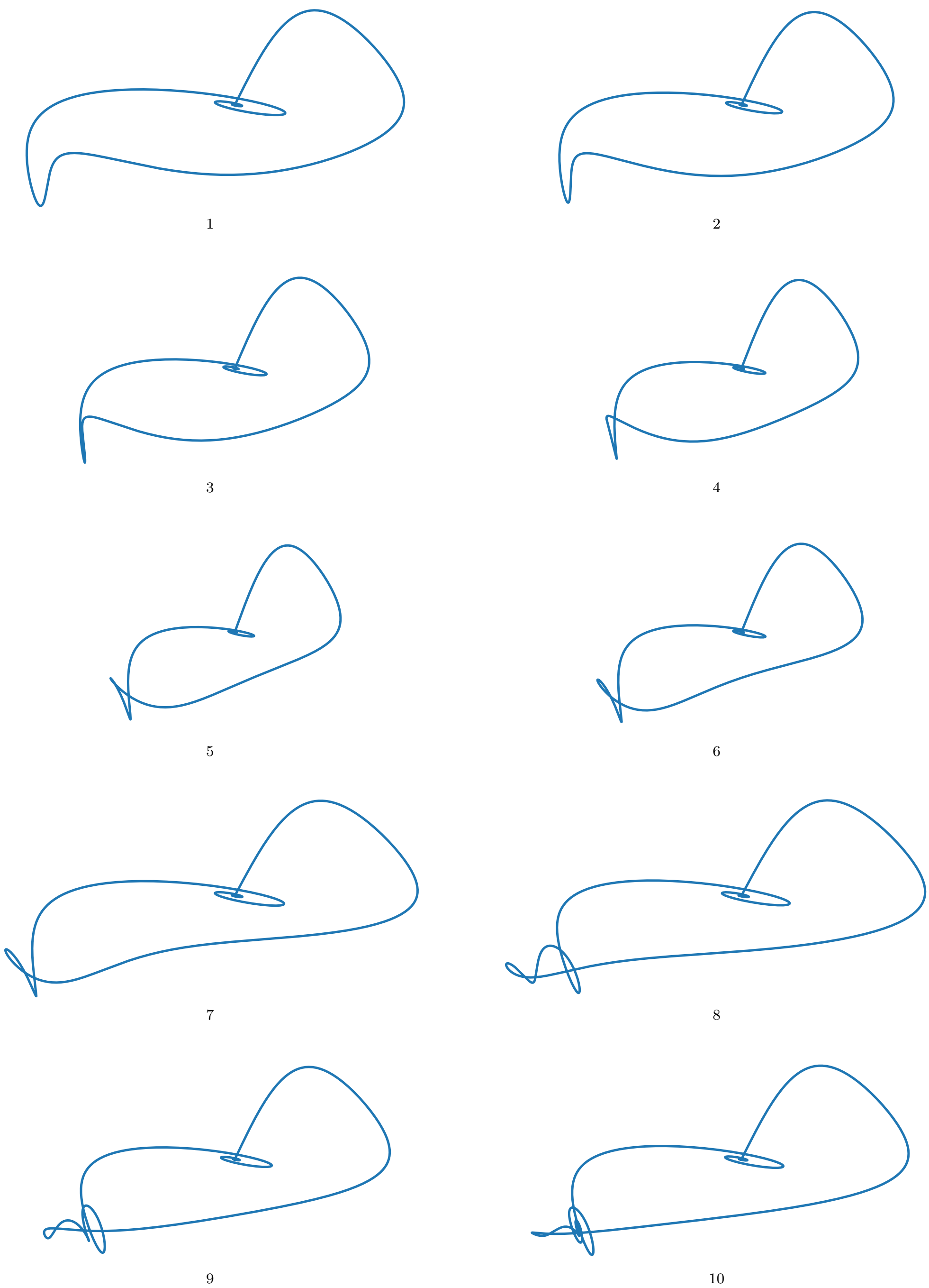

Figure 7. *Evolution of approximate homoclinic orbits to an apparent heteroclinic cycle as the parameters are varied along the standard curve in Figure 6 towards the point of convergence. A three-dimensional view of this cycle is depicted in Figure 8.*

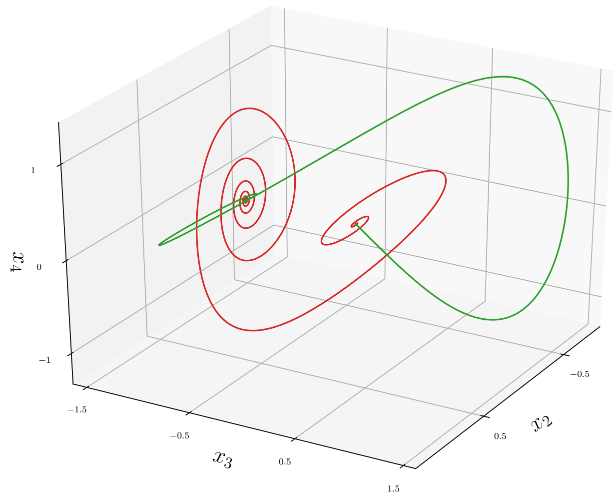

Figure 8. *A putative heteroclinic cycle between the origin and one of the other equilibria for the parameter values* $\kappa = 1.273569335850322$ *and* $\gamma = -0.6554219936232768$. *These parameter values are approximately the coordinates of the point to which the extended standard curve in Figure 6 is converging. The existence of this cycle is yet to be rigorously verified.*

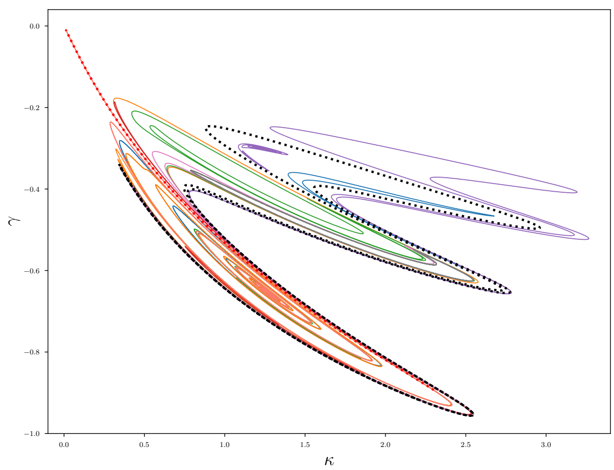

Figure 9. *Other curves in the parameter space along which homoclinic orbits may exist.*

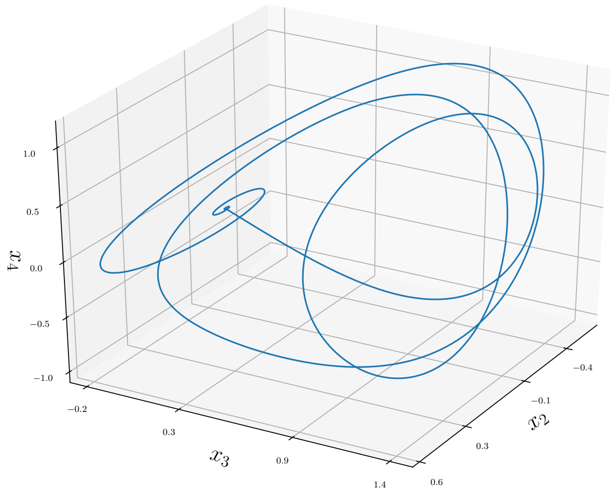

Figure 10. *An approximate homoclinic orbit at the point* $(\kappa, \gamma) = (1.0, -0.2930607170242212)$ *on a nonstandard curve. Near this computed orbit, there exists a true Shilnikov homoclinic orbit satisfying the conditions (D1)-(D4).*

satisfying (D1) with associated homoclinic orbit $q(t) = (x_0, \zeta(t))$ and it is easily verified that (D2) holds but neither (D3) nor (D4) holds.

Next suppose we take $f(x) = Ax$ and $x_0 = 0$, and let A, g, y_0 and $\zeta(t)$ be as in the preceding paragraph. In addition, let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 function satisfying $h(y_0) = 0$. Then $z = (0, y_0)$ is an equilibrium for

$$\dot{x} = Ax + h(y), \quad \dot{y} = g(y)$$

satisfying (D1) with associated homoclinic orbit $q(t) = (u(t), \zeta(t))$, where $u(t)$ is the unique bounded solution of $\dot{x} = Ax + h(\zeta(t))$. Then it is easily verified that (D2) holds but (D4) does not and if $\int_{-\infty}^{\infty} e^{-tA} h'(\zeta(t))\dot{\zeta}(t)dt \neq 0$, that (D3) holds. An explicit example is the system $\ddot{x} - 2a\dot{x} + x = y, \quad \ddot{y} - y + 2y^3 = 0$, where $0 < a < 1$.

System Eq. (33) in our detailed example corresponded to the case where (D1), (D2), (D4) hold but not (D3). There is a more general class of systems which has this property. Let f, A, g, x_0, y_0 and $\zeta(t)$ be as in the first paragraph of this section and suppose $h : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^1 function satisfying $h(x_0) = 0$. Then provided $\int_{-\infty}^{\infty} \psi^*(t)h'(x_0)e^{tA}dt \neq 0$, where $\psi(t)$ is, up to a scalar multiple, the unique nontrivial

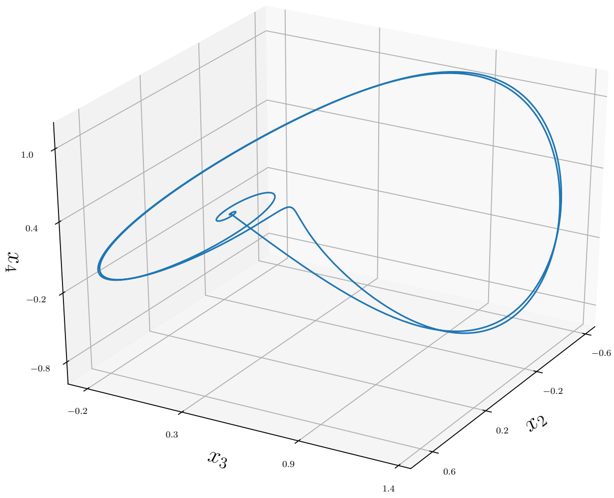

Figure 11. *An approximate "double pulse" homoclinic orbit for the parameter values* $(\kappa, \gamma) = (1.0, -0.5322792300363889)$. *Near this computed orbit, we can prove the existence of a Shilnikov homoclinic orbit satisfying the conditions (D1)-(D3) but not (D4).*

bounded solution of $\dot{y} = -g'(\zeta(t))^* y$, (D1), (D2) and (D4) hold for the equilibrium (x_0, y_0) and associated homoclinic orbit $(x_0, \zeta(t))$ of the system

$$\dot{x} = f(x), \quad \dot{y} = g(y) + h(x)$$

but not (D3).

Appendix 1: Constants and inequalities needed for Existence Theorem

A complete proof of the Existence Theorem for Homoclinic orbits stated in Section 2 is available in Coomes et al. [2016]. For the convenience of the reader, in this Appendix we collect certain quantitative information about the constant C in the Existence Theorem and the conditions which δ and δ_1 need to satisfy.

For each $k \in \mathbb{Z}$, R_k is a positive number such that if $0 \leq t \leq h_k$, then $|\phi^t(y_k, a_0) - y_k| \leq R_k$. Then the rigorous containment region is

$$U = \bigcup_k \{x \in \mathbb{R}^n : |x - y_k| \leq 2R_k\}.$$

Δ_0 is a positive number such that if $|x-z| \le \Delta_0$, then $x \in U$, and the constants M_1, M_2, M_3, M_4, M_5 are bounds on the norms of the derivatives f_x, f_{xx}, f_a, f_{xa}, f_{aa} in $x \in U$, $|a-a_0| \le \Delta_0$. Next we have the quantities

$$\begin{aligned} h_{\max} &= \sup h_k, \quad M_6 = M_2 e^{M_1 h_{\max}}, \quad M_7 = \frac{1}{2} M_2 M_3 h_{\max} e^{M_1 h_{\max}} + M_4, \\ M_8 &= \frac{1}{3} M_2 M_3^2 h_{\max}^2 \cosh(M_1 h_{\max}) + M_3 M_4 h_{\max} + M_5, \\ M &= (M_6 + 2M_7 + M_8) e^{M_1 h_{\max}}. \end{aligned} \tag{35}$$

In view of the eigenvalue assumptions in Hypothesis 1, there exist positive constants K, α and β such that for $t \ge 0$

$$|e^{tA} P| \le K e^{-\alpha t}, \quad |e^{-tA}(I_n - P)| \le K e^{-\beta t}. \tag{36}$$

Next, we have the quantities

$$\begin{aligned} \mu &= \min\{\Delta_0, [4K(\alpha^{-1} + \beta^{-1}) M_2]^{-1}\}, \\ \Delta_1 &= \min\{\Delta_0, [2K(\alpha^{-1} + \beta^{-1}) M_3]^{-1} \mu, [4K(\alpha^{-1} + \beta^{-1}) M_4]^{-1}\}, \end{aligned} \tag{37}$$

and Δ_2 as the largest positive number $\le \Delta_0$ such that for all k

$$[1 + M_3 h_k] e^{M_1 h_k} \Delta_2 \le R_k. \tag{38}$$

Recall that H is the matrix of the eigenvectors of A, and let

$$N = \max\{1, K|H^{-1}|(\beta^{-1} + \alpha^{-1} e^{\alpha h_{\max}})\}, \quad f_a = f_a(z, a_0). \tag{39}$$

Then the constant C is given by

$$C = 2 \max\left\{N\|L^{-1}\|, K\left[N\|L^{-1}\|((\alpha^{-1} + \beta^{-1})|f_a| + |H|) + \alpha^{-1} e^{\alpha h_{\max}} + \beta^{-1}\right]\right\}. \tag{40}$$

Finally, we list the inequalities that δ and δ_1 must satisfy:

$$2MC^2\delta < 1, \quad 2C\delta \le \Delta_1, \qquad 2C\delta < \Delta_2 \tag{41}$$

$$2Ce^{M_1 h_{\max}}(1 + M_3 h_{\max})\delta \le \mu, \quad \delta_1 \le \Delta_2, \quad C(M_6 + M_7) e^{M_1 h_{\max}} \delta_1 \le 1. \tag{42}$$

Appendix 2: Proofs of inequalities (7), (15), (16) and (17) for Section 3.
In these proofs we refer to Appendix 1 for the definitions of the quantities which appear.

First we prove (7). Since $y_k = z$ for large k, $x_k \to z(a^*)$ as $k \to \pm\infty$ and $|x_k - y_k| \leq 2C\delta$ for all k, it follows that

$$|z(a^*) - z| \leq 2C\delta \tag{43}$$

also. Next since by Eq. (38) and Eq. (41), $2C\delta < R_k$ for all k and $y_k = z$ for large $|k|$, it follows that $z(a^*)$ is in the ball of radius R_k, centre $y_k = z$, for large $|k|$ and so we can use the bounds M_i defined before Eq. (35). Then it follows that

$$|f_x(z(a^*), a^*) - f_x(z, a_0)| \leq M_2|z(a^*) - z| + M_4|a^* - a_0| \leq 2(M_2 + M_4)C\delta, \tag{44}$$

So inequality (7) is proved.

Inequalities (15) require us to show that for $t \geq T_1 = t_{N_1}$ and $t \leq -T_2 = t_{-N_2}$

$$\begin{aligned} |\phi^t(x_0, a^*) - z(a^*)| &\leq \mu_1 = (4C\delta + \delta_1)e^{M_1 h_{\max}}, \\ |f_x(\phi^t(x_0, a^*), a^*) - f_x(z, a_0)| &\leq \rho = M_2\mu_1 + 2(M_2 + M_4)C\delta. \end{aligned} \tag{45}$$

The main problem here is controlling what happens to $|\phi^t(x_0, a^*) - z(a^*)|$ when t is between t_k and $t_{k+1} = t_k + h_k$. Note since by Eqs. (5) and (41),

$$|x_k - y_k| \leq 2C\delta < \Delta_2, \quad |a^* - a_0| \leq 2C\delta < \Delta_2,$$

it follows from Lemma 1 in Coomes et al. [2016] that for all k

$$|\phi^t(x_k, a^*) - y_k| \leq 2R_k \quad \text{for} \quad 0 \leq t \leq h_k.$$

Next, using Eqs. (43), (3), (38), (41) and (42), we get

$$|z(a^*) - y_k| \leq |z(a^*) - z| + |y_k - z| \leq 2C\delta + \delta_1 \leq 2\Delta_2 \leq 2R_k \tag{46}$$

if $k \geq N_1$ or $k < -N_2$. It follows that for $t_k \leq t \leq t_{k+1}$ with $k \geq N_1$ and $k < -N_2$, the points $\phi^t(x_k, a^*)$ and $z(a^*)$ belong to a ball of radius $2R_k$ centered at some y_k and therefore we may use the M_i defined before Eq. (35) to obtain for these k and t that

$$\begin{aligned} |f(\phi^t(x_k, a^*), a^*) - f(z(a^*), a^*)| &\leq M_1|\phi^t(x_k, a^*) - z(a^*)|, \\ |f_x(\phi^t(x_k, a^*), a^*) - f_x(z(a^*), a^*)| &\leq M_2|\phi^t(x_k, a^*) - z(a^*)|. \end{aligned} \tag{47}$$

Then if $k \geq N_1$ and $0 \leq t \leq h_k$, it follows that

$$\begin{aligned} |\phi^t(x_k, a^*) - z(a^*)| &= \left| x_k - z(a^*) + \int_0^t [f(\phi^s(x_k, a^*), a^*) - f(z(a^*), a^*)]ds \right| \\ &\leq |x_k - z(a^*)| + M_1 \int_0^t |\phi^s(x_k, a^*) - z(a^*)|ds \end{aligned}$$

and hence, by Gronwall's lemma and using Eqs. (5) and (46), we obtain for $0 \le t \le h_k$

$$|\phi^t(x_k, a^*) - z(a^*)| \le |x_k - z(a^*)|e^{M_1 t} \le (4C\delta + \delta_1)e^{M_1 h_{\max}}.$$

It follows that for $t \ge t_{N_1}$, and similarly for $t \le t_{-N_2}$ that

$$|\phi^t(x_0, a^*) - z(a^*)| \le \mu_1 = (4C\delta + \delta_1)e^{M_1 h_{\max}}.$$

Hence we have derived the first inequality in Eq. (45). Also we conclude that for the same t, using Eq. (47), that

$$|f_x(\phi^t(x_0, a^*), a^*) - f_x(z(a^*), a^*)| \le M_2\mu_1$$

and hence, using Eq. (7), that for $t \ge T_1 = t_{N_1}$ and $t \le -T_2 = t_{-N_2}$

$$|f_x(\phi^t(x_0, a^*), a^*) - f_x(z, a_0)| \le \rho = M_2\mu_1 + 2(M_2 + M_4)C\delta.$$

Thus we have derived the second inequality in Eq. (45).

Next we prove inequality (16). First since $|y_0 - x_0| \le 2C\delta < \Delta_2 < R_k$, y_0 and x_0 are both in the ball of radius $2R_0$, centre y_0. Also $|a^* - a_0| \le 2C\delta \le \Delta_1 \le \Delta_0$. So we may use M_1 and M_3 as Lipschitz constants to get

$$|f(y_0, a_0) - f(x_0, a^*)| \le M_1|y_0 - x_0| + M_3(a_0 - a^*| \le 2C(M_1 + M_3)\delta,$$

thus proving inequality (16).

Finally we prove inequality (17). In fact, using Lemma 2 in Coomes et al. [2016], we obtain for $k = -N_2 + 1, \dots, N_1 - 1$

$$|\phi_x^{h_k}(x_k, a^*) - \phi_x^{h_k}(y_k, a_0)| \le M_6 h_k e^{M_1 h_k}|x_k - y_k| + M_7 h_k e^{M_1 h_k}|a^* - a_0| \le \delta_2,$$

where

$$\delta_2 = 2C(M_6 + M_7)h_{\max}e^{M_1 h_{\max}}\delta,$$

thus establishing Eq. (17).

Appendix 3: Proofs of lemmata

Here we give the proofs of Lemmas 1 to 5 in Section 3.

Proof of Lemma 1. Let E be the Banach space of continuous $\mathbb{R}^n-$valued functions $x(t)$ on $[T_1, \infty)$ equipped with the supremum norm $\|\cdot\|_\infty$. We define $T : E \to E$ according to

$$(Tx)(t) = e^{(t-T_1)A}\xi + \int_{T_1}^{t} e^{(t-s)A} PB(s)x(s)ds - \int_{t}^{\infty} e^{(t-s)A}(I_n - P)B(s)x(s)ds, \quad (48)$$

where $B(t) = f_x(q(t), a^*) - A$. We see that Tx is continuous and, using the inequalities (11), we find that

$$\|Tx\|_\infty \le K|\xi| + K(\alpha^{-1} + \beta^{-1})\rho\|x\|_\infty < \infty \quad (49)$$

so that Tx is in E. Moreover, if x and y are in E, then

$$\|Tx - Ty\|_\infty \le K(\alpha^{-1} + \beta^{-1})\rho\|x - y\|_\infty.$$

Thus T is a contraction and its unique fixed point is the desired solution. The inequality for $\sup_{t\ge T_1} |x(t)|$ follows from Eq. (49) taking $Tx = x$. We denote this solution by $x(t, \xi)$.

From Eq. (48) with $Tx = x$ and $t = T_1$, we see that

$$x(T_1, \xi) = R_+\xi = \xi - \int_{T_1}^{\infty} e^{(T_1-s)A}(I_n - P)B(s)x(s, \xi)ds.$$

R_+ is linear since, by uniqueness, $x(t, \xi)$ is linear in ξ. Then

$$|R_+\xi - \xi| \le K^2\alpha\rho(\alpha\beta - K(\alpha + \beta)\rho)^{-1}|\xi| = \rho_1|\xi|.$$

Define

$$S = R_+P + I_n - P.$$

Then

$$|S\xi - \xi| = |R_+P\xi - P\xi| \le \rho_1|P\xi| \le K\rho_1|\xi|.$$

Since $K\rho_1 < 1$, S is invertible and

$$|S - I_n| \le K\rho_1, \quad |S^{-1}| \le (1 - K\rho_1)^{-1}, \quad |S^{-1} - I_n| \le (1 - K\rho_1)^{-1}K\rho_1.$$

The projection

$$P_+ = SPS^{-1}$$

has the same nullspace as P and its range is the range of R_+. This means that the solutions $x(t)$ of Eq. (18) bounded on $[T_1, \infty)$ are exactly those with $x(T_1)$ in the range of P_+. Note that

$$|P_+ - P| \le |S - I_n|\,|P|\,|S^{-1}| + |P|\,|S^{-1} - I_n| \le 2(1 - K\rho_1)^{-1}K^2\rho_1.$$

This completes the proof of the lemma.

Proof of Lemma 2. This follows from Lemma 1 by reversing time, that is, by applying Lemma 1 to the equation $\dot{y} = -f_x(q(-t), a^*)y$, replacing A, α, β by $-A$, β, α.

Proof of Lemma 3. This follows from Lemma 2 with $f_x(q(t), a^*)$, A, P, α, β replaced by $f_x(q(t), a^*) - \nu I_n$, $A - \nu I_n$, Q, $\nu - \beta$, $\sigma - \nu$, respectively.

Proof of Lemma 4. This is proved as Lemma 1, replacing $f_x(q(t), a^*)$, A, P, α, β by $-(f_x(q(t), a^*)^T - \nu)$, $-(A^T - \nu I_n)$, $I_n - Q^T$, $\sigma - \nu$, $\nu - \beta$ respectively, and using the fact that with $A_1 = A - \nu I_n$, taking transposes in Eq. (14) and using $AQ = QA$, we have for $t \ge 0$,

$$|e^{t(-A_1^T)}(I_n - Q^T)| \le Ke^{-(\sigma-\nu)t}, \quad |e^{-t(-A_1^T)}Q^T| \le Ke^{-(\nu-\beta)t}.$$

Proof of Lemma 5. This is proved as Lemma 2, replacing $f_x(q(t), a^*)$, A, P, α, β by $-f_x(q(t), a^*)^T$, $-A^T$, $I_n - P^T$, β, α respectively and using the fact that taking transposes in Eq. (11) and using $AP = PA$, we have for $t \ge 0$,

$$|e^{t(-A^T)}(I_n - P^T)| \le Ke^{-\beta t}, \quad |e^{-t(-A^T)}P^T| \le Ke^{-\alpha t}.$$

Appendix 4: Computational details for the example

Here we supply more details concerning the computations in the example in Section 4. First in (i) we describe how the approximate homoclinic orbit is found, and how the rigorous containment region and the quantities δ and δ_1 are determined. Next in (ii) we show how the eigenvalues and dichotomy constants are determined, and how (D1) is verified. Finally in (iii) we describe how to verify that $\mathcal{C}$ from Propsition 2 (hence also $\mathcal{B}$ from Proposition 3) has a left inverse and calculate an upper bound on its norm.

(i) *Determination of the approximate homoclinic orbit and parameter value, the rigorous containment region, and δ_1 and δ:* To obtain the approximate homoclinic orbit and parameter value in Eq. (33), we set κ to a small positive value and use Beyn's method (see Eqs. (4.1a), (4.1b), (4.1c), (3.6) in Beyn [1990]) with (0, 0, $\zeta_0(t)$, $\dot{\zeta}_0(t)$)

as our initial orbit and parameter value $\gamma = 0$ as an initial guess for the BVP solver, padding the initial orbit with copies of the appropriate equilibrium at each end as needed so that the output of Beyn's method is suitably close to the equilibrium at each end. The BVP solver we use is the 2012 version of the Fortran 90/95 software developed by Boisvert et al. [2012]. Then Beyn's method yields the corresponding value of γ and the new approximate homoclinic orbit. Then we use these as initial guesses for a slightly higher value of κ. And so on until we reach $\kappa = 1.0$. The raw output of the BVP solver is in general a variable step size orbit. We then use the solver's interpolating routines to produce two constant step size orbits, one for forward time and one for backward time since the time scaling for the BVP is different in these two directions. Because the solver produces times instead of steps and because of round off errors, the step size h_k ends up being not quite uniform.

Thus we obtain finite sequences y_k, $-N_2 = -1221 \le k \le N_1 = 340$ and h_k, $-N_2 \le k < N_1$ and a parameter value

$$\gamma = -0.53237259116071756$$

such that

$$h_{\min} = 0.074999999999988631 \le h_k \le 0.075000000000002842 = h_{\max}.$$

Then if we define $y_k = z_+$ for $k > N_1$, $h_k = h_{\max}$ for $k \ge N_1$ and $y_k = z_-$, $h_k = h_{\max}$ for $k < -N_2$, we obtain infinite sequences y_k, h_k as required in the definition of approximate homoclinic orbit. Moreover we have

$$|y_{N_1}| \le \delta_1 = 2.4647560945643079 \times 10^{-15}.$$

so that Eq. (3) holds.

We follow the procedure described in subsection 6.3 in Coomes et al. [2016] with $m = 3$ to compute the sequence of positive numbers $\{R_k\}_{k=-N_2}^{N_1}$ and hence the rigorous containment region $U = \bigcup_k \{x \in \mathbb{R}^n : |x - y_k| \le 2R_k\}$.

We determine δ in Hypothesis 2 following the procedure given in the last paragraph in subsection 6.2 in Coomes et al. [2016]. The only difference here is that we work in each ball center y_k, radius R_k, rather than in the trapping region which is not available here.

(ii) *Eigenvalues, dichotomy constants and verification of (D1):* For the matrix

$$A = A(\gamma_0) = f_x(0, \gamma_0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & \gamma_0 \end{bmatrix},$$

we use a standard linear algebra routine to compute approximations for the eigenvalues $-\tilde{\alpha} = 1.4059364946668556$, $\tilde{\sigma} = 1.1282487657053040$, $\tilde{\beta} \pm i\tilde{\omega}$ with $\tilde{\beta} = 0.37265756890041568$, $\tilde{\omega} = 1.0592283643884413$. The eigenvectors corresponding to $-\tilde{\alpha}$, $\tilde{\sigma}$ and $\tilde{\beta} + i\tilde{\omega}$ are approximately $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{v} + i\tilde{w}$, where $\tilde{H} = [\tilde{u}_1\ \tilde{v}\ \tilde{w}\ \tilde{u}_2]$, is given by

$$\tilde{H} = \begin{bmatrix} 0.128939 & -0.201547 & 0.572869 & 0.436384 \\ -0.18128 & -0.681907 & 0 & 0.492349 \\ 0.565086 & 0.226242 & -0.149426 & 0.499527 \\ -0.794475 & 0.242587 & 0.183958 & 0.563591 \end{bmatrix}.$$

Now, using the approximate eigenvalues and eigenvectors just found, we want to rigorously estimate the eigenvalues and eigenvectors of A but we also want to estimate the eigenvalues of the matrix $A(\gamma)$, noting that $|A(\gamma) - A|_\infty \leq 2C\delta$. To this end, we use a similar method to that used in Symm and Wilkinson [1980] and Yamamoto [1980, 1982] and apply the Newton method Lemma 4.1 of Coomes et al. [1994] to, in the case of a real eigenvalue λ, the function G from $\mathbb{R} \times \mathbb{R}^4$ to $\mathbb{R}^4$, given by

$$G(\lambda, v) = A(\gamma)v - \lambda v,$$

and in the case of a complex eigenvalue $\alpha + i\beta$ to the function G from $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R}^4$ to $\mathbb{R}^4 \times \mathbb{R}^4$ given by

$$G(\alpha, \beta, w, v) = [\, A(\gamma)v - \alpha v + \beta w \quad A(\gamma)w\beta v - \alpha w \,].$$

We conclude that $A(\gamma)$ has eigenvalues $-\alpha_1$, σ_1, $\beta_1 \pm i\omega_1$, where $\alpha_1 \geq 1.4059364945327910$, $\sigma_1 \geq 1.1282487656074565$, $0.37265756884438972 \leq \beta_1 \leq 0.37265756895644164$, $\omega_1 \geq 1.0592283643324152$. So these eigenvalues satisfy (D1).

In particular, we conclude that A has eigenvalues $-\alpha_2$, σ_2, $\beta_2 \pm i\omega_2$, where $\alpha_2 \geq \alpha = 1.4059364946668425$, $\sigma_2 \geq \sigma = 1.1282487657052955$, $\beta_2 \geq \beta = 0.37265756890039831$, $\omega_2 \geq 1.0592283643884237$. The corresponding eigenvectors are u_1, u_2, $v \pm iw$ where if $H = [u_1\ v\ w\ u_2]$, using a standard a posteriori technique applied to an approximate inverse of $\tilde{H}$, we verify that H is invertible and obtain the rigorous upper bounds

$$|H| \leq 1.1608562757978054, \quad |H^{-1}| \leq 1.9476125853957360.$$

Next, with a view to obtaining the dichotomy constants, note that

$$AH = HD, \quad \text{where} \quad D = \begin{bmatrix} -\alpha_2 & 0 & 0 & 0 \\ 0 & \beta_2 & -\omega_2 & 0 \\ 0 & \omega_2 & \beta_2 & 0 \\ 0 & 0 & 0 & \sigma_2 \end{bmatrix},$$

so that $e^{tA}H = He^{tD}$ and hence for $t \geq 0$,

$$|e^{tA}P| = |He^{tD}P_1H^{-1}| \leq Ke^{-\alpha t}, \quad |e^{-tA}(I-P)| = |He^{tD}(I-P_1)H^{-1}| \leq Ke^{-\beta t},$$

where $P = HP_1H^{-1}$ (recall that $P_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$) and

$$|H^{-1}||H| \leq K = 2.2608982925794296,$$

and

$$|e^{t(A-\nu I)}Q| \leq Ke^{-(\nu-\beta)t}, \quad |e^{-t(A-\nu I)}(I-Q)| \leq Ke^{-(\sigma-\nu)t},$$

where $Q = HP_3H^{-1}$. So our dichotomy constants are

$$\begin{aligned} \alpha &= 1.4059364946668425, \quad & \beta &= 0.37265756890039831, \\ \sigma &= 1.1282487657052955, \quad & K &= 2.2608982925794296. \end{aligned}$$

(iii) *Verification that $\mathcal{C}$ has a left inverse and calculation of an upper bound on its norm:* Here we show how we verify that $\mathcal{C}$ has a left inverse and we find an upper bound for the norm of a left inverse. Denote by $\hat{C}$ the computed $\mathcal{C}$. We have a rigorous upper bound for $|\hat{C} - \mathcal{C}| \leq E$, where here if A is a matrix $[a_{ij}]$, then $|A| = [|a_{ij}|]$ is the matrix of absolute values, E is a matrix and $\leq$ is to be interpreted entrywise. We perform a QR factorization $\hat{C} = QR$, where R is square upper triangular, and set $T = R^{-1}Q^T$ as computed. We obtain a rigorous upper bound $|T| \leq F$ and hence also a rigorous bound for $\|T\|_\infty$. Proceeding as we do with L in subsection 6.7 in Coomes et al. [2016], we find a rigorous ρ_1 such that $\|T\hat{C} - I\|_\infty \leq \rho_1$. Then we estimate

$$|T\mathcal{C} - T\hat{C}| = |T(\hat{C} - \mathcal{C})| \leq |T|\,|\hat{C} - \mathcal{C}| \leq EF$$

and hence, by obtaining an upper bound for $\|EF\|_\infty$, get a rigorous ρ_2 such that

$$\|T\mathcal{C} - T\hat{C}\|_\infty \leq \rho_2.$$

Then

$$\|T\mathcal{C} - I\|_\infty \leq \rho = \rho_1 + \rho_2.$$

If $\rho < 1$, then we conclude that $T\mathcal{C}$ is invertible and $|(T\mathcal{C})^{-1}| \leq (1-\rho)^{-1}$. Then

$$(T\mathcal{C})^{-1}T\mathcal{C} = I$$

so that $(T\mathcal{C})^{-1}T$ is a left inverse for $\mathcal{C}$ (implying that $\mathcal{C}$ is injective) and its infinity norm is bounded by $(1-\rho)^{-1}\|T\|_\infty$ which is a bound for $\|\mathcal{C}^{-1}\|_\infty$.

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