Non-Existence of Black Holes in Certain $\Lambda < 0$ Spacetimes.

G.J. Galloway[a]*, S. Surya[b,c]†, E. Woolgar[c,b] ‡

[a]Dept. of Mathematics
University of Miami, Coral Gables, FL 33124, USA

[b]Theoretical Physics Institute, University of Alberta
Edmonton, AB, Canada T6G 2G1

[c]Dept. of Mathematical Sciences, University of Alberta
Edmonton, AB, Canada T6G 2G1

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Abstract

Assuming certain asymptotic conditions, we prove a general theorem on the non-existence of static regular black holes in spacetimes with a negative cosmological constant, given that the fundamental group of space is infinite. We use this to rule out the existence of regular negative mass AdS black holes with Ricci flat scri. For any mass, we also rule out a class of conformally compactifiable static black holes whose conformal infinity has positive scalar curvature and infinite fundamental group, subject to our asymptotic conditions. In a limited, but important, special case our result adds new support to the AdS/CFT inspired positive mass conjecture of Horowitz and Myers.

1 Introduction

The discovery of new classes of asymptotically Anti-de Sitter (AdS) black holes in the past decade [1, 2, 3, 4, 5, 6] has brought to light the fact that spacetimes with negative cosmological constant can admit black hole event horizons of nonspherical spatial topology. This is possible because these black holes violate neither of the two known obstructions to nontrivial horizon topology. The first of these is the theorem of Hawking [7], which assumes an energy condition incompatible with a negative cosmological constant and so does not apply. The second is topological

*galloway@math.miami.edu
†e-mail: ssurya@pims.math.ca
‡e-mail: ewoolgar@math.ualberta.ca

censorship [8, 9], which applies regardless of the cosmological constant but is not in itself a constraint on horizon topology. It merely enforces a topological relation between the horizon and the Penrose conformal boundary scri (when there is one), which consequently is also spatially nonspherical for black holes with nonspherical horizons. [1]

Despite differing from anti-de Sitter spacetime in the global structure of scri, these spacetimes still satisfy fall-off conditions that are essentially those given by Henneaux and Teitelboim [10, 11] (see also [12, 13, 14]) for asymptotically anti-de Sitter spacetimes, and so serve as examples of what are called asymptotically *locally* AdS (AL-AdS) spacetimes. Spacelike slices of scri are homogeneous manifolds of either positive, zero, or negative Ricci curvature. Interestingly, when these have negative Ricci curvature, the associated black holes can have negative mass [5, 6].

In this paper, we concern ourselves with related black hole existence issues. These are motivated by a curious property of asymptotically locally AdS spacetimes with nonspherical scri, namely that the lowest energy solutions or ground states need not be locally AdS (*i.e.*, constant curvature). Indeed, assuming a generalisation of the AdS/CFT correspondence conjecture [15, 16, 17] (which states that string theory on a certain compactification of ten dimensional supergravity to five dimensional AdS spacetime is equivalent to a certain conformal field theory on scri). In 5 spacetime dimensions, Horowitz and Myers conjectured that when the conformal boundary is Ricci flat, the ground state is a negative mass (but nonsingular!) nonconstant curvature spacetime called the AdS soliton [18]. In [19, 20] we proved (in all spacetime dimensions ≥ 4) that the AdS soliton is the unique negative mass *globally* static vacuum solution satisfying certain asymptotic and topological conditions. In other words, there is a "mass gap" in this class of static solutions. Dropping the requirement of global staticity allows us to include black hole regions in the spacetime (we retain the assumption of an irrotational timelike Killing field in the domain of outer communications: such spacetimes are called static but not globally static). However, as we will show, there exist no regular, static, negative mass black holes with Ricci flat scri, so that our uniqueness result holds more generally. This result is summarised in:

Proposition 1.1 *Let (Σ, h, N) be an $n+1$ dimensional ($n \geq 3$) static AL-AdS spacetime which satisfies the null energy condition $R_{ab}k^a k^b \geq 0\ \forall$ null k^a, has Ricci flat scri, and has Ashtekar-Magnon mass aspect pointwise negative on scri. If $\pi_1(\Sigma)$ is infinite and if the asymptotic condition (S) (Definition 2.1) is satisfied, then (Σ, h, N) does not admit regular black holes.*

Here (and in (1.2) and (1.3)), (Σ, h, N) refers to a spacetime with static domain of outer communications of the form $(\mathbb{R} \times \Sigma, -N^2 dt^2 \oplus h)$. By "regular black hole", we mean one with a smooth, nondegenerate event horizon of any topology, cf., Section 2 for further details.

The uniqueness theorem of [19, 20] can be interpreted as evidence for the Horowitz-Myers conjecture for $n = 4$. The above Proposition strengthens this evidence, by

[1] By "nonspherical scri" we mean "spatially nonspherical"; *i.e.*, slices of scri orthogonal to its timelike conformal Killing field are not topological spheres.

extending the uniqueness to the class of non-nakedly-singular spacetimes with an irrotational Killing field that is timelike near infinity (together with certain asymptotic conditions). This class of spacetimes is important, since if there is a ground state, it should be static (though not necessarily globally static), i.e. should have zero kinetic energy.

Anderson, Chruściel, and Delay [21] have recently established the uniqueness of the AdS soliton and the Lemos toroidal black holes [1] in the $n = 3$ (*i.e.*, spacetime dimension 4) case with fewer asymptotic restrictions. It would be interesting from the point of view of physics to extend their degree of generality and their methods to $n = 4$.

When the conformal boundary of Σ at infinity, $\partial\Sigma_\infty$, has generic topology and geometry, the existence of solutions (let alone a well defined ground state) is largely an open question (cf., however, [21]). If a regular static solution were to exist for a given $\partial\Sigma_\infty$, one might expect it to also admit black holes. The following proposition establishes further restrictions on the models that can occur as static AL-AdS spacetimes.

Proposition 1.2. *Let (Σ, h, N) be an $(n+1)$-dimensional ($n \geq 3$) static AL-AdS spacetimes satisfying the null energy condition. Suppose (a) the conformal boundary at infinity $\partial\Sigma_\infty$ has positive scalar curvature (with respect to the Fermat metric, see Section 2), (b) the fundamental group of Σ is infinite, $|\pi_1(\Sigma)| = \infty$, and (c) the asymptotic condition (S) (Definition 2.1) is satisfied. Then (Σ, h, N) does not admit regular black holes.*

What sort of conformal boundary $\partial\Sigma_\infty$ is compatible with the assumptions (a–c)? In $4+1$ dimensions the prototypical example is $\partial\Sigma_\infty = S^1 \times S^2$, with standard metric. In general, topological censorship [9] requires that $\partial\Sigma_\infty$ have infinite fundamental group whenever Σ does. Moreover, as follows from the boundary analysis in Section 3 (see also Appendix A), assumptions (a) and (c) imply that $\partial\Sigma_\infty$ must have nonnegative Ricci curvature. (Further connections between the convexity condition (S) and the curvature of the conformal boundary are considered in Section 3). Thus Proposition 1.2 rules out, within a certain class of static AL-ADS spacetimes, conformal boundaries which, for example, are products of flat spaces and spaces of positive Ricci curvature.

One may ask whether one can use the above results to rule out certain horizon topologies (once again, our term “black hole” includes nonspherical horizon topologies). This is possible if one restricts to the class of “product” black hole spacetimes, namely ones whose domain of outer communications has the simple product topology $\mathbb{R} \times \partial\Sigma_\infty \times \mathbb{R}^+$. For such spacetimes, scri and the horizon are diffeomorphic. Proposition 1.1 then implies that, under certain asymptotic conditions, there are no (pointwise) negative mass, regular, static, product black hole spacetimes with horizon topology $\mathbb{R}^{n-1}/\Gamma$, with Γ a discrete co-compact isometry group of $\mathbb{R}^{n-1}$. In particular, it rules out static (pointwise) negative mass toroidal horizons in product black hole spacetimes (consistent with [21] in the $n = 3$ case, wherein it is shown that the positive mass toroidal black holes of Lemos [1] are unique amongst vacuum solutions). Similarly, as follows from the discussion above, Proposition 1.2 rules out, for example, a certain family of product spacetimes with horizon topology $U \times V$ where $U = S^{k_1} \times \dots S^{k_m}$ for $k_i \geq 2,\ m \geq 1$ and $V = S^1 \times \dots S^1$. In $n+1=5$ it thus rules out static product black hole spacetimes with $S^2 \times S^1$ horizons.

Our proof makes crucial use of the construction of a complete achronal null geodesic, or *null line* in [19, 20] and the subsequent use of an extension of the null splitting theorem (NST) of Galloway [22] (Theorem 3.1). The NST was used to show that the existence of a null line implies a certain rigidity of the spacetime—indeed, under certain asymptotic restrictions the only vacuum spacetimes with zero and positive cosmological constants admitting null lines are Minkowski and de Sitter spacetimes, respectively [22, 23]. On the other hand, as we will show, null lines always exist in the universal cover of a static AL-AdS spacetime whenever this cover contains non-compact spatial directions, given our asymptotic conditions. The presence of regular black holes regions in the spacetime, however, implies the existence of strictly convex surfaces (in the ambient Fermat metric) in the neighbourhood of the black hole, which can be shown, using the NST, to be incompatible with the presence of a null line. Thus, as we will prove, the existence of a null line in the interior of a static spacetime is incompatible with the existence of regular black holes.

In section 2, we begin by specifying the class of spacetimes which we will consider. Apart from the requirement that the spacetime be AL-AdS and static (in the domain of outer communications), we further require it to satisfy a certain asymptotic convexity condition, which we denote as Condition (C). Condition (C) can be related to the mass when the boundary is Ricci flat, and shown to be equivalent to a weaker condition (S) when the boundary has positive Ricci curvature. This involves a boundary analysis along the lines of [24] which we include in the Appendix. Next, we observe that there exists a neighbourhood of any regular horizon, in which all constant lapse surfaces outside the horizon are strictly convex with respect to the Fermat metric. In section 3 we first show that the null generators of the boundary of the past (respectively future) of the null line are future (respectively past) complete. This fact then allows us to use the null splitting theorem to prove our black hole non-existence theorem:

Theorem 1.3. *Let (Σ, h, N) be an $n+1$ dimensional ($n \geq 3$) static AL-AdS spacetime satisfying the null energy condition. If $\pi_1(\Sigma)$ is infinite and if the asymptotic condition (C) (Definition 2.1) is satisfied, then (Σ, h, N) does not admit regular black holes.*

In Section 4, we show that under the stated assumptions, Propositions 1.1 and 1.2 follow as corollaries of Theorem 1.3.

2 Preliminaries

We will consider $(n+1)$-dimensional, $n \geq 2$, *AL-AdS spacetimes* M, with an event horizon $\mathcal{H}$ (possibly empty). Let $(\mathcal{D}, g)$ be a static connected component of the domain of outer communications, so there is an irrotational timelike Killing vector field on $(\mathcal{D}, g)$, which extends smoothly to $\partial\mathcal{D} = \overline{\mathcal{D}} \cap \mathcal{H}$, where it becomes null. Then we have

$$\mathcal{D} = \mathbb{R} \times \Sigma \quad , \quad g = -N^2 dt^2 \oplus h \quad , \quad N|_{\partial\mathcal{D}} = 0 \quad , \tag{2.1}$$

where h is the induced metric on Σ and N is the lapse, such that the triple (Σ, h, N) is conformally compactifiable. For technical reasons, we will assume that the static black hole horizon is *regular* (nondegenerate), *i.e.*, $dN|_{\mathcal{H}} \neq 0$ pointwise. Thus, there exists a smooth compact manifold with boundary, $\Sigma' = \Sigma \cup \partial\Sigma_H \cup \partial\Sigma_\infty$, where $\partial\Sigma_H = \Sigma' \cap \mathcal{H}$, such that

(a) N and h extend smoothly to the closure $\overline{\Sigma} = \Sigma \cup \partial\Sigma_H$ of Σ in M, with $N|_{\partial\Sigma_H} = 0$, $dN|_{\partial\Sigma_H} \neq 0$,

(b) N^{-1} extends to a smooth function $\widetilde{N}$ on $\widetilde{\Sigma} \equiv \Sigma' \backslash \partial\Sigma_H$, with $\widetilde{N}|_{\partial\Sigma_\infty} = 0$ and $d\widetilde{N}|_{\partial\Sigma_\infty} \neq 0$ pointwise, and

(c) $N^{-2}h$ extends to a smooth Riemannian metric $\widetilde{h}$, the *Fermat metric* on $\widetilde{\Sigma}$.

We will refer to the triple (Σ, h, N) as the *physical spacetime*, and $(\widetilde{\Sigma}, \widetilde{h}, \widetilde{N})$ as the *Fermat conformal gauge*. Note in this gauge, when $\partial\Sigma_H \neq \emptyset$, the surface $N = 0$ gets mapped to infinity. We allow $\partial\Sigma_H$ to have multiple components.

We also require (Σ, h, N) to satisfy the static field equations,

$$R_{ab} = N^{-1}\nabla_a\nabla_b N + \frac{2\Lambda}{n-1}h_{ab} + \mathcal{T}_{cd}h^c_a h^d_b, \tag{2.2}$$

$$\Delta N = -\frac{2\Lambda}{n-1}N + \frac{1}{N}\mathcal{T}_{00}, \tag{2.3}$$

where ∇_a and R_{ab} are respectively the covariant derivative and Ricci tensor on (Σ, h), the Laplacian is $\Delta = h^{ab}\nabla_a\nabla_b$, the cosmological constant is $\Lambda < 0$, and

$$\mathcal{T}_{ab} = T_{ab} - \frac{1}{n-1}g_{ab}g^{cd}T_{cd}, \tag{2.4}$$

with T_{ab} the matter stress-energy tensor, such that $N^{n-1}T_{ab}$ admits a smooth limit to $\partial\Sigma_\infty$.

It will be useful for the purposes of Section 3 and the appendix to write the field equations in terms of the Fermat metric $\widetilde{h}$ and associated $\widetilde{\nabla}_a$ and $\widetilde{R}_{ab}$,

$$\widetilde{R}_{ab} = -\frac{(n-1)}{\widetilde{N}}\widetilde{\nabla}_a\widetilde{\nabla}_b\widetilde{N} + 8\pi\mathcal{T}_{cd}\widetilde{h}^c_a\widetilde{h}^d_b \quad , \tag{2.5}$$

$$\widetilde{N}\widetilde{\Delta}\widetilde{N} = \left(\frac{2\Lambda}{n-1} + n\widetilde{W}\right) - 8\pi\widetilde{N}^2\mathcal{T}_{00} \quad , \tag{2.6}$$

where

$$\widetilde{W} := \widetilde{h}^{ab}\widetilde{\nabla}_a\widetilde{N}\widetilde{\nabla}_b\widetilde{N} = \frac{1}{N^2}h^{ab}\nabla_a N\nabla_b N. \tag{2.7}$$

For our theorem we will need to examine the behaviour of geodesics in the neighbourhood of the compact boundary $\partial\Sigma' = \partial\Sigma_H \cup \partial\Sigma_\infty$. Our analysis will make use of the existence of a "collar neighbourhood" of the boundary $U = \partial\Sigma' \times [0, \epsilon)$ whose interior is foliated by constant lapse surfaces. Such a collaring exists for any function f satisfying (i) $f|_{\partial\Sigma'} = const$ and (ii) $df|_{\partial\Sigma'} \neq 0$. Putting $f = N$ for the regular black hole boundary $\partial\Sigma_H$, and $f = \widetilde{N}$ for the boundary at infinity $\partial\Sigma_\infty$ and using conditions (a), (b) and (c) above, we may therefore construct collar neighbourhoods of both $\partial\Sigma_H$ and $\partial\Sigma_\infty$ whose interiors are foliated by constant lapse surfaces $\mathcal{V}_c = \{N = c\}$.

We will require a particular convexity assumption for the constant lapse surfaces in a neighbourhood of scri in the Fermat conformal gauge (see also [20]). Let us consider a collar neighbourhood of $\partial\Sigma_\infty$ which is foliated by the level surfaces $\mathcal{V}_c = \{N = c\}$ of the lapse N. The second fundamental form of the $\mathcal{V}_c$ in the Fermat conformal gauge, with respect to the unit normal vector $\widetilde{n}^a$ pointing towards scri, is given by $\widetilde{H}_{ab} = (\widetilde{\nabla}_a \widetilde{n}_b)_\perp$ where $\perp$ denotes the projection to the $\mathcal{V}_c$. The eigenvalues of $\widetilde{H}_{ab}$ are called the *principal curvatures* of the $\mathcal{V}_c$.

Definition 2.1. We say that (Σ, h, N) *satisfies condition (S)* provided that the second fundamental form $\widetilde{H}_{ab}$ of each level surface $N = c$ is semi-definite (equivalently, provided that the principal curvatures of each level surface $N = c$ are either all non-negative or all non-positive) whenever c is sufficiently large (*i.e.*, near scri). If $\widetilde{H}_{ab}$ is positive semi-definite (equivalently, if the principal curvatures are all non-negative) for each of the level surfaces in this neighbourhood of scri, we say that (Σ, h, N) *satisfies condition (C)*, and the level surfaces of N in this neighbourhood are said to be *weakly convex*.

As in [20], Condition (C) will be used to control the behaviour of certain geodesics near scri as follows. Suppose Condition (C) holds, so that the level surfaces $\mathcal{V}_c := \{N = c\}$ are weakly convex, in the sense of the definition, for all c sufficiently large. Let $\mathcal{V}_0 = \{N = c_0\}$ be such a level surface; $\mathcal{V}_0$ has a well-defined "inside" ($N < c_0$) and "outside" ($N > c_0$). Then, as follows from the maximum principle, if γ is a geodesic segment with endpoints inside $\mathcal{V}_0$, all of γ must be contained inside $\mathcal{V}_0$. Thus, Condition (C) provides "barrier surfaces" for the construction of certain minimizing geodesics, as will be seen in the next subsection. Condition (S) on the other hand allows the level surfaces $N = c$ near scri to be either weakly convex (H_{ab} positive semi-definite) or weakly concave (H_{ab} negative semi-definite). All the relevant examples known to us obey condition (S), even when Condition (C) fails. As we will see in Section 4, Condition (S) along with certain extra boundary conditions may be used to obtain Condition (C).

We also need to understand the behaviour of geodesics near the black hole boundary $\partial\Sigma_H$. In the following we will call a surface $\mathcal{K} \subset \Sigma$ *strictly convex* with respect to a choice of normal if the second fundamental form $\widetilde{H}_{ab}$ in the Fermat conformal gauge is positive definite (equivalently, its principal curvatures are strictly positive) with respect to this normal.

Lemma 2.2. *Let (Σ, h, N) be a static spacetime with regular black hole boundary $\partial\Sigma_H \neq \emptyset$. Then for each $c > 0$ sufficiently small, the constant lapse surface $\mathcal{V}_c = \{N = c\}$ is diffeomorphic to $\partial\Sigma_H$ and strictly convex in $(\widetilde{\Sigma}, \widetilde{h})$ with respect to the normal pointing towards $\partial\Sigma_H$. Thus, any Fermat geodesic with endpoints in the region $N > c$ cannot intersect $\mathcal{V}_c$.*

Proof. Let us consider the collar neighbourhood $\mathcal{U} \approx [0, c_0) \times \partial\Sigma_H$ of $\partial\Sigma_H$ foliated by the constant lapse surfaces $\mathcal{V}_c = \{N = c\} \approx \partial\Sigma_H$, $0 < c < c_0$. The second fundamental forms of each $\mathcal{V}_c$ in the physical metric and in the Fermat metric are related by

$$\widetilde{H}_{ab}(c) = c^{-1}H_{ab}(c) - c^{-2}h_{ab}n^d\partial_d N, \tag{2.8}$$

where n^d is the unit normal field to the $\mathcal{V}_c$ in the physical metric, chosen to be pointing towards surfaces of decreasing lapse, *i.e.*, towards $\partial\Sigma_H$.

Let X^a be a unit vector field in the physical metric, defined in a neighbourhood $\mathcal{O} \subset \mathcal{U}$ of any point $p \in \partial\Sigma_H$, such that X^a is tangent to each $\mathcal{V}_c$ meeting $\mathcal{O}$. On $\mathcal{O} \setminus \partial\Sigma_H$, set $\tilde{X}^a = NX^a$; $\tilde{X}^a$ is a unit vector field in the Fermat metric also tangent to each $\mathcal{V}_c$. Contracting equation (2.8) with $\tilde{X}^a$, we obtain

$$\widetilde{H}_{ab}(c)\tilde{X}^a\tilde{X}^b = cH_{ab}(c)X^aX^b - n^d\partial_d N\,. \tag{2.9}$$

Thus, in the limit as $c \to 0$, $\widetilde{H}_{ab}(c)\tilde{X}^a\tilde{X}^b \to dN(n)|_{\partial\Sigma_H} > 0$. Using the compactness of $\partial\Sigma_H$, it follows that $\widetilde{H}_{ab}(c)$ is positive definite for all c sufficiently small. The statement regarding Fermat geodesics follows from the discussion after Definition 2.1. □

Remark. In the above proof, the condition of regularity, $n^d\partial_d N(0) \neq 0$, is crucial in obtaining a neighbourhood of the horizon in which $\widetilde{H}_{ab}(c)$ is strictly positive definite. For degenerate horizons, $n^d\partial_d N(0) = 0$, and hence our lemma does not extend in a trivial manner to this case. However, for familiar examples such as the extremal Reissner Nordstrom and the extremal charged AdS black holes, one can explicitly construct a strictly convex surface $\mathcal{K}$ in the neighbourhood of the horizon.

3 Black hole non-existence theorem.

We begin by reminding the reader of the definition of a line. In a Riemannian manifold, a *line* is an inextendible geodesic, each segment of which has minimal length, while in a spacetime, a *timelike line* is an inextendible timelike geodesic, each segment of which has maximal length. Motivated by these more standard cases, a *null line* in spacetime is defined to be an inextendible null geodesic which is globally achronal, *i.e.*, for which no two points can be joined by a timelike curve. (Hence, each segment of a null line is maximal with respect to the Lorentzian arc length functional.) In static spacetimes, spacelike Fermat geodesics in $(\widetilde{\Sigma}, \widetilde{h})$ can be lifted, via the timelike Killing field, to null geodesics in the physical spacetime (Σ, h, N) under suitable reparameterisations (see [20]). Indeed, there is essentially a one-to-one correspondence between affinely parameterised Fermat geodesics in $(\widetilde{\Sigma}, \widetilde{h})$ and affinely parameterised, future directed null geodesics in (Σ, h, N). Moreover, a Fermat line in $(\widetilde{\Sigma}, \widetilde{h})$ lifts to a unique future directed null line in (Σ, h, N) through a fixed basepoint [20].

Since our proof will make crucial use of a slight generalisation of the NST, we state it here:

Theorem 3.1 (Null Splitting Theorem, Galloway [22]). *If a null geodesically complete spacetime obeys $R_{ab}X^aX^b \geq 0$ for all null vectors X^a and also contains a null line η, then η lies in a smooth, edgeless, totally geodesic null hypersurface.*

For our purposes, null geodesic completeness is too strong a requirement, since we would like to allow for the presence of black holes in the spacetime. As was pointed out in [22], the NST is still valid if null geodesic completeness is dropped and instead one imposes the less stringent requirement that the null generators of $S^+ = \partial J^+(\eta)$ and $S^- = \partial J^-(\eta)$ be past and future complete, respectively. Indeed, it is only in order to show this latter property of the null generators of $S^\pm$ that null geodesic completeness was used in the proof. Thus, in order to use the NST in the absence of null geodesic completeness, one requires an alternative method to prove this. For our purposes it suffices to show:

Lemma 3.2. *Let (Σ, h, N) be a static spacetime, and let Σ_0 be an open subset of Σ. Suppose that*

(a) $\overline{\Sigma}_0$, the closure of Σ_0 in Σ, is a complete Riemannian manifold-with-boundary in the Fermat metric $\widetilde{h} = N^{-2}h$,

(b) the boundary $\partial\overline{\Sigma}_0$ of $\overline{\Sigma}_0$ is weakly convex (with respect to the outward normal) in $(\Sigma, \widetilde{h})$, and

(c) N has a positive lower and upper bound on Σ_0.

If η is a complete null line in the "truncated spacetime" $M_0 = (\Sigma_0, h, N)$, then the null generators of $S^+ = \partial J^+(\eta, M_0)$ are past complete, and the null generators of $S^- = \partial J^-(\eta, M_0)$ are future complete.

In the proof of Theorem 1.3, Lemma 3.2 will be applied to the *universal cover* of the domain of outer communications.

Proof. In what follows, pasts and futures will refer to the spacetime $M_0 = (\Sigma_0, h, N)$, and we identify Σ_0 with the time slice $t = 0$ in M_0.

We begin with the hypersurface S^+ and a point $q \in S^+$. Without loss of generality, we may assume that q is not a future endpoint of a null generator of S^+. We express $q = (t_0, p) \in (\Sigma_0, h, N)$, where $p \in \Sigma_0$. Let $p_0 \in \eta$ be the point where η meets Σ_0. Let γ be the projection of η into Σ_0; γ is a complete line in $(\Sigma_0, \widetilde{h})$ passing through p_0. Let $\{q_i\}$ be a sequence of points which exhaust η to the past. The points $q_i = (t_i, p_i)$ with $p_i \in \Sigma_0$ then project to the sequence $\{p_i\} \in \gamma$ such that the Fermat distance from p to p_i in $(\Sigma_0, \widetilde{h})$ for successive i tends to infinity. The convexity assumption implies that any two points in Σ_0 can be joined by a minimal Fermat geodesic contained in Σ_0. Thus, for each i, there exists a minimal Fermat geodesic $\sigma_i \subset \Sigma_0$ from p to p_i. By passing to a subsequence if necessary, the σ_i converge to a ray (geodesic half-line) σ in $(\Sigma_0, \widetilde{h})$ starting at p; σ is referred to as an asymptote of γ.

Each σ_i lifts to a unique null geodesic μ_i in the physical spacetime with $q_i = (t_i, p_i)$ as its past endpoint. The future endpoint of each μ_i is some $x_i = (\tau_i, p)$. We now show that the τ_i lie in a finite interval. In $(\Sigma_0, \widetilde{h})$, the triangle inequality allows us to write

$$|\tilde{d}(p_0, p_n) - \tilde{d}(p, p_n)| \leq \tilde{d}(p, p_0) = l_0, \tag{3.1}$$

where $\tilde{d}(p_1, p_2)$ denotes the Fermat distance between the points p_1 and p_2. Let $\gamma_i \subset \gamma$ be a segment of the Fermat line γ with endpoints p_0 and p_i. Since γ_i and σ_i are minimal with respect to the Fermat metric, $\tilde{d}(p_0, p_i) = \tilde{L}(\gamma_i)$ and $\tilde{d}(p, p_i) = \tilde{L}(\sigma_i)$, where $\tilde{L}(x)$ represents the Fermat arc length. In a static spacetime, the time taken to traverse a null geodesic in the physical metric is given by the arc length of the projected Fermat metric. Thus, $\tilde{L}(\gamma_i) = t_i$ and $\tilde{L}(\sigma_i) = t_i - \tau_i$, which, from (3.1) implies that $|\tau_i| \leq l_0$. Thus, by passing to a subsequence if necessary, the $x_i = (\tau_i, p)$ converge to some $x = (\tau, p)$. This means that, passing to a further subsequence if necessary, the μ_i converge to a null geodesic μ with future endpoint x, such that the projection of μ onto $(\Sigma_0, \widetilde{h})$ is the ray σ. Since the q_i exhaust η to the past, μ is past inextendible. If u denotes an affine parameter along μ then it follows that $du = KN^2 d\tilde{s}$, $K =$ const., where $d\tilde{s}$ is arc length along σ. Since σ has infinite length, and N^2 has a positive lower bound on Σ_0, it follows that μ is past complete in (Σ_0, h, N).

Finally, we show that μ is in fact a generator of S^+ with future endpoint q. Since $\mu_n \subset J^+(\eta)$, this means that $\mu \subset \overline{J^+(\eta)} = \overline{I^+(\eta)}$. Let us assume that $\mu \cap I^+(\eta) \neq \phi$. Thus, there exists an $a \in \mu$ and $b \in \eta$ such that $a \in I^+(b)$. Consider a neighbourhood U_a of a such that $U_a \subset I^+(b)$. Since $a \in \mu$ and μ is a limit curve to the μ_i there exists an N_1 such that $\forall i > N_1$, there exists an $a_i \in \mu_i$ such that $a_i \in U_a$. Moreover, since the q_i exhaust η to the past, for any $b \in \eta$, $\exists\ N_2$ such that $\forall i > N_2$, q_i is to the past of b on η. Thus, $\exists\ N$ such that $\forall i > N$, q_i lies to the past of b on η and $a_i \in I^+(b)$ which implies that $a_i \in I^+(q_i)$, which is not possible since μ_i is achronal. Hence $\mu \subset \partial J^+(\eta)$. That its future endpoint $x = q$ follows, since $x \in S^+$ and x and q lie along the same orbit of the time-like Killing vector field $(\frac{\partial}{\partial t})^a$. A similar argument shows that the null generators of $S^- = \partial J^-(\eta)$ are future complete. $\square$

Remark. If, further, $R_{ab}X^aX^b \geq 0$ for all null vectors X^a then the proof of the NST [22] implies that $S^+ = S^- = S$ is a smooth, edgeless, totally geodesic null hypersurface. This ultimately leads to a contradiction in the presence of black holes as will be shown in the following theorem:

Theorem 1.3. *Let M be an $n+1$ dimensional ($n \geq 3$) AL-AdS spacetime with static domain of outer communications (Σ, h, N) satisfying the null energy condition. If $\pi_1(\Sigma)$ is infinite and if the asymptotic condition (C) (Definition 2.1) is satisfied, then (Σ, h, N) does not admit regular black holes, i.e., $\partial\Sigma_H = \emptyset$.*

Proof. Suppose to the contrary that (Σ, h, N) is conformally compactifiable with conformal boundary $\partial\Sigma_\infty$, and that it has a regular black hole horizon $\partial\Sigma_H \neq \emptyset$, as described in Section 2.

Recall, by definition, $\Sigma' = \Sigma \cup \partial\Sigma_\infty \cup \partial\Sigma_H$ is a compact Riemannian manifold-with-boundary. We truncate Σ' by removing small collared neighbourhoods (foliated by constant lapse slices) $\mathcal{U}_H \approx [0, \epsilon] \times \Sigma_H$ and $\mathcal{U}_\infty \approx [0, \epsilon] \times \Sigma_\infty$ of $\partial\Sigma_H$ and $\partial\Sigma_\infty$, respectively. In this way we obtain an open set Σ_0 in Σ whose closure $\overline{\Sigma}_0 \subset \Sigma$ is a compact manifold with boundary $\partial\overline{\Sigma}_0 = \mathcal{V}_A \cup \mathcal{V}_B$, where $\mathcal{V}_A$ is a constant lapse slice near $\partial\Sigma_H$ and $\mathcal{V}_B$ is a constant lapse slice near $\partial\Sigma_\infty$. By Lemma 2.2, we may assume that $\mathcal{V}_A$ is strictly convex. By Condition C we may assume that $\mathcal{V}_B$, and all constant lapse slices near $\mathcal{V}_B$, are weakly convex.

Let $(\overline{\Sigma}_0^*, \widetilde{h}^*)$ be the Riemannian universal cover of $(\overline{\Sigma}_0, \widetilde{h})$. Since $\overline{\Sigma}_0$ is a deformation retract of Σ', its fundamental group is isomorphic to that of Σ', and hence to that of Σ, and so $|\pi_1(\overline{\Sigma}_0)| = \infty$. Thus, $(\overline{\Sigma}_0^*, \widetilde{h}^*)$ is a complete noncompact Riemannian manifold with boundary $\partial\overline{\Sigma}_0^* = \mathcal{V}_A^* \cup \mathcal{V}_B^*$, where $\mathcal{V}_A^*$ covers $\mathcal{V}_A$ and $\mathcal{V}_B^*$ covers $\mathcal{V}_B$. The convexity conditions on $\mathcal{V}_A$ and $\mathcal{V}_B$ lift to $\mathcal{V}_A^*$ and $\mathcal{V}_B^*$.

Σ_0^*, the manifold interior of $\overline{\Sigma}_0^*$, corresponds to the universal cover of Σ_0. We now show that $(\Sigma_0^*, \widetilde{h}^*)$ contains a Fermat line. This construction is a simple extension of that used in the Lorentzian structure theorem of [20]. Let $p \in \Sigma_0^*$ and $\{q_i\} \subset \Sigma_0^*$ be a sequence of points uniformly bounded away from $\partial\overline{\Sigma}_0^*$ such that the distance from p to q_i tends to infinity. For each i, let γ_i be a length minimizing geodesic in Σ_0^* from p to q_i . Our convexity conditions imply that such γ_i exist, and are uniformly bounded away from $\partial\overline{\Sigma}_0^*$. Since $\overline{\Sigma}_0$ is compact, the universal cover $\overline{\Sigma}_0^*$ admits a compact fundamental domain D. For every point in $\overline{\Sigma}_0^*$ there exists a covering space transformation, $\mathfrak{g} \in \pi_1(\overline{\Sigma}_0^*)$, mapping it to a point in D. Let $\mathfrak{g}_i \in \pi_1(\overline{\Sigma}_0^*)$ map the midpoint r_i of γ_i into D. Since the $\mathfrak{g}_i$ are isometries, the new curves $\gamma_i' = \gamma_i \circ \mathfrak{g}_i$ are minimal geodesics which meet D, and are bounded away from $\partial\overline{\Sigma}_0^*$, since the γ_i are. Moreover, the lengths of the γ_i' are now unbounded in both directions. It follows that some subsequence of $\{\gamma_i'\}$ converges to a complete line γ in $(\Sigma_0^*, \widetilde{h}^*)$.

Consider the static spacetime $M_0^* = (\Sigma_0^*, h^*, N^*)$, where $N^* = N \circ \rho$, and $\rho : \Sigma_0^* \to \Sigma_0$ is the covering map; M_0^* is the universal covering spacetime of $M_0 = (\Sigma_0, h, N)$. Now, γ lifts to a complete null line η in (Σ_0^*, h^*, N^*). By Lemma 3.2 and the remark following its proof, the NST implies that η is contained in a smooth totally geodesic, edgeless achronal null hypersurface S in (Σ_0^*, h^*, N^*). As S is edgeless, and hence closed as a subset of (Σ_0^*, h^*, N^*), and as its generators are complete, it can be shown that S maps diffeomorphically onto Σ_0^* (viewed as the slice $t = 0$) via the flow lines of the timelike Killing field $(\frac{\partial}{\partial t})^a$; cf., [20] for details. The null generators of S, being complete null lines, project to complete Fermat lines in Σ_0^*. It follows that $(\Sigma_0^*, \widetilde{h}^*)$ is ruled by complete Fermat lines. Fix a point x on the boundary component $\mathcal{V}_A$, and let $\{x_i\}$ be a sequence of points in Σ_0^* such that $x_i \to x$. Let σ_i be a complete Fermat line in Σ_0^* passing through x_i. By passing to a subsequence if necessary, σ_i will converge to a complete Fermat geodesic σ contained in $\overline{\Sigma}_0^*$ passing tangentially through $x \in \mathcal{V}_A$. But this contradicts the fact that $\mathcal{V}_A$ is strictly convex. Thus we conclude that (Σ, h, N) does not admit regular black holes.
□

4 Applications: Proofs of Propositions 1.1 and 1.2

In special cases, condition (C) can be weakened to condition (S) by using a boundary analysis along the lines of [24, 20] where one considers a foliation of a neighbourhood of scri with constant lapse surfaces. Consider a collar neighbourhood $\mathcal{O}$ of the boundary $\partial\Sigma_\infty$ at $x \equiv \widetilde{N} = 0$. In the coordinates $x^1 = x, x^2, \cdots, x^n$ the Fermat metric $\widetilde{h}$ takes the form

$$\widetilde{h} = \widetilde{W}^{-1} dx^2 + \widetilde{b}_{\alpha\beta} dx^\alpha dx^\beta, \tag{4.1}$$

where $\widetilde{b}_{\alpha\beta} = \widetilde{b}_{\alpha\beta}(x, x^\gamma)$ is the induced metric on the constant x slices $\mathcal{V}_x$, which are diffeomorphic to $\partial\Sigma_\infty$.

Let us first consider the class of spacetimes with Ricci flat scri, $\partial\Sigma_\infty = \mathbb{R}^{n-1}/\Gamma$, with Γ a discrete co-compact isometry group of $\mathbb{R}^{n-1}$. The boundary analysis of [20] shows that the mean curvature of a constant lapse surface in a neighbourhood of scri is

$$\widetilde{H}(x) = -nx^{n-1}\mu + \mathcal{O}(x^n), \tag{4.2}$$

where the mass aspect μ is related to the the Ashtekar-Magnon mass [13] via

$$M_{AM} = \frac{1}{16\pi}\int_{\partial\widetilde{\Sigma}} \mu\sqrt{b}\; dS, \tag{4.3}$$

When the mass aspect is pointwise negative, the mean curvature is positive. This, along with Condition (S) implies Condition (C). Then it is an immediate corollary of Theorem 3.1 that:

Proposition 1.1. *Let M be an $n+1$ dimensional ($n \geq 3$) AL-AdS spacetime, with static domain of outer communications (Σ, h, N), which satisfies the null energy condition and has a Ricci flat scri, and whose Ashtekar-Magnon mass aspect on scri is pointwise negative. If $\pi_1(\Sigma)$ is infinite and if the asymptotic condition (S) is satisfied, then (Σ, h, N) does not admit regular black holes, i.e., $\partial\Sigma_H = \emptyset$.*

Next, we consider the case in which $\partial\Sigma_\infty$ has positive scalar curvature in the induced metric $\widetilde{b}_{ab}(0)$. In the collar neighbourhood $\mathcal{O}$ of $\partial\Sigma_\infty$, the mean curvature of the $\mathcal{V}_x$ can be expressed as

$$\widetilde{H}(x) = x\partial_x\widetilde{H}(0) + O(x^2), \tag{4.4}$$

where we have used the fact that the extrinsic curvature $\widetilde{H}_{\alpha\beta}(0) = 0$ (see (A.7)). By taking the projection of the field equation (2.5) tangent to $\mathcal{V}_x$ and differentiating once (i.e. putting $k = 1$ in (A.8)) we obtain

$$\partial_x\widetilde{H}_{\alpha\beta}(0) = \frac{\ell}{(n-2)}\widetilde{\mathcal{R}}_{\alpha\beta}(0), \tag{4.5}$$

where $\widetilde{\mathcal{R}}_{\alpha\beta}$ is the Ricci curvature with respect to $\widetilde{b}_{\alpha\beta}$. Tracing the above, and substituting into (4.4) we obtain

$$\widetilde{H}(x) = \frac{\ell}{(n-2)}\,\widetilde{S}\; x + O(x^2), \tag{4.6}$$

where $\widetilde{S}$ is the scalar curvature $\partial\Sigma_\infty$ with respect to $\widetilde{b}_{\alpha\beta}$. $\widetilde{H}(x)$ is clearly positive for small enough x, under the assumption $\widetilde{S} > 0$. Thus, Condition (C) reduces to Condition (S) in this case, and we obtain another corollary of Theorem 1.3:

Proposition 1.2. *Let M be an $n+1$ dimensional ($n \geq 3$) AL-AdS spacetime, with static domain of outer communications (Σ, h, N) satisfying the null energy condition. Suppose (a) the conformal boundary at infinity $\partial\Sigma_\infty$ has positive scalar* *curvature with respect to the Fermat metric, (b) the fundamental group of Σ is infinite, $|\pi_1(\Sigma)| = \infty$, and (c) the asymptotic condition (S) is satisfied. Then (Σ, h, N) does not admit regular black holes.*

To conclude this section, we examine Condition (S) for a class of conformal boundaries $\partial\Sigma_\infty$ which naturally have positive scalar curvature and infinite fundamental group, such as $S^2 \times S^1$.

Consider the class of spacetimes with "mixed" boundary $\partial\Sigma_\infty = U \times V$, such that U and V are compact, with $\dim(U) \geq 2$ and $\dim(V) \geq 1$. In a collar neighbourhood of $\partial\Sigma_\infty$ the constant lapse surfaces $\mathcal{V}_x$ are diffeomorphic to $U \times V$. Let $\mathbf{u}(\vec{u}, \vec{v}, x)$ and $\mathbf{v}(\vec{u}, \vec{v}, x)$ be the induced metrics with respect to $\widetilde{b}_{\alpha\beta}$ on U and V respectively, where $\vec{u}$ and $\vec{v}$ are the coordinates on U and V respectively. We further assume that metric has the simple product form at $x = 0$

$$\widetilde{b}_{\alpha\beta}(\vec{u}, \vec{v}, 0) = \mathbf{u}_{\alpha\beta}(\vec{u}, 0) + \mathbf{v}_{\alpha\beta}(\vec{v}, 0), \tag{4.7}$$

such that

$${}^{u}\widetilde{\mathcal{R}}_{\alpha\beta}(\vec{u}, 0) = \frac{n(n-1)}{2\ell^2}\mathbf{u}_{\alpha\beta}(\vec{u}, 0); \quad {}^{\mathbf{v}}\widetilde{\mathcal{R}}_{\alpha\beta}(\vec{v}, 0) = 0. \tag{4.8}$$

where ${}^{\mathbf{u}}\widetilde{\mathcal{R}}_{\alpha\beta}$ and ${}^{\mathbf{v}}\widetilde{\mathcal{R}}_{\alpha\beta}$ are the Ricci tensors associated with $\mathbf{u}$ and $\mathbf{v}$, respectively. Our boundary data are thus "mixed" in the sense that it is a product of a positive Ricci curvature Einstein manifold and a zero curvature Einstein manifold. We leave the details of the boundary analysis for this class of boundary data to the appendix, only stating the relevant results here.

From (4.5) and (4.8), the projection along U of the second fundamental form $\widetilde{P}_{\delta\rho} = \mathbf{u}^{\alpha}{}_{\delta}\mathbf{u}^{\beta}{}_{\rho}\widetilde{H}_{\alpha\beta}$ of $\mathcal{V}_x$ has the expansion

$$\widetilde{P}_{\alpha\beta}(\vec{u}, \vec{v}, x) = \frac{n(n-1)}{2(n-2)\,\ell}\; x\; \mathbf{u}_{\alpha\beta}(\vec{u}, 0) + O(x^2), \tag{4.9}$$

which, by itself, for small enough x, has positive eigenvalues. From Proposition A.3, the remaining components of the second fundamental form $\widetilde{Q}_{\delta\beta} = \mathbf{v}^{\alpha}{}_{\delta}\widetilde{H}_{\alpha\beta}$ can be expanded as

$$\widetilde{Q}_{\alpha\beta}(\vec{u}, \vec{v}, x) = \frac{1}{(n-1)!}\; x^{n-1}\widetilde{Q}^{(n-1)}_{\alpha\beta}(\vec{u}, \vec{v}, 0) + O(x^n). \tag{4.10}$$

Since the boundary data $\widetilde{H}^{(n-1)}_{\alpha\beta}(\vec{u}, \vec{v}, 0)$ are independent of $\widetilde{b}_{\alpha\beta}(\vec{u}, \vec{v}, 0)$, we are free to choose $\widetilde{Q}^{(n-1)}_{\alpha\beta}(\vec{u}, \vec{v}, 0)$ so that $\widetilde{H}_{ab}$ has only non-negative eigenvalues in a small enough neighbourhood of scri. Thus, Condition (S) can indeed be satisfied in a neighbourhood of the conformal boundary when it is of the type under consideration here. [2]

[2] If the "mixed" boundary were instead such that ${}^{\mathbf{v}}\widetilde{\mathcal{R}}_{\alpha\beta}(\vec{v}, 0)$ was Einstein with *negative* curvature, then it can be seen that Condition (S) could never be satisfied.

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A Boundary analysis

Before we proceed, it is useful to compare our analysis with that of [24]. While the work of [24] is general and includes metrics of both Lorentzian and Euclidean signature, it needs to be modified to suit the purpose of our work. In [24] a particular choice of conformal gauge is made, namely, $h = y^{-2}\widetilde{h}_1 = y^{-2}(dy^2 \oplus \widetilde{b}_1)$ and the boundary data $\{\partial_y^k \widetilde{b}_1(0)\}$ is then analysed. However, the choice of a Fermat conformal gauge is crucial to the method employed in our work, since we specifically require conditions (C) and (S) to be satisfied with respect to the Fermat metric. From (4.1) it is clear that the Fermat conformal gauge coincides with the conformal gauge used in [24] only when $\widetilde{W} = 1$. Moreover, while $\widetilde{b}(0) = \widetilde{b}_1(0)$ the boundary data $\{\partial_x^k \widetilde{b}(0)\}$ and $\{\partial_y^k \widetilde{b}_1(0)\}$ will in general differ.

From Definition 2.1 and (4.1) $x^{1-n}T_{ab}$ should have a smooth limit to scri. Thus in a neighbourhood of scri we have

$$T_{ab}(x, x^\alpha) = A_{ab}(x^\alpha)x^{n-1} + O(x^n). \tag{A.1}$$

Moreover, the second fundamental form of $\mathcal{V}_x$ can be expressed as

$$\widetilde{H}_{\alpha\beta} = -\frac{1}{2\psi}\partial_x \widetilde{b}_{\alpha\beta}, \tag{A.2}$$

where $\psi = \frac{1}{\sqrt{\widetilde{W}}}$. Taking the projections of (2.5) tangent and normal to the $\mathcal{V}_x$ we obtain

$$\widetilde{R}_{\alpha\beta} = \frac{(n-1)}{x\psi}\widetilde{H}_{\alpha\beta} + 8\pi[\mathcal{A}_{\alpha\beta}x^{n-1} + \mathcal{O}(x^n)], \tag{A.3}$$

$$\widetilde{R}_{xx} = \frac{(n-1)}{x\psi}\partial_x\psi + 8\pi[\mathcal{A}_{xx}x^{n-1} + \mathcal{O}(x^n)], \tag{A.4}$$

where we have used (A.1) and defined $\mathcal{A}_{ab} = A_{ab} - \frac{1}{n-1}\widetilde{h}_{ab}\widetilde{h}^{cd}A_{cd}$. As in [20] we may expand the left-hand sides of the above equations to obtain

$$\begin{aligned}
\widetilde{H}_{\alpha\beta} &= \frac{x}{n-1}\Big(\partial_x \widetilde{H}_{\alpha\beta} + 2\widetilde{H}_{\alpha\gamma}\widetilde{H}^{\gamma}_{\beta}\psi - \widetilde{H}\widetilde{H}_{\alpha\beta}\psi + \widetilde{\mathcal{R}}_{\alpha\beta}\psi - D_\alpha D_\beta \psi \\
&\quad + 8\pi\psi[\mathcal{A}_{\alpha\beta}x^{n-1} + \mathcal{O}(x^n)]\Big), \qquad \text{(A.5)}\\
\partial_x \psi &= \frac{x}{n-1}\psi^2\Big(\partial_x \widetilde{H} - D^2\psi - \widetilde{H}_{\alpha\beta}\widetilde{H}^{\alpha\beta}\psi + \frac{1}{\psi^2}8\pi[\mathcal{A}_{xx}x^{n-1} + \mathcal{O}(x^n)]\Big), \qquad \text{(A.6)}
\end{aligned}$$

where D_α is the connection compatible with $\widetilde{b}_{\alpha\beta}$, and $\widetilde{\mathcal{R}}_{\alpha\beta}$ is the associated Ricci curvature. Assuming the conformal metric is C^2 at $x = 0$, this yields

$$\widetilde{H}_{\alpha\beta}(0) = \partial_x \widetilde{b}_{\alpha\beta}(0) = \partial_x\psi(0) = 0; \quad \psi(0) = \ell \qquad \text{(A.7)}$$

Assuming C^k regularity of the metric for $1 \le k < n-1$, (A.5) and (A.6) may be differentiated k times to obtain

$$\begin{aligned}
\widetilde{H}^{(k)}_{\alpha\beta}(0) &= \frac{k}{(n-k-1)}\partial_x^{k-1}(2\widetilde{H}_{\alpha\gamma}\widetilde{H}^{\gamma}_{\beta}\psi - \widetilde{H}\widetilde{H}_{\alpha\beta}\psi + \psi\widetilde{\mathcal{R}}_{\alpha\beta} - D_\alpha D_\beta\psi)\Big|_0 \qquad \text{(A.8)}\\
\psi^{(k+1)}(0) &= \frac{k}{n-1}\partial_x^{k-1}(\psi^2\partial_x\widetilde{H} - \psi^2 D^2\psi - \psi^3\widetilde{H}_{\alpha\beta}\widetilde{H}^{\alpha\beta})\Big|_0, \qquad \text{(A.9)}
\end{aligned}$$

where $X^{(k)} \equiv \partial_x^k X \equiv \partial^k X/\partial x^k$ and $X^{(0)} = X$. Notice that the matter stress-energy terms do not contribute to these expressions because of the assumed fall-off rates (A.1).

It is handy to write down the expressions relating $\widetilde{H}^{(k)}_{\alpha\beta}(0)$ and $\widetilde{b}^{(k+1)}_{\alpha\beta}(0)$, which we obtain by differentiating (A.2) k times and using (A.7),

$$\widetilde{H}^{(k)}_{\alpha\beta}(0) = -\frac{1}{2}\sum_{i=0}^{k-1}\binom{k}{i}\ \partial_x^i(\psi^{-1})(0)\widetilde{b}^{(k+1-i)}_{\alpha\beta}(0), \qquad \text{(A.10)}$$

or,

$$\widetilde{b}^{(k+1)}_{\alpha\beta}(0) = -2\sum_{i=0}^{k-1}\binom{k}{i}\ \psi^{(i)}(0)\widetilde{H}^{(k-i)}_{\alpha\beta}(0). \qquad \text{(A.11)}$$

By inspection, we notice that

Lemma A.1. *For $k > 0$ even, every term in (A.10) will contain either a $\widetilde{b}^{(m+2)}_{\alpha\beta}(0)$, or a $\psi^{(m)}(0)$ where m is odd and $1 \le m \le k-1$. Similarly, for k even, every term in (A.11) will contain either an $\widetilde{H}^{(m+1)}_{\alpha\beta}(0)$, or a $\psi^{(m)}(0)$ where m is odd and $1 \le m \le k-1$.*

Proposition A.2 *Assume that (Σ, h, N) is a C^n differentiable static spacetime which is AL-AdS. For even $k < n-1$, $\widetilde{b}^{(k+1)}_{\alpha\beta}(0) = \widetilde{H}^{(k)}_{\alpha\beta}(0) = \psi^{(k+1)}(0) = 0$. For*

$k = n-1$, $\widetilde{b}^{(k+1)}_{\alpha\beta}(0)$ and $\widetilde{H}^{(k)}_{\alpha\beta}(0)$ cannot be determined from $\widetilde{b}_{\alpha\beta}(0)$ and constitute independent boundary data.

Remark. This matches the results of [24] for their choice of conformal gauge. Notice that unlike [20] we do not require that the conformal boundary be Ricci flat.

Proof. As in [24, 20] the proof is iterative. We take k to be even and $< n-1$ throughout the proof.

Step A: Consider the expression for $\psi^{(k+1)}(0)$ in (A.9). By inspection, and using Lemma A.1 and the Leibniz rule, every term in the expression can be seen to contain either a (i) $\psi^{(m)}(0)$ for m odd and $1 \leq m \leq k-1$ or a (ii) $\widetilde{b}^{(l)}_{\alpha\beta}(0)$ for l odd and $1 \leq l \leq k+1$ (or both). Applying this recursively to $\psi^{(m)}(0)$, and using $\partial_x \psi(0) = 0$ from (A.7), this simplifies to the fact that every term in the right hand side of (A.9) must contain a $\widetilde{b}^{(m)}_{\alpha\beta}(0)$ for m odd and $1 \leq m \leq k+1$.

Step B: Next, consider the expression for $\widetilde{H}^{(k)}_{\alpha\beta}(0)$ in (A.8). Again, using the Leibniz rule and Lemma A.1, we see that every term in the expression contains at least a (i) $\psi^{(m)}(0)$ or a (ii) $\widetilde{b}^{(m)}_{\delta\rho}(0)$ for m odd and $m \leq k-1$. From step A above, this implies that every term in the expansion for $\widetilde{H}^{(k)}_{\alpha\beta}$ contains a $\widetilde{b}^{(m)}_{\delta\rho}(0)$ for m odd and $1 \leq m \leq k-1$. Using this recursively with Lemma A.1, this means that $\widetilde{b}^{(k+1)}_{\alpha\beta}$ can be expanded such that each term in the expansion contains a $\widetilde{b}^{(m)}_{\delta\rho}(0)$ for m odd and $1 \leq m \leq k-1$.

Thus, if $\widetilde{b}^{(m)}_{\delta\rho}(0) = 0 \; \forall \; m$ odd and $1 \leq m \leq k-1$, then from steps A and B we may conclude that $\widetilde{b}^{(k+1)}_{\alpha\beta}(0) = 0 \; \forall \; k$ even and $0 < k < n-1$. Since this is true for $k = 0$, i.e., $\widetilde{b}^{(1)}_{\alpha\beta} = 0$ (A.7), by iteration we may conclude that $\widetilde{b}^{(k+1)}_{\gamma\delta}(0) = 0 \; \forall \; k$ even and $0 < k < n-1$. It follows from step A, and Lemma A.1 that $\psi^{(k+1)}(0) = \widetilde{H}^{(k)}_{\alpha\beta}(0) = 0$ $\forall \; k$ even and $0 < k < n-1$. □ .

Next, we consider the class of boundaries (4.7, 4.8). However, to prove Proposition A.3 below, it suffices to merely require that

$$^{\mathbf{v}}\widetilde{\mathcal{R}}_{\alpha\beta}(0) = 0, \tag{A.12}$$

with $\mathbf{u}(\vec{u}, 0)$ left arbitrary. As in Section 4, we will use the shorthand

$$\widetilde{P}_{\alpha\beta}(\vec{u}, \vec{v}, x) = (\mathbf{u}^{\delta}{}_{\alpha} \mathbf{u}^{\rho}{}_{\beta} \widetilde{H}_{\delta\rho})(\vec{u}, \vec{v}, x) \tag{A.13}$$

to denote the projection of the second fundamental form of the constant lapse surfaces along U, and

$$\widetilde{Q}_{\alpha\beta}(\vec{u}, \vec{v}, x) = (\mathbf{v}^{\delta}{}_{\alpha} \, \widetilde{H}_{\beta\delta})(\vec{u}, \vec{v}, x). \tag{A.14}$$

to denote the remaining components.

Proposition A.3. *Let (Σ, h, N) be a C^n differentiable, static spacetime which is AL-AdS. Let the conformal boundary at $x = 0$ be of the form described in (4.7, A.12). Then, $\widetilde{Q}^{(k)}_{\delta\beta}(0) = 0$, $\forall \; k < n-1$.*

Proof. From Proposition A.2 we know that $\widetilde{b}^{(k+1)}_{\alpha\beta} = \widetilde{H}^{(k)}_{\alpha\beta}(0) = 0$, $\forall k$ even and $0 \leq k \leq n-1$. Thus, we need to only consider the case k odd. For the rest of the proof we will assume that k is odd and $k < n-1$.

Step A: We begin by differentiating $\widetilde{P}^{(k)}_{\alpha\beta}(0)$ along V, $\mathbf{v} \cdot (\partial\, \widetilde{P}^{(k)}_{\alpha\beta})(0)$, where $\mathbf{v} \cdot \partial \equiv \mathbf{v}^{\alpha}{}_{\gamma}\partial_{\alpha}$. Using the Leibniz rule, Proposition A.2, (A.10), (A.11), (A.7) and (4.7), we see that every term on the right hand side of the expression contains either a (i) $\widetilde{Q}^{(m-1)}_{\delta\rho}(0)$ or a (ii) $\mathbf{v} \cdot \partial\, (\psi^{(m)})(0)$ or a (iii) $\mathbf{v} \cdot \partial\, (P^{(m)}_{\delta\rho})(0)$ for $1 \leq m \leq k-1$. Applying this recursively to (iii), this implies that every term in the expression for $\mathbf{v} \cdot \partial\, (\widetilde{P}^{(k)}_{\alpha\beta})(0)$ contains either a (i) $\widetilde{Q}^{(m-1)}_{\delta\rho}(0)$ or a (ii)$\mathbf{v} \cdot \partial\, (\psi^{(m)})(0)$ for $1 \leq m \leq k-1$.

Step B: Next, differentiate (A.9) along V, to obtain an expression for $\mathbf{v} \cdot \partial\, (\psi^{(k+1)})(0)$. Again, using the Leibniz rule, Proposition A.2, (A.10), (A.11), (A.7) and (4.7), we see that every term in the expression contains either a (i) $\widetilde{Q}^{(m-1)}_{\delta\beta}(0)$ or a (ii) $\mathbf{v} \cdot \partial\, (\widetilde{P}^{(m)}_{\delta\eta})(0)$ or a (iii) $\mathbf{v} \cdot \partial(\psi^{(m)})(0)$ for $1 \leq m \leq k-1$. Applying this recursively along with the results of step A, we see that every term in the expression for $\mathbf{v} \cdot \partial\, (\psi^{(k+1)})(0)$ contains a $\widetilde{Q}^{(m)}_{\delta\beta}(0)$ for $1 \leq m \leq k-2$.

Step C: Finally, we consider the projection of (A.8) along U, $\widetilde{Q}^{(k)}_{\delta\beta}(0)$. Again, using the Leibniz rule, Proposition A.2, (A.10), (A.11), (A.7) and (4.7), we see that every term in the projection of the right hand side of (A.8) contains either a (i) $\widetilde{Q}^{(m-1)}_{\delta\beta}(0)$ or a (ii) $\mathbf{v} \cdot \partial\, (P^{(m)}_{\delta\rho})(0)$ or a (iii) $\mathbf{v} \cdot \partial(\psi^{(m)})(0)$ for $1 \leq m \leq k-1$. Applying this recursively along with the results of steps A and B above, we see that every term in the expansion of $\widetilde{Q}^{(k)}_{\delta\beta}(0)$ contains a $\widetilde{Q}^{(m)}_{\delta\beta}(0)$ for $1 \leq m \leq k-2$.

Thus, if $\widetilde{Q}^{(m)}_{\alpha\beta}(0) = 0$ $\forall m$ satisfying $1 \leq m \leq k-2$ then $\widetilde{Q}^{(k)}_{\alpha\beta}(0) = 0$. From (A.8) and (A.7), $\partial_x \widetilde{H}_{\alpha\beta}(0) = \frac{\ell}{(n-2)} \widetilde{\mathcal{R}}_{\alpha\beta}(0)$. Since ${}^{\mathbf{v}}\widetilde{\mathcal{R}}_{\alpha\beta}(0) = 0$ and $\mathbf{v}^{\delta}{}_{\alpha}\widetilde{\mathcal{R}}_{\delta\beta}(0) = 0$ from (4.7), $\widetilde{Q}^{(1)}_{\alpha\beta}(0) = \frac{\ell}{(n-2)}\, \mathbf{v}^{\delta}{}_{\alpha}\widetilde{\mathcal{R}}_{\delta\beta}(0) = 0$. Since $\widetilde{Q}^{(k)}_{\alpha\beta}(0) = 0$ for $k = 1$, by induction, we can conclude that $\widetilde{Q}^{(k)}_{\alpha\beta}(0) = 0$ $\forall k < n-1$. □ .