Marginally outer trapped surfaces and scalar curvature rigidity

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1 Introduction

A key development in the study of manifolds of positive scalar curvature is the following fundamental observation of Schoen and Yau [41].

Theorem 1.1. *Let (M, g) be an n-dimensional, $n \geq 3$, Riemannian manifold with positive scalar curvature, $R > 0$. If N is a stable, closed two-sided minimal hypersurface in M then N admits a metric of positive scalar curvature.*

Moreover, by refinements of the arguments in [41], one obtains the *infinitesimal* rigidity statement that if $R \geq 0$, and N does not admit a metric of positive scalar curvature then N must be totally geodesic and Ricci flat, and R must vanish along N (cf. [18, 21]). In [9], M. Cai proved the following *splitting theorem* by assuming N is area-minimizing, rather than just being stable (see also [21] for a simplified proof).

Theorem 1.2. *Let (M, g) be an n-dimensional, $n \geq 3$, Riemannian manifold with nonnegative scalar curvature, $R \geq 0$, and suppose N^{n-1} is a closed two-sided minimal hypersurface which locally minimizes area. If N does not admit a metric of positive scalar curvature then there exists a neighborhood V of N such that $(V, g|_V)$ is isometric to $((-\delta, \delta) \times N, dt^2 \oplus h)$.*

This result extends to higher dimensions the torus splitting result in [10] for 3-manifolds of nonnegative scalar curvature. Over the years a number of related rigidity results have been obtained under various assumptions on the ambient scalar curvature and the topology of the minimal surface, see, e.g. [8, 35, 34, 1, 11, 12]. Indeed, the study of scalar curvature rigidity continues to be an active area research; see e.g. [30, 44, 43].

From the point of view of initial data sets in general relativity, such results are often referred to as *time-symmetric* results, meaning more specifically that they are purely Riemannian results. An initial data set in general relativity consists of a smooth manifold M equipped wtih a Riemanian metric g and a symmetric covariant 2-tensor K. Physically, such an initial data set corresponds to a spacelike hypersurface M in an enveloping spacetime (time-oriented Lorenzian manifold) $(\bar{M}, \bar{g})$, together with its induced metric g and second fundamental form K. A time-symmetric initial data set is then one in which $K = 0$.

In this paper we wish to present some rigidity results for, in general, non-time-symmetric initial data sets, in which the assumption of nonnegative scalar curvature is replaced by the dominant energy condition (see Section 2 for basic definitions). The results we consider are motivated in part by the positive mass theorem.

In [17], Eichmair, Huang, Lee and Schoen proved the following spacetime version of the positive mass theorem.

Theorem 1.3 (EHLS). *Let (M, g, K) be an n-dimensional, $3 \leq n \leq 7$, asymptotically flat initial data set satisfying the dominant energy condition. Then $E \geq |P|$, where (E, P) is the ADM energy-momentum vector of (M, g, K).*

While the proof involves some interesting new ideas, in broad terms, it generalizes to the spacetime setting the proof of the Riemannian positive mass theorem of Schoen and Yau [40, 38], where now *marginally outer trapped surfaces* (see Section 2) play a role analogous to minimal surfaces in the Schoen-Yau proof.

An interesting development in the proof of the Riemannian positive mass theorem is based on the following observation of Lohkamp [31]: To show that the mass in this case is nonnegative, it is sufficient to consider asymptotically flat Riemannian manifolds (M^n, g) with nonnegative scalar curvature, which are exactly Euclidean outside a compact set. In other words, by assuming the mass is negative, on can reduce to this situation. One may then compactify (M^n, g) to obtain a manifold of nonnegative scalar curvature with topology $T^n \# N^n$, where $T^n = n$-torus, and N^n is compact. Then one reaches a contradiction by applying known obstructions to the existence of positive scalar curvature metrics on manifolds with this topology; cf. Schoen and Yau [42], and Gromov and Lawson [25] (in the case N is spin).

In [32], Lohkamp presents a proof of Theorem 1.3 (in all dimensions) using a similar 'compactification' strategy. His approach naturally leads to the presentation of our first initial data rigidity result. This result, and its connection to Lokamp's approach, are discussed in Section 3. Some consequences, and related results, including some pure scalar curvature rigidity results, are also presented in Section 3. As our results rely on properties of marginally outer trapped surfaces, we begin with a discussion of these generalizations of minimal surfaces in Section 2.

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2 Marginally outer trapped surfaces

Let (M, g, K) be an n-dimensional initial data set.[1] For the purpose of making certain definitions we find it convenient to assume that (M, g, K) is embedded in an $n+1$ dimensional spacetime $(\bar{M}, \bar{g})$, by which we mean M is a spacelike hypersurface in $\bar{M}$ with induced metric g and second fundamental form K. (By our sign conventions, $K(X,Y) = \bar{g}(\bar{\nabla}_X u, Y)$ where u is the future directed unit normal field to M in $\bar{M}$.) Such an embedding can always be arranged; see e.g. [6, Section 3]. However, all essential objects introduced here will depend only on the initial data.

The initial data set (M, g, K) is said to satisfy the dominant energy condition (DEC) provided

$$\mu \geq |J|\,,$$

along M, where $\mu =$ *local energy density* $= G(u,u)$, and $J =$ *local momentum density* $=$ 1-form $G(u,\cdot)$ on M, where G is the Einstein tensor, $G = \mathrm{Ric}_{\bar{M}} - \frac{1}{2}R_{\bar{M}}\bar{g}$. Using the Gauss-Codazzi equations, μ and J can be expressed solely in terms of the initial data:

$$\mu = \frac{1}{2}\left(R + (\mathrm{tr}\,K)^2 - |K|^2\right)\,,$$
$$J = \mathrm{div}\,K - d(\mathrm{tr}\,K)\,,$$

where R is the scalar curvature of M. These are the so-called Einstein constraint equations. Note that in the time-symmetric case ($K = 0$), the DEC becomes the requirement that M have nonnegative scalar curvature.

Let Σ be a closed embedded 2-sided hypersurface in M. Let ν be a smooth unit normal field along Σ in M, which, by convention, we shall refer to as *outward* pointing. Σ admits two future directed null normal vector fields along Σ, $\ell^+ = u + \nu$ (future directed outward pointing) and $\ell^- = u - \nu$ (future directed inward pointing).

Associated to ℓ^+ and ℓ^- are the two *null second fundamental forms*, χ^+ and χ^-, respectively, defined as,

$$\chi^{\pm} : T_p\Sigma \times T_p\Sigma \to \mathbb{R}, \qquad \chi^{\pm}(X,Y) = \bar{g}(\bar{\nabla}_X \ell^{\pm}, Y)\,. \tag{2.1}$$

These null second fundamental forms completely determine the second fundamental form of Σ, viewed as a codimension two submanifold of spacetime. In terms of our

[1] We will always assume M is conenected. Moreover, although not needed for all results, we will also assume M is orientable.

initial data (M, g, K),

$$\chi^{\pm} = K|_{\Sigma} \pm A \tag{2.2}$$

where A is the 2nd fundamental form of Σ in M, $A(X,Y) = g(\nabla_X \nu, Y)$.

Tracing the null second fundamental forms, we obtain the *null expansion scalars* (also referred to as null mean curvatures) θ^+, θ^-:

$$\theta^{\pm} = \mathrm{tr}_{\Sigma}\chi^{\pm} = \mathrm{div}_{\Sigma}\,\ell^{\pm}\,.$$

Physically, θ^+ (resp., θ^-) measures the divergence of the outgoing (resp., ingoing) light rays emanating from Σ. In terms of our initial data (M, g, K),

$$\theta^{\pm} = \mathrm{tr}_{\Sigma}K \pm H\,,$$

where H is the mean curvature of Σ within M. In particular, in the time-symmetric case, $K = 0$, θ^+ is just the mean curvature of Σ in M.

For round spheres in Euclidean slices of Minkowski space, or, more generally, for large 'radial' spheres in asymptotically flat initial data sets, one has $\theta^- < 0$ and $\theta^+ > 0$ (with the obvious choice of inside and outside). However, in regions of spacetime where the gravitational field is strong, one may have both $\theta^- < 0$ and $\theta^+ < 0$, in which case Σ is called a *trapped surface*. The concept of a trapped surface was introduced by Penrose [36], and plays a key role in the Penrose singularity theorem.

Focusing attention on the outward null normal only, we say that Σ is an outer trapped surface if $\theta^+ < 0$. Finally, we define Σ to be a marginally outer trapped surface (MOTS) if θ^+ vanishes identically. MOTSs arise naturally in a number of situations. For example, cross sections of the event horizon in stationary (i.e. steady state) black hole spacetimes are MOTSs. MOTSs also arise as the boundary of the so-called trapped region. Oldeer heuristic arguments for their existence in this case were made rigorous first by Andersson and Metzger [6] for three dimensional initial data sets, and then by Eichmair [14, 15] for initial data sets up to dimension seven. For these reasons MOTSs are often used to model the surface of a black hole.

Note that in the time-symmetric case, a MOTS is simply a minimal hypersurface in M. It is in this sense that MOTSs may be viewed as spacetime analogues of minimal surfaces. Despite the absence in general of a variational characterization of MOTSs, like that for minimal surfaces, MOTSs have been shown to satisfy a number of analogous properties.

2.1 Stability and local rigidity of MOTS.

MOTSs admit an important notion of stability, as introduced by Andersson, Mars and Simon [4, 5]. This is based on variations of the null expansion, as we now discuss.

Let Σ be a MOTS in our initial data set (M, g, K) with outward normal ν. Consider a normal variation $t \to \Sigma_t$ of $\Sigma = \Sigma_0$ in M, with variation vector field

$$\mathcal{V} = \left.\frac{\partial}{\partial t}\right|_{t=0} = \phi\nu, \quad \phi \in C^{\infty}(\Sigma)\,. \tag{2.3}$$

Let $\theta^+(t)$ denote the null expansion of Σ_t with respect to $l_t^+ = u + \nu_t$, where u is the future directed timelike unit normal to M and ν_t is the outward unit normal to Σ_t in M. A computation shows,

$$\left.\frac{\partial \theta^+}{\partial t}\right|_{t=0} = L(\phi) := -\triangle\phi + 2\langle X, \nabla\phi\rangle + \left(Q + \operatorname{div} X - |X|^2\right)\phi\,, \tag{2.4}$$

where,

$$Q = \frac{1}{2}R_\Sigma - (\mu + J(\nu)) - \frac{1}{2}|\chi^+|^2\,. \tag{2.5}$$

Here Δ, ∇, and div are the Laplacian, gradient, and divergence operators, respectively, and R_Σ is the scalar curvature, of Σ with respect to the induced metric $\langle\,,\,\rangle$. Moreover, and X is the vector field on Σ dual to the one form $K(\cdot,\nu)|_{T\Sigma}$.

A MOTS Σ is said to be *stable* if there exists $\phi \in C^\infty(\Sigma)$, $\phi > 0$, such that $L(\phi) \geq 0$. Hence, in view of (2.4), a MOTS Σ is stable if there exists an outward variation of Σ such that the null expansion is 'infinitesimally nonincreasing'. In the time-symmetric case, θ^+ becomes the mean curvature H, the vector field X vanishes and L reduces to the stability (or Jacobi) operator of minimal surface theory. As shown in [5], although in general L is not self-adjoint, it nevertheless admits a real principal eigenvalue $\lambda_1(L)$ and an associated eigenfunction ϕ, $L(\phi) = \lambda_1(L)\phi$, which is strictly positive. Morever, one has that Σ is stable if and only if $\lambda_1(L) \geq 0$.

A basic criterion for stability is the following. Let Σ be a MOTS in (M, g, K), and let $U \subset M$ be a neighborhood of Σ that is separated by Σ. (Since Σ is 2-sided, such a neighborhood always exists.) We say the Σ is weakly outermost in U if there are no outer trapped ($\theta^+ < 0$) surfaces in U^+ (the part of U to the outside of Σ) homologous to Σ. When this situation holds, we say that Σ is *locally* weakly outermost. If, however, $U = M$, we simply say that Σ is weakly outermost. It then follows that (locally) weakly outermost MOTSs are necessarily stable. Indeed, if $\lambda_1(L) < 0$, equation (2.4), with ϕ a positive eigenfunction, would then imply that Σ could be perturbed outward to an outer trapped surface.

Let Σ be a stable MOTS in (M, g, K); hence, there exists $\phi > 0$ such that $L(\phi) \geq 0$. As shown in [24], one may derive from this inequality, the following *MOTS stability inequality*,

$$\int_\Sigma |\nabla\psi|^2 + \left(\frac{1}{2}S - (\mu + J(\nu)) - \frac{1}{2}|\chi|^2\right)\psi^2 \geq 0\,, \tag{2.6}$$

for all $\psi \in C^2(\Sigma)$. This inequality is remarkably similar to the well-known stability inequality of minimal surface theory, and in fact reduces to the latter in the time-symmetric case. Of particular significance for applications is the fact that the vector field X in (2.4) does not appear. From (2.6), one readily obtains the following result.

Theorem 2.1 (infinitesimal rigidity). *Let (M, g, K) be an n-dimensional, $n \geq 3$, initial data set that satisfies the DEC, $\mu \geq |J|$. If Σ is a connected stable MOTS in M that does not admit a metric of positive scalar curvature then*

(1) Σ *is Ricci flat and has vanishing null second fundamental form,* $\chi^+ = 0$.

(2) $\mu + J(\nu) = 0$ *along* Σ.

This theorem was formulated in a slightly different way in [24], and interpreted to mean that, apart from certain exceptional circumstances, a stable MOTS must admit a metric of positive scalar curvature. This gives rise to certain topological restrictions, and, in particular, generalizes to higher dimensions Hawking's black hole topology theorem [27].

By strengthening the stability assumption, namely by requiring the MOTS Σ to be (locally) weakly outermost, we obtain additional rigidity. The following was proved in [22] (see also [20]).

Theorem 2.2. *Let* (M, g, K) *be an* n*-dimensional,* $n \geq 3$*, initial data set that satisfies the DEC. If* Σ *is a connected locally weakly outermost MOTS in* M *that does not admit a metric of positive scalar curvature then there is a neighborhood* $U \cong [0, \delta) \times \Sigma$ *of* Σ *in* M *such that the following hold for each* $t \in [0, \delta)$*:*

(1) $\Sigma_t = \{t\} \times \Sigma$ *is a MOTS. In fact,* Σ_t *has vanishing outward null second fundamental form.*

(2) Σ_t *is Ricci flat with respect to the induced metric.*

(3) $\mu + J(\nu_t) = 0$ *for every* $t \in [0, \delta)$*, where* ν_t *is the unit normal of* Σ_t *pointing towards increasing values of* t*.*

Properties (1), (2), (3) follow from Theorem 2.1 and the fact that each Σ_t is stable.

2.2 A basic MOTS existence result

We will make use of the following fundamental existence result for MOTSs. It was obtained by L. Andersson and J. Metzger [6] in dimension $n = 3$ and then, using different techniques, by the Eichmair [14, 15] in dimensions $3 \leq n \leq 7$. The regularity theory developed by Eichmair relies on aspects of geometric measure theory, which is responsible for the dimension restriction. Both approaches are based on an idea of R. Schoen [39] to construct a MOTS between suitably trapped hypersurfaces by forcing a blow up of the Jang equation. See also [3] for an excellent survey of these existence results.

Theorem 2.3. *Let* (M, g, K) *be an* n*-dimensional,* $3 \leq n \leq 7$*, compact-with-boundary initial data set. Suppose that the boundary* ∂M *can be expressed as a disjoint union of hypersurfaces,* $\partial M = \Sigma_{\rm in} \cup \Sigma_{\rm out}$*, such that* $\theta^+ < 0$ *along* $\Sigma_{\rm in}$ *with respect to the normal pointing into* M*, and* $\theta^+ > 0$ *along* $\Sigma_{\rm out}$ *with respect to the normal pointing out of* M*. Then there is an outermost MOTS* Σ *in* (M, g, K) *homologous to* $\Sigma_{\rm out}$*.*

Remarks.

1. Here outermost means that (i) Σ is separating and (ii) there are no *weakly* outer trapped ($\theta^+ \le 0$) surfaces in the region outside of (and including) Σ that are homologous to Σ, other than Σ itself.

2. In the time-symmetric case, the boundary conditions simply mean that M has mean convex boundary. One can then minimize area in the homology class determined by Σ_{out}, to obtain a compact minimal hypersurface homologous to Σ_{out}. This direct variational approach is not in general available for MOTSs. As noted above, a quite different approach to the proof of existence is taken.

3. Theorem 2.3 remains valid under the slightly weaker boundary condition: $\theta^+ \le 0$ along Σ_{in} (cf. [6, Section 5]). In this case the MOTS Σ guaranteed by theorem may have some components in common with Σ_{in}. However, by the maximum principle for MOTSs [7], Σ will not meet Σ_{out}.

3 Global rigidity results

In [32] Lohkamp presented a proof of the EHLS spacetime positive mass theorem, Theorem 1.3, in the introduction (in all dimensions $n \ge 3$), by extending his compactification approach to the general spacetime setting. By this approach, the proof reduces to establishing the following result (cf. [32, Theorem 2]:

Nonexistence of $\mu - |J| > 0$ - islands: Let (M, g, K) be an initial data set that is isometric to Euclidean space, with $K = 0$, outside some bounded set U. Then one cannot have $\mu > |J|$ on U.

In particular, in the case of general interest, in which (M, g, K) satisfies the DEC, there must be a point in U at which $\mu = |J|$. The goal of our first result is to show that a much stronger conclusion holds in dimensions $3 \le n \le 7$.

By placing U in a large box, and identifying all but one pair of sides, we obtain a compact manifold, which we still refer to as M, with two compact boundary components Σ_0 and S, each of which is a flat, totally geodesic $(n-1)$-torus in M. Moreover, since we are in a region in which K vanishes, each are MOTS; in fact, each has vanishing null second fundamental form, with respect to either choice of normal ν (cf. (2.2)). This is the basic configuration for our first result. However, we want to generalize the setting some.

The homotopy condition. Roughly speaking, the compactified manifold M has 'almost product' topology, $M \cong ([0,1] \times \Sigma_0) \# N$. Generalizing this, we will assume that M satisfies the *homotopy condition* with respect to Σ_0, namely that there exists a continuous map $\rho : M \to \Sigma_0$ such that $\rho \circ i : \Sigma_0 \to \Sigma_0$ is homotopic to id_{Σ_0}, where $i : \Sigma_0 \hookrightarrow M$ is the inclusion map. Since M is connected by assumption, this condition implies that Σ_0 is connected. (We don't require that S be connected.) Note that if $\rho \circ i$ actually equals id_{Σ_0}, this simply says that ρ is a retraction of M onto Σ_0.

The cohomology condition. Further, we will assume that Σ_0 satisfies the *cohomology condition*, namely that there exist $\omega_1, \ldots, \omega_{n-1} \in H^1(\Sigma_0, \mathbb{Z})$ such that

$$\omega_1 \smile \cdots \smile \omega_{n-1} \neq 0 . \tag{3.7}$$

This condition insures that Σ_0 does not carry a metric of positive scalar curvature (and, in particular, is satisfied if Σ_0 is a torus). This follows from [42, Theorem 5.2], although the form of the condition used here is that given in [29, Theorem 2.28].

We are finally prepared to state the following theorem, cf. [16, Theorem 1.2].

Theorem 3.1. *Let (M, g, K) be an n-dimensional, $3 \leq n \leq 7$, compact-with-boundary initial data set. Suppose that (M, g, K) satisfies the DEC and that the boundary ∂M can be expressed as a disjoint union of hypersurfaces, $\partial M = \Sigma_0 \sqcup S$, such that the following conditions hold:*

(a) $\theta^+ \leq 0$ *along* Σ_0 *with respect to the normal pointing into* M, *and* $\theta^+ \geq 0$ *along* S *with respect to the normal pointing out of* M,

(b) M *satisfies the homotopy condition with respect to* Σ_0 *and* Σ_0 *satisfies the cohomology condition.*

Then M is diffeomorphic to $[0, \ell] \times \Sigma_0$, such that the following hold for each $t \in [0, \ell]$:

(i) $\Sigma_t = \{t\} \times \Sigma$ *is a MOTS. In fact, each* Σ_t *has vanishing outward null second fundamental form,* $\chi_t^+ = 0$.

(ii) Σ_t *is a flat torus, with respect to the induced metric.*

(iii) $\mu = |J|$ *on* M *and* $J|_{T\Sigma_t} = 0$.

We note that Theorem 3.1 provides a relatively simple proof of Theorem 2 in [32] in dimensions $3 \leq n \leq 7$, without requiring any strictness in the dominant energy condition.

We would like to make some comments about the proof of Theorem 3.1; for details see [16]. The theorem follows from the next two results. The first is a global version of Theorem 2.2.

Proposition 3.2. *Let (M, g, K) be an n-dimensional, $n \geq 3$, compact-with-boundary initial data set satisfying the DEC. Suppose that ∂M can be expressed as a disjoint union of hypersurfaces, $\partial M = \Sigma_0 \sqcup S$, such that Σ_0 is a (connected) MOTS and S has null expansion $\theta^+ \geq 0$, with repect to the normal pointing out of M. If Σ_0 is weakly outermost in M and does not admit a metric of positive scalar curvature then $M \cong [0, \ell] \times \Sigma_0$, such that properties $(1), (2), (3)$ of Theorem 2.2 hold for each $t \in [0, \ell]$.*

Comments on the proof. Applying Theorem 2.2, we obtain a neighborhood $U \cong [0,\delta) \times \Sigma_0$ satisfying $(1), (2), (3)$. The vanishing of the null second fundamental forms of the Σ_t's implies a uniform bound, in terms of K, on the second fundamental forms of the Σ_t's within M (cf. (2.2)). Area bounds for the Σ_t's also follow from this. Then by known compactness results, the Σ_t's converge smoothly to an immersed MOTS Σ_δ as $t \to \delta$. A surgery argument of Andersson and Metzger [5, Section 6] ensures that Σ_δ does not touch itself to the outside, and hence is, in fact, embedded, as otherwise the weakly outermost assumption would be violated. It follows that the foliation of U be MOTSs, satisfying $(1), (2), (3)$, extends to $t = \delta$. Then, by a continuity argument, the foliation will extend all the way to S. By the maximum principle for MOTS [7], the "last leaf" Σ_ℓ must agree with S. □

The next result establishes conditions under which Σ_0 is a weakly outermost MOTS (cf. [16, Lemma 3.2]). It is here that the homotopy and cohomology conditions are used.

Proposition 3.3. *Let (M, g, K) be an n-dimensional, $3 \le n \le 7$ compact-with-boundary initial data set. Suppose that (M, g, K) satisfies the DEC and that ∂M can be expressed as a disjoint union of hypersurfaces, $\partial M = \Sigma_0 \sqcup S$, such that the following conditions hold:*

(a) $\theta^+_K \le 0$ along Σ_0 with respect to the normal pointing into M, and $\theta^+_K \ge 0$ along S with respect to the normal pointing out of M,

(b) M satisfies the homotopy condition with respect to Σ_0 and Σ_0 satisfies the cohomology condition.

Then Σ_0 is a weakly outermost MOTS in (M, g, K).

Here we have introduced subscript notation on θ^+, to show its dependence on the given initial data set, as a different (but related) initial data set, is used in the proof.

Comments on the proof. Assume for the moment that Σ_0 is a MOTS. We want to show it is weakly outermost. If it isn't, then there exists an outer trapped surface ($\theta^+_K < 0$) Σ homologous to Σ_0 strictly between Σ and S. Let W be the region bounded by Σ and S. Now reverse the time orientation, i.e. consider the initial data set $(W, g, -K)$. S has null expansion $\theta^+_{-K} \le 0$ with respect to the normal pointing into W and Σ has null expansion $\theta^+_{-K} > 0$ with respect to the normal pointing out of W. Hence, with these boundary conditions we can apply Theorem 2.3 (see also point 3 of the remarks following its statement) to obtain an outermost MOTS Σ' in the initial data set $(W, g, -K)$ homologous to Σ_0. (It's precisely here that the dimension restriction is used.) Now one can use the homotopy condition, and the fact that Σ_0 satisfies the cohomology condition, to show that Σ' satisfies the cohomology condition. In particular, some component of Σ' does not admit a metric of positive scalar curvature. But then Theorem 2.2 implies that Σ' can't be outermost.

Finally, Σ_0 must be a MOTS. If not, then we have $\theta^+ \leq 0$ along Σ_0, and $\theta^+ < 0$ somewhere. In this case one can use null mean curvature flow to perturb Σ_0 to a strictly outer trapped surface [6, Section 5], and we can run the same argument again to get a contradiction. □

We can now complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Propositions 3.2 and 3.3 imply that properties (1), (2), (3) of Theorem 2.2 hold for each $t \in [0, \ell]$. Hence property (i) holds. The DEC and property (3) easily imply that (iii) holds. Finally, we need to observe that each Σ_t is a flat torus. The cohomology condition, Poincaré duality, and the fact that $H^n(M, \mathbb{Z})$ is torsion free imply the first Betti number estimate, $b_1(\Sigma_t) \geq n$. However, by a classical result of Bochner (see e.g. [37, p. 208]) it follows from the Ricci flatness of Σ_t that $b_1(\Sigma_t) \leq n$ with equality if and only if Σ_t is isometric to a flat torus.

One may ask if it is possible to obtain greater rigidity in Theorem 3.1. We briefly describe an example which shows that there is still a fair amount of flexibility in the initial data sets covered by Theorem 3.1.

Example. For notational simplicity, we restrict to 3-dimensional initial data sets. Consider the box $\mathcal{B} : 0 \leq x, y, z \leq 1$ in the $t = 0$ slice of 4-dim Minkowski space. Let $f : \mathcal{B} \to \mathbb{R}$ be any smooth function which vanishes near the boundary of $\mathcal{B}$ and whose graph is spacelike. Let M be the manifold obtained from the graph of f by identifying the x-sides and the y-sides of $\mathcal{B}$. Note that $M \cong [0, 1] \times \mathbb{T}^2$. Now, consider the initial data set (M, g, K), where g and K are the induced metric and second form fundamental of graph f. One sees that (M, g, K) satisfies all the assumptions of Theorem 3.1; in particular, since the graph of f sits in Minkowski space, it is a *vacuum* initial data set, $\mu = 0$, $J = 0$, and so the DEC holds trivially. Hence, the conclusions of Theorem 3.1 must hold, as well. Where, then, is the foliation by MOTSs ? In fact, it is obtained by intersecting graph f with the null hyperplanes $\mathcal{H}_c : t = z + c$. These are totally geodesic null hypersurfaces, and by interesting features of null geometry, the intersections with graph f produce a foliation of M by flat tori with vanishing null second fundamental forms; see Figure 1. This example is still quite special in certain ways. In particular, (M, g, K) is embedded in a flat spacetime. It would be interesting to find other, more general, examples.

In order to obtain results with stronger rigidity, we make use of the notion of a 'k-convex' hypersurface (k an integer between 1 and the dimension of the hypersurface), which has been well-studied in Riemannian geometry; see e.g. [19, 28, 26]. This concept, as applied initial data sets, was first considered by Mendes in [33].

Let (M, g, K) be a given initial data set. We say that a symmetric covariant 2-tensor P on M is $(n-1)$-convex if, at every point, the sum of the smallest $n-1$ eigenvalues of P, with respect to g, is non-negative. It can be shown that if P is $(n-1)$-convex then the partial trace of P along every hypersurface Σ in M is nonnegative, $\mathrm{tr}_\Sigma P \geq 0$.

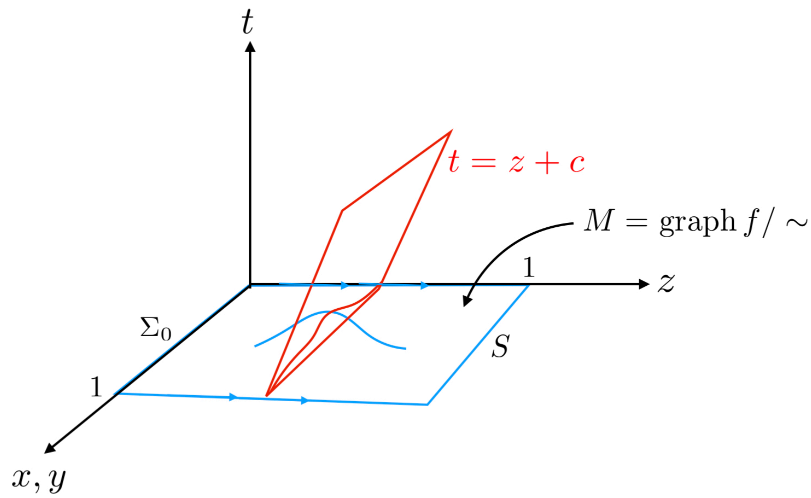

Figure 1: Example for Theorem 3.1

With this definition at hand we may state the following theorem (see [16, Theorem 1.3]).

Theorem 3.4. *Let (M, g, K) be an n-dimensional, $3 \leq n \leq 7$, compact-with-boundary initial data set. Suppose that (M, g, K) satisfies the DEC and that the boundary ∂M can be expressed as a disjoint union of hypersurfaces, $\partial M = \Sigma_0 \sqcup S$, such that the following conditions hold:*

(i) $\theta_K^+ \leq 0$ *along* Σ_0, *and* $\theta_K^- \leq -2(n-1)\epsilon$ *along* S, *where* $\epsilon = 0$ *or* 1.

(ii) M *satisfies the homotopy condition with respect to* Σ_0 *and* Σ_0 *satisfies the cohomology condition.*

(iii) $K + \epsilon\, g$ *is* $(n-1)$*-convex.*

Then

(a) (M, g) *is isometric to the (warped) product* $([0, \ell] \times \Sigma_0, dt^2 + e^{2\epsilon t} g_0)$, *where* (Σ_0, g_0) *is a flat torus.*

(b) $K = (1-\epsilon)\, a(t)\, dt^2 - \epsilon\, g$ *on* M.

(c) $\mu_K = 0$ *and* $J_K = 0$ *on* M. *(vacuum IDS).*

In the $\epsilon = 0$ case, the assumptions are the same as in Theorem 3.1 (except for a small change in the boundary condition at S), but now with the added assumption that K is $(n-1)$-convex. With the "parameter choice" $\epsilon = 1$, Theorem 3.4 becomes relevant to asymptotically hyperbolic (or hyperboloidal) initial data sets. It is shown in [16], that, in either case ($\epsilon = 0$ or $\epsilon = 1$), the resulting initial data set can be embedded into a quotient of Minkowski space. We make a few remarks about the proof.

- The $(n-1)$-convexity condition, and the inequality $\theta_K^- \le -2(n-1)\epsilon$ along S imply $\theta_K^+ \ge 0$ along S. Then (M, g, K) satisfies all the assumptions of Theorem 3.1, and hence is foliated by MOTS, $\theta_K^+ = 0$.

- Now consider the initial data set (M, g, P), where $P = -K - 2\epsilon\, g$. Using the $(n-1)$-convexity condition and properties of (M, g, K), one shows that (M, g, P) also satisfies all the assumptions of Theorem 3.1, and hence is foliated by MOTS, $\theta_P^+ = 0$.

- Again using the $(n-1)$-convexity condition, one may apply the maximum principle for MOTS to conclude that these two foliations agree. One then makes use of the fact that $\theta_K^+ = \theta_P^+ = 0$ along each leaf Σ_t of the common foliation, in conjunction with equation (2.4), adapted to each initial data set, to derive the result. The scalar ϕ in (2.4) is determined by the foliation, and is ultimately shown to be constant along each Σ_t.

3.1 Scalar curvature rigidity

In this final section, we present some rigidity results for Riemannian manifolds with scalar curvature suitably bounded from below. We first consider the consequences of Theorem 3.4 from setting $K = -\epsilon g$. With this choice for K, the DEC reduces to the scalar curvature inequality,

$$R \ge -n(n-1)\epsilon\,, \quad \epsilon = 0, 1\,. \tag{3.8}$$

This choice for K then leads to the following result (see [16, Corollary 1.4]).

Theorem 3.5. *Let (M, g) be an n-dimensional, $3 \le n \le 7$, compact Riemannian manifold with boundary. Suppose that the scalar curvature of (M, g) satisfies $R \ge -n\,(n-1)\,\epsilon$, where $\epsilon = 0$ or 1. Suppose further that the boundary ∂M can be expressed as a disjoint union of hypersurfaces, $\partial M = \Sigma_0 \sqcup S$, such that the following hold.*

(i) The mean curvature of Σ_0 satisfies $H \le (n-1)\,\epsilon$ with respect to the normal pointing into M, and the mean curvature of S satisfies $H \ge (n-1)\,\epsilon$ with respect to the normal pointing out of M.

(ii) Σ_0 satisfies the cohomology condition and M satisfies the homotopy condition with respect to Σ_0.

Then (M, g) is isometric to $([0, \ell] \times \Sigma_0, dt^2 + e^{2\epsilon t} g_0)$, where (Σ_0, g_0) is a flat torus.

By setting $K = -\epsilon\, g$ in Theorem 3.4, one easily verifies that all the assumptions of Theorem 3.4 are satisfied. Hence the result follows.

Theorem 3.5 bears a similarity to a (warped) product splitting theorem of Croke and Kleiner [13, Theorem 1], in which they assume the corresponding lower bound on Ricci curvature. We also note that Theorem 3.5, with $\epsilon = 1$, contains the hyperbolic space rigidity result, Theorem 1.1 in [2], as a special case. The proof of Theorem 3.5 may also be approached via variational methods, e.g. area minimization in the case $\epsilon = 0$. The MOTS methodology described here gives a way of treating both cases in a unified way.

We would like to consider one further scalar curvature rigidity result that is a conseqeunce of the MOTS methodology. Let M be a complete *noncompact* Riemannian manifold, with compact connected boundary Σ_0. Suppose Σ_0 has constant mean curvature H_0 with respect to the normal pointing into M. We say that Σ_0 is *weakly outermost* in M if there does not exist a compact hypersurface $\Sigma \subset M \setminus \Sigma_0$ homologous to Σ_0 satisfying the (strict) mean curvature inequality, $H_\Sigma < H_0$.

Theorem 3.6. *Let (M, g) be a complete, noncompact n-dimensional ($n \geq 3$) Riemannian manifold with compact connected boundary Σ_0. Suppose that the scalar curvature of (M, g) satisfies $R \geq -n\,(n-1)\,\epsilon$, where $\epsilon = 0$ or 1. Suppose, further, that the following hold:*

(i) Σ_0 has mean curvature $H = \epsilon(n-1)$.

(ii) Σ_0 does not carry a metric of positive scalar curvature and is weakly outermost.

Then (M, g) is isometric to $[0, \infty) \times N$, with (warped) product metric $dt^2 + e^{2\varepsilon t}h$, where (N, h) is Ricci flat.

This is essentially Theorem 3.1 in [23]. The slightly weaker assumption made there, $H \leq \epsilon(n-1)$, together with the weakly outermost assumption, in fact forces Σ_0 to have constant mean curvature $\epsilon(n-1)$. Well-known examples discussed in [23] show that Theorem 3.6 fails if, in (ii), either one of the two conditions is dropped. Theorem 3.6 is a consequence of the following variation of Proposition 3.2.

Proposition 3.7. *Let (M, g, K) be an n-dimensional, $n \geq 3$, initial data set, where (M, g) is a complete noncompact manifold with compact connected MOTS boundary Σ_0. Suppose that (M, g, K) satisfies the DEC. If Σ_0 is weakly outermost in M and does not admit a metric of positive scalar curvature then $M \cong [0, \infty) \times \Sigma_0$, such that properties $(1), (2), (3)$ of Theorem 2.2 hold for each $t \in [0, \infty)$.*

The proof of this is almost identical to that of Proposition 3.2. To establish Theorem 3.6, one applies Proposition 3.7 to the initial data set (M, g, K), with $K = -\epsilon g$. One easily verifies in this case that all assumptions of Proposition 3.7 are satisfied. One may now make use of properties $(1), (2), (3)$, of Theorem 2.2, and equation (2.4) (adapted to the present setting), with $\theta = 0$, to derive the result. See [23, Theorem 3.1] for further details.

In this section we have described some of the main results from [16, 23] (or small variations thereof). The interested reader may consult these papers for further results.

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