

# **Cyclic Sieving of Multisets with Bounded Multiplicity and the Frobenius Coin Problem**

Séminaire Lotharingien de Combinatoire 93, Pocinho

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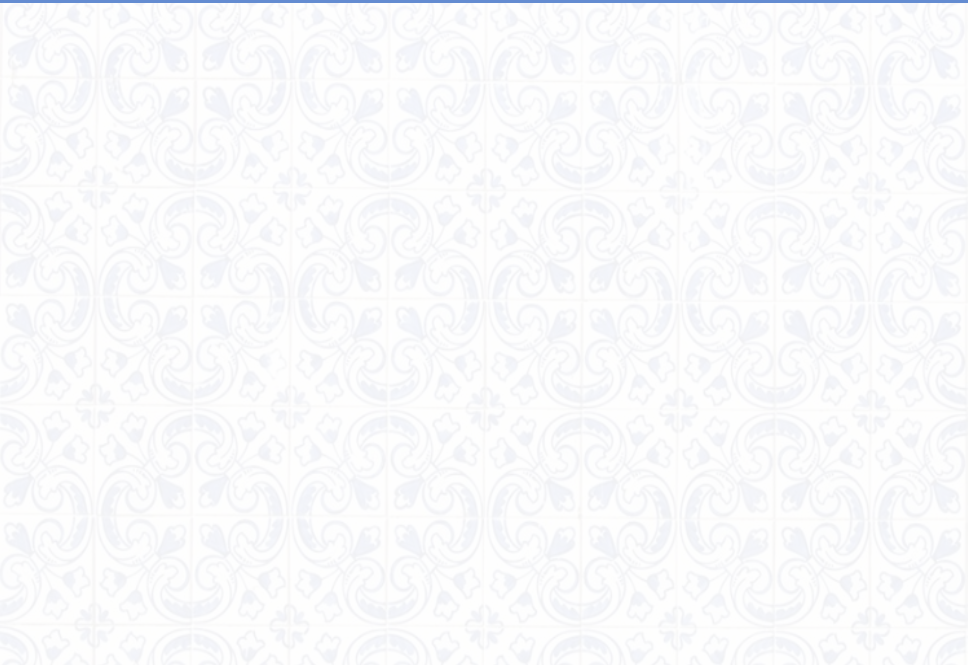
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March 25, 2025

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## 1. Generalized Binomial Coefficients



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Two interpretations of binomial coefficients:

$$E(n; t) = \prod_{i=1}^n (1 + t) = \sum_{k \geq 0} \binom{n}{k} t^k,$$

$$H(n; t) = \prod_{i=1}^n (1 + t + t^2 + \dots) = \sum_{k \geq 0} \binom{n + k - 1}{k} t^k.$$

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We consider the following interpolation:

$$H^{(b)}(n; t) = \prod_{i=1}^n (1 + t + \dots + t^{b-1}) = \sum_{k \geq 0} \binom{n}{k}^{(b)} t^k.$$

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Note that  $\binom{n}{k}^{(2)} = \binom{n}{k}$  and  $\binom{n}{k}^{(b)} = \binom{n+k-1}{k}$  when  $b > k$ . We will write

$$\binom{n}{k}^{(\infty)} = \binom{n+k-1}{k}.$$

## 1. Generalized Binomial Coefficients

Example:  $n = 3$  and  $b = 4$ . The generating function is

$$\begin{aligned}H^{(4)}(3; t) &= (1 + t + t^2 + t^3)^3 \\ &= 1 + 3t + 6t^2 + 10t^3 + 12t^4 + 12t^5 + 10t^6 + 6t^7 + 3t^8 + t^9.\end{aligned}$$

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The coefficients are

$k$	0	1	2	3	4	5	6	7	8	9
$\binom{3}{k}^{(4)}$	1	3	6	10	12	12	10	6	3	1

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In general we have  $\binom{n}{k}^{(b)} = 0$  for  $k > (b-1)n$  and

$$\binom{n}{0}^{(b)} + \binom{n}{1}^{(b)} + \cdots + \binom{n}{(b-1)n}^{(b)} = b^n.$$

Remark:  $\binom{n}{k}^{(b)} / b^n$  is the probability of getting a sum of  $k$  in  $n$  rolls of a fair  $b$ -sided die with sides labeled  $\{0, 1, \dots, b-1\}$ .

# 1. Generalized Binomial Coefficients

## Remarks:

- The numbers  $\binom{n}{k}^{(b)}$  occur often but they don't have a standard name.
- We roughly follow Euler's (1778) notation:  $\left(\frac{n}{k}\right)^b$ .
- Belbachir and Igueroufa (2020) compiled a historical bibliography.

## 2. Symmetric Functions and $q$ -Analogues



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Recall the generating functions for *elementary* and *complete* symmetric polynomials:

$$E(z_1, \dots, z_n; t) = \prod_{i=1}^n (1 + z_i t) = \sum_{k \geq 0} e_k(z_1, \dots, z_n) t^k,$$

$$H(z_1, \dots, z_n; t) = \prod_{i=1}^n (1 + z_i t + (z_i t)^2 + \dots) = \sum_{k \geq 0} h_k(z_1, \dots, z_n) t^k.$$

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We consider the following interpolation:

$$H^{(b)}(z_1, \dots, z_n; t) = \prod_{i=1}^n (1 + z_i t + \dots + (z_i t)^{b-1}) = \sum_{k \geq 0} h_k^{(b)}(z_1, \dots, z_n) t^k.$$

Note that  $h_k^{(2)} = e_k$  and  $h_k^{(b)} = h_k$  when  $b > k$ . We will write  $h_k^{(\infty)} = h_k$ .

## 2. Symmetric Functions and $q$ -Analogues

We can view  $h_k^{(b)}(z_1, \dots, z_n)$  as a generating function for lattice points in a diagonal slice of the integer box  $\{0, 1, \dots, b-1\}^n$ :

$$X := \{(x_1, \dots, x_n) \in \{0, 1, \dots, b-1\}^n : x_1 + x_2 + \dots + x_n = k\}.$$

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Then we have

$$h_k^{(b)}(z_1, \dots, z_n) = \sum_{\mathbf{x} \in X} \mathbf{z}^{\mathbf{x}} = \sum_{\mathbf{x} \in X} z_1^{x_1} z_2^{x_2} \cdots z_n^{x_n}.$$

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We can also view these lattice points as  $k$ -multisubsets of  $\{1, 2, \dots, n\}$  with multiplicities bounded above by  $b$ :

$$(x_1, x_2, \dots, x_n) \longleftrightarrow \underbrace{\{1, \dots, 1\}}_{x_1 \text{ times}}, \underbrace{\{2, \dots, 2\}}_{x_2 \text{ times}}, \dots, \underbrace{\{n, \dots, n\}}_{x_n \text{ times}}.$$

$$b = 2 : k\text{-subsets of } \{1, \dots, n\},$$

$$b = \infty : k\text{-multisubsets of } \{1, \dots, n\}.$$

## 2. Symmetric Functions and $q$ -Analogues

Example:  $n = 3$  and  $k = 3$  for various values of  $b$ :

			030		
		120		021	
	210		111		012
300		201		102	003

$$h_3^{(2)}(z_1, z_2, z_3) = z_1 z_2 z_3,$$

$$h_3^{(3)}(z_1, z_2, z_3) = z_1 z_2 z_3 + z_1^2 z_2 + \cdots + z_2 z_3^2,$$

$$h_3^{(4)}(z_1, z_2, z_3) = z_1 z_2 z_3 + z_1^2 z_2 + \cdots + z_2 z_3^2 + z_1^3 + z_2^3 + z_3^3,$$

$$h_3^{(5)}(z_1, z_2, z_3) = z_1 z_2 z_3 + z_1^2 z_2 + \cdots + z_2 z_3^2 + z_1^3 + z_2^3 + z_3^3,$$

⋮

## 2. Symmetric Functions and $q$ -Analogues

A natural  $q$ -analogue of  $\binom{n}{k}^{(b)}$  is given by the **principal specialization** of  $h_k^{(b)}$ :

$$\left[ \begin{matrix} n \\ k \end{matrix} \right]_q^{(b)} := h_k^{(b)}(\mathbf{1}, q, \dots, q^{n-1}) = \sum_{\mathbf{x} \in X} q^{0x_1 + 1x_2 + 3x_2 + \dots + (n-1)x_n}.$$

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This generalizes the standard  $q$ -binomial coefficients in the following sense:

$$\begin{aligned} \left[ \begin{matrix} n \\ k \end{matrix} \right]_q^{(2)} &= q^{k(k-1)/2} \left[ \begin{matrix} n \\ k \end{matrix} \right]_q, \\ \left[ \begin{matrix} n \\ k \end{matrix} \right]_q^{(\infty)} &= \left[ \begin{matrix} n+k-1 \\ k \end{matrix} \right]_q. \end{aligned}$$

Opinion: This is the reason why sometimes we multiply  $\left[ \begin{matrix} n \\ k \end{matrix} \right]_q$  by  $q^{k(k-1)/2}$  and sometimes we don't.

## 2. Symmetric Functions and $q$ -Analogues

Example:  $n = 3$  and  $k = 3$  for various values of  $b$ :

0	1	2	3	4	5	6
			030			
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$$\begin{bmatrix} 3 \\ 3 \end{bmatrix}_q^{(2)} = q^3,$$

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix}_q^{(3)} = q^3 + q^1 + 2q^2 + 2q^4 + q^5,$$

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix}_q^{(\infty)} = q^3 + q^1 + 2q^2 + 2q^4 + q^5 + 1 + q^3 + q^6.$$

## 2. Symmetric Functions and $q$ -Analogues

### Remarks:

- Like the numbers  $\binom{n}{k}^{(b)}$ , the polynomials  $h_k^{(b)}(z_1, \dots, z_n)$  don't have a standard name or notation.
- Doty and Walker (1992) used  $h'_k(n)$  and called them *modular complete symmetric polynomials*.
- Fu and Mei (2020) used  $h_k^{[b-1]}$  and called them *truncated complete*.
- Grinberg (2022) used  $G(b, k)$  and called them *Petrie symmetric functions*. He would not have chosen the name *Petrie* if he had known about the previous occurrences (personal communication).
- Since the definition is simple I believe that the name should be simple. In the paper I called them  ***$b$ -bounded symmetric polynomials***.

## 2. Symmetric Functions and $q$ -Analogues

### Remarks:

- Doty and Walker (1992) mention the following generalization of Newton's identities, which they attribute to **Macdonald**:\*

$$h_k^{(b)}(z_1, \dots, z_n) = \frac{1}{k!} \det \begin{pmatrix} p_1^{(b)} & p_2^{(b)} & \dots & \dots & p_k^{(b)} \\ -1 & p_1^{(b)} & p_2^{(b)} & & \vdots \\ & -2 & p_1^{(b)} & p_2^{(b)} & \vdots \\ & & \ddots & \ddots & p_2^{(b)} \\ & & & -(k-1) & p_1^{(b)} \end{pmatrix}$$

where

$$p_m^{(b)} = \begin{cases} (1-b)(z_1^m + \dots + z_n^m) & b \mid m, \\ z_1^m + \dots + z_n^m & b \nmid m. \end{cases}$$

\* They did not express it as a determinant.

## 2. Symmetric Functions and $q$ -Analogues

### Remarks:

- This has an interesting consequence when  $z_1 = \cdots = z_n = 1$ :

$$\binom{n}{k}^{(b)} = \sum_{\lambda \vdash k} \frac{1}{z_\lambda} (1-b)^{l_b(\lambda)} n^{l(\lambda)},$$

where the sum is over  $(\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$  with  $\sum_i \lambda_i = k$ , and

$$l(\lambda) = \#\{i : \lambda_i \neq 0\},$$

$$l_b(\lambda) = \#\{i : b \mid \lambda_i\},$$

$$m_j = \#\{j : m_j = i\},$$

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!.$$

## 2. Symmetric Functions and $q$ -Analogues

### Remarks:

- In a recent paper (*Lattice points and  $q$ -Catalan*, 2024) I proved that

$$\frac{1}{[n+1]_q} \sum_{k=\ell}^m q^k \begin{bmatrix} n \\ k \end{bmatrix}_q^{(n+1)} \in \mathbb{Z}[q]$$

whenever  $\gcd(n+1, \ell-1) = \gcd(n+1, m) = 1$ , and I conjectured that the coefficients are positive. I called these  *$q$ -Catalan germs*.

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- I don't know how this generalizes to  $b \neq n+1$ .

### 3. A Bit of Galois Theory



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Our main theorem will compute

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q^{(b)} \quad \text{when } q \rightarrow \text{roots of unity.}$$

Before stating the theorem, it is worthwhile to mention a **very general phenomenon**, which follows from some basic Galois theory. This phenomenon is surely well known but I have not seen it written down.

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#### Observation

Let  $f(z_1, \dots, z_n) \in \mathbb{Z}[z_1, \dots, z_n]$  be **symmetric polynomial in  $n$  variables** and let  $\omega$  be a **primitive  $d$ th root of unity** for some  $d$ .

- (a) If  $d|n$  then  $f(1, \omega, \dots, \omega^{n-1}) = f(\omega, \dots, \omega^n)$  is an integer.\*
- (b) If  $d|(n-1)$  then  $f(1, \omega, \dots, \omega^{n-1})$  is an integer.
- (c) If  $d|(n+1)$  then  $f(\omega, \dots, \omega^n)$  is an integer.

\* If  $\deg(f) = k$  and  $d \nmid k$  then this integer is zero.

### 3. A Bit of Galois Theory

Proof Sketch: (1) Let  $\omega$  be a primitive  $d$ th root of unity and consider the field extension  $\mathbb{Q}(\omega)/\mathbb{Q}$ . The Galois group is

$$\text{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = \{\varphi_r : \gcd(r, d) = 1\},$$

where  $\varphi_r : \mathbb{Q}(\omega) \rightarrow \mathbb{Q}(\omega)$  is defined by  $\varphi_r(\omega) := \omega^r$ . If  $\alpha \in \mathbb{Z}[\omega]$  satisfies  $\varphi_r(\alpha) = \alpha$  for all  $\gcd(r, d) = 1$  then Galois theory tells us that  $\alpha \in \mathbb{Z}$ .

(2) Consider the sequence  $\omega := (\omega, \dots, \omega^{d-1})$ . If  $\gcd(r, d) = 1$  then  $\varphi_r$  permutes the sequence  $\omega$ , hence it permutes sequences of the following four types:

$$(1, \omega, \dots, \omega, 1),$$

$$(\omega, 1, \dots, \omega, 1),$$

$$(1, \omega, 1, \omega, \dots, \omega, 1),$$

$$(\omega, 1, \omega, 1, \dots, 1, \omega).$$

□

### 3. A Bit of Galois Theory

#### Corollary

Let  $\omega$  be a primitive  $d$ th root of unity.

(a) If  $d|n$  then\*

$$\left[ \begin{matrix} n \\ k \end{matrix} \right]_{\omega}^{(b)} = h_k^{(b)}(1, \omega, \dots, \omega^{n-1}) = h_k^{(b)}(\omega, \dots, \omega^n) \in \mathbb{Z}.$$

(b) If  $d|(n-1)$  then

$$\left[ \begin{matrix} n \\ k \end{matrix} \right]_{\omega}^{(b)} = h_k^{(b)}(1, \omega, \dots, \omega^{n-1}) \in \mathbb{Z}.$$

(c) If  $d|(n+1)$  then

$$\omega^k \left[ \begin{matrix} n \\ k \end{matrix} \right]_{\omega}^{(b)} = h_k^{(b)}(\omega, \dots, \omega^n) \in \mathbb{Z}.$$

\* If  $d \nmid k$  then this integer is zero.

Our main theorem will compute these integers.

## 4. The Main Theorem and Cyclic Sieving



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(a) If  $d|n$  then  $\sum_k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = (1 + t^d + \dots + (t^d)^{b-1})^{n/d}$ , i.e.,

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \binom{n/d}{k/d} \geq 0.$$

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(b) If  $d|(n-1)$  then

$$\sum_k \left[ \begin{matrix} n \\ k \end{matrix} \right]_{\omega}^{(b)} t^k = (1 + t + \dots + t^{b-1})(1 + t^d + \dots + (t^d)^{b-1})^{(n-1)/d}, \text{ i.e.,}$$

$$\left[ \begin{matrix} n \\ k \end{matrix} \right]_{\omega}^{(b)} = \sum_{\ell} \binom{(n-1)/d}{(k-\ell)/d} \geq 0.$$

## 4. The Main Theorem and Cyclic Sieving

### Main Theorem (in three parts)

Let  $\omega$  be a **primitive  $d$ th root of unity** with  $\gcd(b, d) = 1$ .

(c) If  $d|(n+1)$  then

$$\sum_k \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{(1 + t^d + \dots + (t^d)^{b-1})^{(n+1)/d}}{1 + t + \dots + t^{b-1}} \in \mathbb{Z}[t].$$

These coefficients are **sometimes negative** and are more difficult to describe. We will give an explicit formula below in terms of the Frobenius Coin Problem.

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Remark: My paper also gives explicit generating functions for (a),(b),(c) when  $\gcd(b, d) \neq 1$ , which are more complicated.

## 4. The Main Theorem and Cyclic Sieving

Parts (a) and (b) have a nice combinatorial interpretation, in terms of **cyclic sieving** (Reiner-Stanton-White, 2004). Again, consider the set of points in a diagonal slice of the integer box  $\{0, 1, \dots, b-1\}^n$ :

$$X = \{(x_1, \dots, x_n) \in \{0, 1, \dots, b-1\}^n : x_1 + x_2 + \dots + x_n = k\}.$$

This set is closed under permutations. Consider the following two permutations:

$$\rho \cdot (x_1, \dots, x_n) := (x_2, \dots, x_n, x_1),$$

$$\tau \cdot (x_1, \dots, x_n) := (x_2, \dots, x_{n-1}, x_1, x_n).$$

Note that  $\langle \rho \rangle \cong \mathbb{Z}/n\mathbb{Z}$  and  $\langle \tau \rangle \cong \mathbb{Z}/(n-1)\mathbb{Z}$ . Recall that we can identify  $X$  with  $k$ -subsets and  $k$ -multisubsets of  $\{1, \dots, n\}$  when  $b = 2$  and  $b = \infty$ .

## 4. The Main Theorem and Cyclic Sieving

### Corollary of Main Theorem

Let  $\omega$  be a primitive  $d$ th root of unity with  $\gcd(b, d) = 1$ .

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(a) If  $d|n$  then we have

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_{\omega}^{(b)} = \#\{\mathbf{x} \in X : \rho^{n/d}(\mathbf{x}) = \mathbf{x}\}.$$

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I find the condition  $\gcd(b, d) = 1$  surprising!

## 4. The Main Theorem and Cyclic Sieving

### Remarks:

- This result generalizes the **prototypical examples** of cyclic sieving (Theorem 1.1 in RSW) for  $k$ -subsets (when  $b = 2$ ) and  $k$ -multisubsets (when  $b = \infty$ ).
- I find it surprising that it was not already known to the experts.
- Our Main Theorem (a),(b) generalizes Prop 4.2 in RSW, which appears there as a random collection of identities.
- Main Theorem (c) has no analogue in RSW.
- It may be interesting to look at the integers  $f(\omega, \dots, \omega^n) \in \mathbb{Z}$  when  $d|(n+1)$  for other classes of symmetric polynomials.

## 5. The Frobenius Coin Problem



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Let  $\omega$  be a primitive  $d$ th root of unity with  $d|(n+1)$  and  $\gcd(b, d) = 1$ . Recall that

$$\sum_k \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{(1 + t^d + \dots + (t^d)^{b-1})^{(n+1)/d}}{1 + t + \dots + t^{b-1}} \in \mathbb{Z}[t].$$

The integers  $\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)}$  are not directly related to cyclic sieving.

## 5. The Frobenius Coin Problem

Let  $\omega$  be a primitive  $d$ th root of unity with  $d|(n+1)$  and  $\gcd(b, d) = 1$ . Recall that

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We want to study the coefficients of the polynomial

$$\frac{[b]_{t^d}}{[b]_t} \in \mathbb{Z}[t].$$

It turns out these coefficients are related to the **Frobenius Coin Problem**.

## 5. The Frobenius Coin Problem

Given integers  $\gcd(b, d) = 1$ , consider the function  $\nu_{b,d} : \mathbb{N} \rightarrow \mathbb{N}$ ,

$$\nu_{b,d}(n) := \#\{(k, \ell) \in \mathbb{N}^2 : bk + d\ell = n\}.$$

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The set of **non-representable numbers** is finite, called the **Sylvester set**:

$$S_{b,d} = \{n \in \mathbb{N} : \nu_{b,d}(n) = 0\}.$$

For example,  $S_{3,5} = \{1, 2, 4, 7\}$ . Sylvester (1882) proved that

$$\#S_{b,d} = (b-1)(d-1)/2 \quad \text{and} \quad \max(S_{b,d}) = bd - b - d.$$

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Let us define the **Sylvester polynomial**

$$S_{b,d}(t) := \sum_{s \in S_{b,d}} t^s.$$

For example,  $S_{3,5}(t) = t + t^2 + t^4 + t^7$ .

## 5. The Frobenius Coin Problem

Brown and Shiue (1993) attribute the following result to Ozluk.

### Theorem (OzluK)

If  $\gcd(b, d) = 1$  then we have  $[b]_{t^d} / [b]_t = 1 + (t - 1)S_{b,d}(t)$ , i.e.,

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### Corollary

If  $\omega$  is a primitive  $d$ th root of unity with  $d|(n + 1)$ , it follows that

$$\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_{\omega}^{(b)} = \binom{(n+1)/d-1}{k/d}^{(b)} + \sum_{s \in S_{b,d}} \binom{(n+1)/d-1}{(k-1-s)/d}^{(b)} - \sum_{s \in S_{b,d}} \binom{(n+1)/d-1}{(k-s)/d}^{(b)}.$$

It is not clear from this formula when  $\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_{\omega}^{(b)}$  is positive or negative.

## 5. The Frobenius Coin Problem

Here is a cute formula, which allows us to be much more precise.

### Theorem

Let  $\gcd(b, d) = 1$ . For any  $r \in \mathbb{N}$ , let  $0 \leq \beta_r < b$  and  $0 \leq \delta_r < d$  satisfy

$$\beta_r \equiv rd^{-1} \pmod{b} \quad \text{and} \quad \delta_r \equiv rb^{-1} \pmod{d}.$$

Then

$$\frac{[b]_{t^d}}{[b]_t} = \frac{[d]_{t^b}}{[d]_t} = [\beta_1]_{t^d} [\delta_1]_{t^b} - t[b - \beta_1]_{t^d} [d - \delta_1]_{t^b}.$$

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### Corollary

Let  $\gcd(b, d) = 1$ . If  $\omega$  is a primitive  $d$ th root of unity and  $d|(n+1)$  then

$$\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} \text{ is } \begin{cases} \geq 0 & \text{when } \delta_k < \delta_1, \\ \leq 0 & \text{when } \delta_k \geq \delta_1. \end{cases}$$

## 5. The Frobenius Coin Problem

I really like this theorem because it has a geometric interpretation.

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14	19	24	.	.	.	.	.	.	.	.	.
7	12	17	.	.	.	.	.	.	.	.	.
0	5	10	15	20	25	30	35	.	.	.	.
		3	8	13	18	23	28	.	.	.	.
			1	6	11	16	21	.	.	.	.
					4	9	14	19	24	.	.
						2	7	12	17	.	.
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Example: Let  $(b, d) = (7, 5)$ . Draw an infinite array starting at 0, adding 5 for each right step and subtracting 7 for each down step.

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The [Sylvester set](#) forms a triangle:

$$S_{7,5} = \{1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23\}.$$

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In this case we have  $(\beta_1, \delta_1) = (3, 3)$ , which tells us that the label 1 occurs in position  $(\beta_1, \delta_1 - d) = (3, -2)$ .

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The cute formula describes two rectangles with bottom corners at **0** and **1**.

$$[b]_{t^d} / [b]_t = [\beta_1]_{t^d} [\delta_1]_{t^b} - t[b - \beta_1]_{t^d} [d - \delta_1]_{t^b}$$

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And this leads to a precise description of  $\omega^k [n]_{\omega}^{(7)}$  when  $\omega^5 = 1$ .

Obrigado!



Thanks to DeepSeek for suggesting the azulejos background image.